SWALLOWTAILS AND CONE-LIKE SINGULARITIES ON A MAXFACE

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ABSTRACT. When a connected component of the set of singular points of the maxface X consists of only generalized cone-like singular point, we construct a sequence of maxfaces X_n, with an increasing number of swallowtails, converging to the maxface X. We include the general discussion toward this.

1. INTRODUCTION

Maximal surfaces in the Lorentz-Minkowski space $\mathbb{E}_1^3$ are space-like immersions that maximize area locally. They are similar to the minimal surfaces in $\mathbb{R}^3$, as both are zero mean curvature surface and can be constructed using many similar methods but in the maximal surfaces non-isolated singularities appear.

Estudillo and Romero [2] named generalized maximal immersion for the maximal surfaces with singularities. They are of two types, branched and non branched. Non-branched maximal immersions are the one when limiting tangent plane contains a light-like vector. The Maxface, introduced by Umehara and Yamada [12], are non branched generalized maximal immersions.

Umehara and Yamada [12] have shown that all the maxface, as a map to $\mathbb{R}^3$, become frontal, and a few of them near the singularity become front. As a front and frontal, in [4], [5], [6], [12], [11], we see authors have discussed various singularities: swallowtails, cuspidal-edge, cuspidal cross cap, cuspidal butterflies, cuspidal S_1^-. There is another important singularity, called the cone-like singularity. Cone-like singularities, introduced by O. Kobayashi [10], are non isolated and discussed in [3], [5] etc.

In [7], Kim and Yang gave an example of family of maxfaces for each natural number n, that has swallowtails in an increasing order. We call it (a name given by the authors in [5]) Kim and Yang's toroidal maxface. The authors discussed the global properties of this toroidal maxface. The family of toroidal maxfaces seems to "converge" to the cone-like. Moreover, in [5], the authors talked about trinoids with swallowtails whose computer graphics look cone-like. All these poses a question, if we start from a maxface having a cone-like singularity (or generalized cone-like) then,
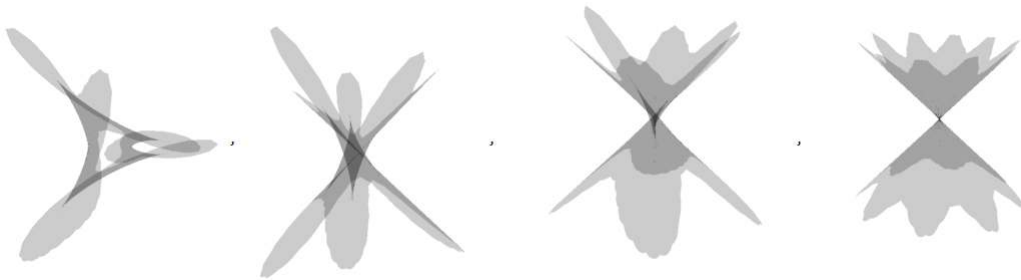
FIGURE 1. Maxfaces converging to the Lorentzian Catenoid

can we find a sequence of maxfaces having an increasing number of swallowtails, and converging to the former one?

In this article, we discuss the general situation, we start from a maxface with generalized cone-like singularity, and we construct the sequence in various cases. Let λ be a non-constant real analytic curve, whose trace is compact, when λ satisfies certain conditions, in section 4, we prove the following theorem.

Theorem (Theorem: 4.4). *Let X be a maxface having generalized cone-like singularity on trace of λ, with singular Björling data $\{\lambda, \alpha, \beta\}$. Moreover the data $\{\lambda, \alpha, \beta\}$ has a sequence of the scaling functions as in the definition 4.1. Then there is a sequence of maxfaces X_n defined on a neighborhood Ω of trace of λ, having an increasing number of swallowtails, and sequence of maxfaces converges (in the norm $\| \cdot \|_\Omega$) to X, having cone-like.*

A significant class of maxface (defined locally) can be constructed using the singular Björling problem introduced by Kim and Yang in [8] and also discussed in [1]. Given a null curve γ and a null vector field on the real axis or the unit circle, in [1], [8], authors have constructed the maxface. In section 3, we start with the singular Björling problem when we expect singularities lie on the trace of a non-constant smooth curve λ. Further in section 3, we give necessary and sufficient conditions (in propositions: 3.1, 3.2) on the singular Björling data (defined on the trace of λ) such that singularity is of swallowtail or generalized cone-like.

In section 4, we prove theorem 4.4 and an example of a sequence of maxfaces having an increasing number of swallowtails converging to the Lorentzian Catenoid. All the discussion is local that is near a singular curve.

2. Preliminaries

The Lorentz-Minkowski space $\mathbb{E}_1^3$ is a vector space $\mathbb{R}^3$ with metric $\langle \ , \ \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $\langle (a_1, b_1, c_1), (a_2, b_2, c_2) \rangle := a_1 a_2 + b_1 b_2 - c_1 c_2$, where (a_1, b_1, c_1) and (a_2, b_2, c_2) are two vectors in $\mathbb{R}^3$. In the following, we write the Weierstrass-Enneper representation of the maxface given by Umehara and Yamada [12].

2.1. **Weierstrass Enneper representation.** Let M be a Riemann surface, the map $X : M \to \mathbb{E}_1^3$ be a maxface if and only if there is a pair (g, f) of meromorphic function and a holomorphic 1-form on M such that $(1 + |g|^2)^2 |f|^2$ gives a positive definite Riemannian metric on M, and $|g|$ is not identically equal 1. Moreover for the map $\Phi := \left((\frac{1}{2}(1 + g^2), \frac{i}{2}(1 - g)^2, -g)f \right)$, $Re \int_{\gamma_j} \Phi = 0$, $(j = 1, .., N)$ for loops $\{\gamma_j\}_{j=1}^N$ such that $[\gamma_j]$ are generators of $\pi_1(M)$ and $X(p) = Re \int_o^p \Phi$.

2.2. **Singularities.** Various singularities appear on the maxface. Here we review a few of these. We start by recalling the definition of $\mathcal{A}-$ equivalence as in [6], [12].

Two smooth maps $X : \Omega \subset \mathbb{R}^2 \to \mathbb{E}_1^3$ and $Y : \mathcal{V} \subset \mathbb{R}^2 \to \mathbb{E}_1^3$ are said to be $\mathcal{A}-$equivalent at the points $p \in \Omega$ and $q \in \mathcal{V}$ if there exists a local diffeomorphism η of $\mathbb{R}^2$ with $\eta(p) = q$ and a local diffeomorphism Φ of $\mathbb{E}_1^3$ with $\Phi(X(p)) = Y(q)$ such that $Y = \Phi \circ X \circ \eta^{-1}$.

Definition 2.1 (Swallowtails [6], [12]). *A maxface $X : \Omega \subset \mathbb{C} \to \mathbb{E}_1^3$ is said to have a swallowtail at $p \in \Omega$ if at $p \in \Omega$ and $(0, 0) \in \mathbb{R}^2$, X is $\mathcal{A}$-equivalent to $f_{sw}(u, v) = (3u^4 + u^2v, 4u^3 + 2uv, v)$; $(u, v) \in \mathbb{R}^2$.*

Definition 2.2 (Shrinking singularity [8]). *For a maxface $X : \Omega \to \mathbb{E}_1^3$, we call $p \in \Omega$ a shrinking singular point if there is some $U_p \subset \Omega$ and a regular embedded curve $\gamma : I \subset U_p \to \Omega$, such that every point of $\gamma[I]$ singularity and $X \circ \gamma$ is a single point.*

Definition 2.3 (Generalized cone-like singularity and cone-like singularity [5]). *Let $\sum_0$ be the connected component of the set of all admissible singular points on a generalized maximal immersion $X : \Omega \to \mathbb{E}_1^3$. Each point of $\sum_0$ is called a generalized cone-like singular point if $\sum_0$ is compact and the image $X(\sum_0)$ is a single point. Moreover, if there is a neighborhood U of $\sum_0$ such that $X(U \setminus \sum_0)$ is embedded, then each point of $\sum_0$ is called a cone-like singular point.*

If the trace of λ is compact and it is a connected component of the singularity, then the shrinking singularity becomes generalized cone-like. For examples and detailed discussion, we refer to [6], [8], [12] etc.

For a maxface $X : \Omega \to \mathbb{E}_1^3$ with the Weierstrass data $\{g, f\}$, the authors in [5], [8], [12] have given a very useful criterion to check the nature of singularity at a particular point. For a maxface, if p is a singularity, then there exists a curve λ on a neighborhood of p such that every point on the trace of λ is a singularity. The curve λ is said to be a singular curve and the vector η along λ is said to null curve if $dX_{\lambda(t)}(\eta) = 0$. Umehara and Yamada [12] have shown that in a maxface, such direction is uniquely determined and calculated in terms of Weierstrass data.

Following [12], in terms of the Weierstrass data $\{g, f\}$, and in terms of η, λ, we recall the functions:

Definition 2.4 ([12]). *On a neighborhood U_p of a singularity p, we define $A(z) := \frac{g'}{g^2 f}$.*

Since λ is a singular curve parameterized on I, we define $H(t) = Det(\lambda'(t), \eta(t))$ for all $t \in I$.

Moreover[12] a maxface becomes front on a neighborhood of singularity if $Re(A(\lambda(t))) \neq 0$ at the singular points and as a front, singularity at $p = \lambda(t)$ is swallowtail if and only if $H'(t) \neq 0$.

3. Singular Björling problem and the necessary and sufficient conditions for few singularities

In this section, we will discuss the singular Björling problem in a general case when we expect the singularities to lie on the trace of a smooth non-constant curve. It is similar to the discussion in [1], [8]. Further, we will find the necessary and sufficient conditions on the singular Björling data so that singularities are of generalized cone-like or swallowtails.

3.1. Singular Björling problem on a curve.

We start by defining the singular Björring data similar to the way Kim and Yang have given in [8].

Definition 3.1 (Singular Björling data)**.** *This is a triplet $\{\lambda, \alpha, \beta\}$, where*
  (1) *$\lambda : I \to \mathbb{C}$ be a smooth curve such that for all $t \in I$, $\lambda'(t) \neq 0$.*
  (2) *α be a null curve and β be a null vector field defined on I and for all $t \in I$, $\langle \alpha'(t), \beta(t) \rangle = 0$. Moreover, either $\alpha'(t) \neq 0$ or $\beta(t) \neq 0$ for all $t \in I$.*
  (3) *The map given by $\phi(\lambda(t)) = \alpha'(t) - i\beta(t)$ has an extension $\phi(z)$ on the trace of λ.*
  (4) *Analytic extension $g(w)$ of $g(\lambda(t)) = G(t)$ where*

$$(3.1) \qquad G(t) := \begin{cases} \frac{\alpha_1'(t) + i\alpha_2'(t)}{\alpha_3'(t)} & \text{if } \alpha'(t) \neq 0 \\ \frac{\beta_1(t) + i\beta_2(t)}{\beta_3(t)} & \text{if } \beta_3(t) \neq 0 \end{cases}$$

  and $|g(w)|$ is not identically equal to 1.

For the data $\{\lambda, \alpha, \beta\}$, if $Re \int_{\gamma_j} \phi(z)dz$ vanishes for all loops γ_j on a neighborhood Ω of trace of λ. Then the map $X := (X_1, X_2, X_3) : \Omega \to \mathbb{E}_1^3$, given by

$$(3.2) \qquad X_{\lambda,\alpha,\beta}(z) = Re\left(\int_{\lambda(t_0)}^{z} \phi(w)dw \right)$$

is well defined and it gives a generalized maximal immersions on Ω, containing the trace of λ. Moreover the trace of λ is a subset of the singular set. The generalized maximal immersion given in the equation 3.2 is unique up to translations (we have many choices of $\lambda(t_0)$).

Example 3.2. *Let $\lambda(t) = (t, 0)$, and α and β be real analytic curve and vector field respectively. Then $\phi(z) = \alpha'(z) - \beta(z)$ is analytic and it gives a generalized maximal immersion. This is the case of the singular Björling problem as introduced and discussed by Kim and Yang in [8].*

Example 3.3. *Let $\lambda(t) = e^{it}$, $0 \le t \le 2\pi$. Let $\alpha'(t) = (\cos(t), \sin(t), 1)$ and $\beta(t) = (0, 0, 0)$, then*

$$\phi(e^{it}) = (\cos(t), \sin(t), 1)$$
$$= \left(\frac{e^{it} + e^{-it}}{2}, \frac{e^{it} - e^{-it}}{2i}, 1 \right)$$

it has analytic extension $\phi(z) = (\frac{z}{2} + \frac{1}{2z}, i(\frac{1}{2z} - \frac{z}{2}), 1)$ in a annular region containing unit circle. It gives a generalized maximal immersion in an annular region that contains the unit circle.

The generalized maximal immersion as in the equation 3.2 is a maxface on a neighborhood of trace of λ. Moreover, for $z = \lambda(t)$, $f(z) = \phi_1(z) - i\phi_2(z) = (\alpha'_1(t) + \beta_2(t)) - i(\alpha'_2(t) + \beta_1(t))$. Therefore f is never zero on Trace of λ.

The Gauss map is given by

$$g(z) = -\frac{\phi_3(z)}{\phi_1(z) - i\phi_2(z)} = -\frac{\phi_1(z) + i\phi_2(z)}{\phi_3(z)},$$

and $g(\lambda(t)) = G(t)$, as in the equation 3.1.

3.2. **Null direction.** Let $X_{\lambda,\alpha,\beta} : \Omega \to \mathbb{E}_1^3$ be the maxface for a fixed t_0 as in the equation 3.2. Since $dX = \frac{1}{2}X_z dz + \frac{1}{2}X_{\bar{z}} d\bar{z} = \phi(z)dz + \overline{\phi(z)}d\bar{z}$. At $z = \lambda(t)$, we have

$$dX_{\lambda(t)} = (\alpha'(t) - i\beta(t))dz_{\lambda(t)} + (\alpha'(t) + i\beta(t))d\bar{z}_{\lambda(t)}$$

So let $v = \eta \frac{\partial}{\partial \bar{z}} + \bar{\eta} \frac{\partial}{\partial \bar{z}}$ such that $dX_{\lambda(t)}(v) = 0$, we have $(\alpha'_3(t) - i\beta_3(t))\eta + (\alpha'_3(t) + i\beta_3(t))\bar{\eta} = 0$, that is:

$$\alpha'_3(t).Re(\eta) + \beta_3(t)Im(\eta) = 0.$$

Therefore the null direction at each $\lambda(t)$ is given by

$$(3.3) \qquad\qquad \eta(t) = -\beta_3(t) + i\alpha'_3(t).$$

3.3. **Calculation for $A(t)$ and $H'(t)$.** In the following, we will calculate $A(\lambda(t))$ and $H'(t)$ as given in the equation 2.4. We write it here again

$$A(\lambda(t)) = \frac{G'(t)}{G^2(t)\lambda'(t)f(\lambda(t))}, \text{ and } H(t) = Det(\lambda'(t), \eta(t)).$$

We define

$$(3.4) \qquad D(\beta_{12}, \beta'_{12}) := \beta_1\beta'_2 - \beta_2\beta'_1, \quad D(\alpha'_{12}, \alpha''_{12}) := \alpha'_1\alpha''_2 - \alpha'_2\alpha''_1.$$

We have $g'(\lambda(t)) = \frac{G'(t)}{\lambda'(t)}$, where G is given by the equation 3.1.

For the case when $\beta_3(t) \ne 0$, a straight calculation gives,

$$\frac{G'(t)}{G^2(t).f(\lambda(t))} = \frac{-D(\beta_{12}, \beta'_{12})}{(\alpha'^2_3 + \beta^2_3)\beta_3} + i\frac{\alpha'_3 D(\beta_{12}, \beta'_{12})}{(\alpha'^2_3 + \beta^2_3)\beta^2_3}.$$

Therefore we

$$A(\lambda(t)) = \left(\frac{g'(\lambda(t)}{g^2(\lambda(t)f(\lambda(t))}\right) = \frac{G'(t)}{G^2(t).f(\lambda(t))}.\frac{\lambda_1'(t) - i\lambda_2'(t)}{\lambda_1'^2(t) + \lambda_2'^2(t)}$$

$$= \frac{D(\beta_{12}, \beta_{12}')}{(\alpha_3'^2 + \beta_3^2)\beta_3^2.(\lambda_1'^2(t) + \lambda_2'^2(t))}.(-\beta_3 + i\alpha_3').(\lambda_1'(t) - i\lambda_2'(t))$$

$$= \frac{D(\beta_{12}, \beta_{12}')}{(\alpha_3'^2 + \beta_3^2)\beta_3^2.(\lambda_1'^2 + \lambda_2'^2)}\left((-\beta_3\lambda_1' + \alpha_3'\lambda_2') + i.(\alpha_3'\lambda_1' + \beta_3\lambda_2')\right)$$

Similarly, we can calculate for the case when $\alpha_3'(t) \neq 0$ and we get the following.
(3.5)

$$A(\lambda(t)) := \begin{cases} \frac{D(\alpha_{12}', \alpha_{12}'')}{(\lambda_1'^2 + \lambda_2'^2)(\alpha_3'^2 + \beta_3^2)\alpha_3'^2}\left((-\beta_3\lambda_1' + \alpha_3'\lambda_2') + i.(\alpha_3'\lambda_1' + \beta_3\lambda_2')\right) & \text{if } \alpha_3'(t) \neq 0 \\ \frac{D(\beta_{12}, \beta_{12}')}{(\alpha_3'^2 + \beta_3^2)\beta_3^2.(\lambda_1'^2 + \lambda_2'^2)}\left((-\beta_3\lambda_1' + \alpha_3'\lambda_2') + i.(\alpha_3'\lambda_1' + \beta_3\lambda_2')\right) & \text{if } \beta_3(t) \neq 0. \end{cases}$$

Now we calculate the function $H(t)$. The null direction is given by the equation 3.3. We get $H(t) = \lambda_1'\alpha_3' + \lambda_2'\beta_3$ and therefore we have

$$(3.6) \qquad\qquad H'(t) = \lambda_1''\alpha_3' + \lambda_1'\alpha_3'' + \lambda_2'\beta_3' + \lambda_2''\beta_3$$

3.4. **For shrinking and generalized cone-like.** The maxface $X_{\lambda,\alpha,\beta}$ as in the equation 3.2 has shrinking singularity on the trace of λ if and only if $X_{\lambda,\alpha,\beta}(\lambda(t))$ is constant and $g'(\lambda(t)) \neq 0$ for all $t \in I$, where g is the Gauss map.

This implies $\forall t \in I, DX_{\lambda(t)}(\lambda'(t)) = 0$ if and only if $\forall t \in I, \alpha'(t)\lambda_1'(t) + \beta(t)\lambda_2'(t) = 0$ if and only if for all $t \in I, Im(A(t) = 0$.

As $g'(\lambda(t)) = \dfrac{G'(t)}{\lambda'(t)} \neq 0$ and $Im(A(t)) = 0$, if and only if $\forall t \in I, Re(A(t)) \neq 0$.

From above we get, $\forall t \in I, -\beta_3\lambda_1' + \alpha_3'\lambda_2' \neq 0$. Therefore $Re(A(t)) \neq 0$ if and only if $D(\beta_1, \beta_2') \neq 0$ when $\beta_3 \neq 0$, and $D(\alpha_{12}', \alpha_{12}'') \neq 0$ when $\alpha_3'(t) \neq 0$.

Summarising everything here, we get the following.

Proposition 3.1. *The solution of Björling problem (as in the equation 3.2) on a neighborhood of the trace of λ, consists of only shrinking singularities on the trace of λ if and only if*

(1) $\forall t \in I, \alpha_3'(t)\lambda_1'(t) + \beta_3(t)\lambda_2'(t) = 0$
(2) $D(\beta_1, \beta_2') \neq 0$ when $\beta_3(t) \neq 0$, and $D(\alpha_{12}', \alpha_{12}'') \neq 0$ when $\alpha_3'(t) \neq 0$. Here $D(.\,,\,.)$ as in the equation 3.4.

In particular, if the trace of λ is compact, then the trace of λ consists of only generalized cone-like singularities.

Example 3.4. *When $\lambda(t) = e^{it}$, $0 \leq t < 2\pi$, Let $\alpha'(t) = \cos t(\cos t, \sin t, -1)$, and $\beta(t) = \sin t(\cos t, \sin t, -1)$. For this singular Björling data, the maxface as in the equation 3.2, satisfies conditions of the proposition 3.1. Moreover the trace of λ is unit circle. Therefore every point of the unit circle is a generalized cone-like singularity (in fact it is cone-like singularity).*

The Weierstrass data $\{g, w\}$ for this maxface is $\{g(z) = z, w(z) = \frac{1}{z^2}\}$. This is the Lorentzian Catenoid [9].

Example 3.5. *Let $\lambda(t) = (t, 0), a < t < b$, and $\alpha(t)$ be a constant curve. Let β be a real null analytic vector field defined on I such that $\forall t, (\beta_1' \beta_2 - \beta_2 \beta_1') \neq 0$ then solution as in equation 3.2 gives shrinking singularity.*

In particular, if $\beta(t) = (1 - t^2, 2t, 1 + t^2)$ then we see that for all $t \in \mathbb{R}$, $(\beta_1 \beta_2' - \beta_2 \beta_1') \neq 0$. Therefore, this β with any constant α gives a maxface.

Moreover, the Weierstrass data for this maxface (equation 3.2) is

$$\left\{ g(z) = -\frac{1 - z^2 + 2iz}{1 + z^2}, w(z) = -2z - i(1 - z^2) \right\}$$

To treat separately the case, when $\lambda(t) = (t, 0)$ and every point $(t, 0)$ is a shrinking singularity, we write the following particular case as the corollary.

Corollary 3.6. *Let $\lambda(t) = (t, 0)$ then the solution of singular Björling problem as in the equation 3.2 consist of shrinking singularity on a neighborhood of trace of λ if and only if*

(1) α is a constant curve.
(2) and for all $t, (\beta_1' \beta_2 - \beta_2 \beta_1') \neq 0$

3.5. **For Swallowtails.** Here we give necessary and sufficient conditions on the singular Björling data as in the definition 3.1, such that $X_{\lambda, \alpha, \beta}$ has swallowtails at some points on the trace of λ

Let $\{\lambda, \alpha, \beta\}$ be the singular Björling data and $X_{\lambda, \alpha, \beta}$ be its solution as in the equation 3.2 on a neighborhood of trace of λ. We denote it by X.

We know at $p = \lambda(t)$, X has swallowtails if and only if at $p = \lambda(t)$, $\mathrm{Re}(A(\lambda(t)) \neq 0$, $\mathrm{Im}(A(\lambda(t))) = 0$, and $H'(t) \neq 0$. Here A and H are functions as in the definition 2.4. At point $p = \lambda(t)$, $A(\lambda(t))$ and $H'(t)$ are given by equations 3.5 and 3.6

Putting everything together, we have the following criterion.

Proposition 3.2. *At $\lambda(t)$, solution of the Singular Björling problem as in the equation 3.2 has Swallowtails if and only if at t*

(1) $\alpha_3' \lambda_1' + \beta_3 \lambda_2' = 0$ and
(2) $\lambda_1'' \alpha_3' + \lambda_1' \alpha_3'' + \lambda_2' \beta_3' + \lambda_2'' \beta_3 \neq 0$
(3) when $\alpha'(t) \neq 0$, $D(\alpha_{12}', \alpha_{12}'') \neq 0$, and when $\beta'(t) \neq 0$, $D(\beta_{12}, \beta_{12}') \neq 0$

In particular, for $\lambda(t) = (t, 0)$ and trace of λ has shrinking singularity, we have

Corollary 3.7. *Solution of the singular Björling problem , when $\lambda(t) = (t, 0)$, has swallowtails at $\lambda(t)$ if and only if at t*
$\alpha_3' = 0$, β_3 and α_3'' are not zero and $D(\beta_{12}, \beta_{12}') \neq 0$

With the conditions in the propositions 3.1, 3.2, in the following, we have given a family of maxfaces X_n defined in an annular region containing the unit circle as

the singularity. Moreover for each n there is an increasing number of swallowtails on the unit circle.

Example 3.8. *For $n \geq 1$, consider the Björling data on $\lambda(t) = e^{it}$, $0 \leq t \leq 2\pi$. as $\alpha'_n(t) = \cos t(\cos t, \sin t, -1)$ and $\beta_n(t) = \left(\sin t + \frac{\cos nt}{n}\right)(\cos t, \sin t, -1)$.*
We have at each $z = e^{it}$

$$\alpha'_{n3}\lambda'_1 + \beta_{n3}\lambda'_2 = -\frac{1}{n}\cos t \cos nt$$

$$\lambda''_1\alpha'_{n3} + \lambda'_1\alpha''_{n3} + \lambda'_2\beta'_{n3} + \lambda''_2\beta_{n3} = \cos t \sin nt + \frac{1}{n}\sin t \cos nt.$$
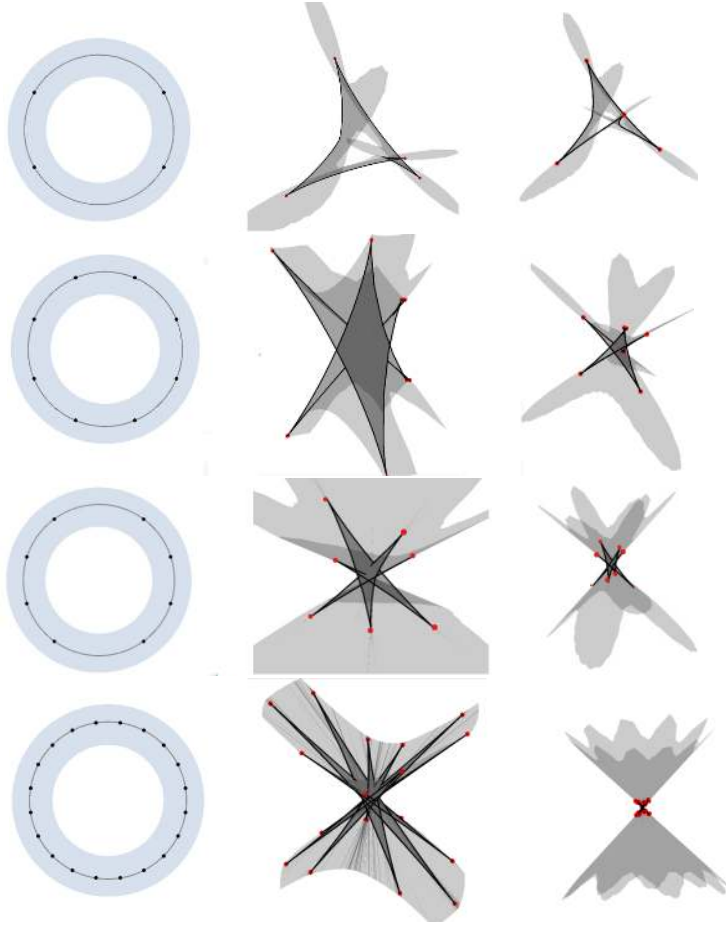
From the proposition 3.2, we see

FIGURE 2. A sequence of maxfaces with an increasing number of swallowtails as in the example 3.8, for n=3, 4, 5 and 10.

- *when n is even then $\forall z = e^{it_k}$ where $t_k = \dfrac{(2k+1)\pi}{2n}$, $k = 0, 1, ..., 2n-1$, are swallowtails for these maxfaces and*
- *when n is odd then $\forall z = e^{it_k}$ where $t_k = \dfrac{(2k+1)\pi}{2n}$, $k = \{0, 1, ..., 2n-1\} \setminus \{\frac{n-1}{2}, \frac{3n-1}{2}\}$ are swallowtails, So there are in total 2n-2 swallowtails.*

The Weierstrass data for each $X_{\lambda, \alpha_n, \beta_n}$ is given by

$$\left\{ g_n(z) = z, w_n(z) = \left(\frac{1}{z^2} - \frac{i}{2nz}(z^n + z^{-n}) \right) \right\}$$

*In the figure **??**, for various n, we have shown the maxfaces in an annular region where these are defined. The dotted points on the unit circle are the points where the respective maxface has swallowtails. In the middle, we have shown the curve where $|z| = 1$ is mapped in $\mathbb{E}_1^3$ with singular points. The last column is for the maxfaces for $n = 3, 4, 5$, and 10. This figure captures the essence of this article.*

From the above picture, we see this sequence of maxfaces "converges" to the Lorentzian Catenoid. This example is similar (at least graphically) to Kim and Yang's Toroidal maxface [7] and Trinoids as in [5]. All these motivate us to study the general problem, we talked in the introduction.

4. Sequence of swallowtails converging to the generalized cone-like

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $E = \mathbb{R}^n$ or $\mathbb{C}^n$ and $X \in C(\overline{\Omega}, E)$, the space continuous maps. For each $z \in \overline{\Omega}$, we denote, $\|X(z)\| = \max\{|X_i(z)|: i = 1 \ldots n\}$ and $\|X\|_\Omega := \sup_{z \in \overline{\Omega}} \|X(z)\|$. The space $C(\overline{\Omega}, E)$ becomes a Banach space under the norm $\| \cdot \|_\Omega$. For convergence of sequence of maxfaces, we will be using these norms.

4.1. **For the shrinking singularities on $\lambda(t) = (t, 0)$.** In the following, we see that for a maxface X with the Björling data $\{\lambda(t) = (t, 0), \alpha, \beta\}$ and having shrinking singularities on each $(t, 0)$, we have a sequence of maxfaces X_n, with an increasing number of swallowtails, converging to X in a domain containing $\lambda(t)$. Precisely we prove the following-

Proposition 4.1. *Let $X : \mathcal{U} \to \mathbb{E}_1^3$ be a maxface, for a singular Burling data $\{\lambda, \alpha, \beta\}$ defined on $\lambda : [a, b] \to \mathcal{U}$, $t \to (t, 0)$, and X has shrinking singularities on $(t, 0)$. Then there is a domain Ω containing trace of λ and a sequence of maxfaces X_n, such that*

- *(1) X and X_n defined on Ω*
- *(2) Each X_n has n swallowtails on the trace of λ, moreover*
- *(3) X_n converges to X in $\| \cdot \|_\Omega$*

Proof. Without loss of generality, we take $I = [0, 1]$. The maxface $X_{\lambda,\alpha,\beta}$ for the data $\{\lambda, \alpha, \beta\}$ has shrinking singularities each $(t, 0)$, therefore α and β satisfy the conditions given in Proposition 3.1.

We take Ω such that $I \subset \Omega \subset \overline{\Omega} \subset \mathcal{U}$ and $\overline{\Omega} \subset \{z \in \mathbb{C} : |z| < r\}$ for some $r > 0$. Since I is closed interval, such Ω exits.

For natural number $n \geq 1$, we take a polynomial,

$$(4.1) \qquad\qquad f_n(z) := \frac{1}{(r+1)^n . n} \prod_{i=1}^{n} (z - \frac{1}{i})$$

We see that for each $z \in \overline{\Omega}$, $|f_n(z)| < \frac{1}{n}$. Therefore $f_n \to zero$ uniformly on $\overline{\Omega}$. For each $t \in [0, 1]$, we define

$$\alpha_n(t) = \int_{t_0}^{t} f_n(t)\beta(t), \;\; \beta_n(t) = \beta(t).$$

For each n, $\{t \to (t, 0), \alpha_n, \beta_n\}$ satisfies to be the data of singular Björling problem. Moreover at each $t_i = \frac{1}{i}$, these satisfies all the conditions in the corollary 3.7.

Therefore each maxface X_n (the solution of singular Björling problem for $\{\alpha_n, \beta_n\}$) has swallowtails at each $t_i = \frac{1}{i}$.

Moreover for $z \in \overline{\Omega}$, $\alpha'_n(z) = f_n(z)\beta(z)$ and each $\beta_i(z)$ is bounded on $\overline{\Omega}$. Hence on $\overline{\Omega}$, $\|\alpha'_n\|_\Omega \to zero$.

Let fix some point $t_0 \in I$ and for $z \in \overline{\Omega}$, let $|\int_{t_0}^{z} dw| < L$. We denote h^k for k-th co-ordinate function for $h = (h^1, h^2, h^3)$.

We have , $\forall z \in \overline{\Omega}$,

$$|X_n^k(z) - X^k(z)| = \left| Re \int_{t_0}^{z} (\alpha_n^{k\prime} - i\beta_n^k) - (\alpha^{k\prime} - i\beta^k(t))dw \right|$$

$$\leq \left| \int_{t_0}^{z} (\alpha_n^{k\prime} - \alpha^{k\prime})dw \right| + \left| \int_{t_0}^{z} (\beta_n^k - \beta^k)dw \right|$$

$$\leq L . (\|\alpha'_n - \alpha'\|_\Omega + \|(\beta_n - \beta)\|_\Omega)$$

Therefore we have

$$\|X_n - X\|_\Omega \leq L . (\|\alpha'_n - \alpha'\|_\Omega + \|(\beta_n - \beta)\|_\Omega) \leq L\|\alpha'_n\|_\Omega \to zero.$$

This proves that X_n converges to X. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Now in the following, we will discuss a similar situation for arbitrary λ. For that, we give the following definition.

Definition 4.1. *We say a sequence $\{f_n\}$, of functions defined on I,* ***a sequence of scaling functions for the singular Björling data*** $\{\lambda, \alpha, \beta\}$ *if for each n, there are points $t_{n_k} \in I$, $k = 0, 1 \ldots a_n$, and $a_n \geq n$ such that,*

(1) *On the set $\{t_{n_k} : k = 0, 1, .., a_n\}$, λ is 1-1.*
(2) *$\forall k = 0, 1, .. a_n$, $f_n(t_{n_k}) = 0$ and $f'_n(t_{n_k}) \neq 0$.*

(3) $g_n(\lambda(t)) = f_n(t)(\alpha'(t) - i\beta(t))$ has analytic extension on a neighborhood Ω, of the trace of λ.

(4) $Re \int_{\gamma_i} g_n = 0$ for each closed curve γ_i in the domain Ω as in (3).

(5) $\|g_n\|_\Omega \to zero$.

For arbitrary data $\{\lambda, \alpha, \beta\}$, it is not direct to say whether such function exists or not, but for the particular cases, we can say.

Example 4.2. *Let the singular Björling data be as in the example 3.5 with $\lambda(t) = (t, 0)$ on $[0, 1]$. Then the sequence of functions as defined in the equation 4.1 on I with $t_{n_k} = \frac{1}{k}$; $k = 1, .., n$ turns out to be the sequence of scaling functions.*

Example 4.3. *Let $\lambda(t) = e^{it}$, $0 \le t \le 2\pi$ and α, β as in the example 3.4. We take $f_n(t) = \frac{1}{2^{n+2} \cdot n} \cos nt$. Let $\Omega = Ann(0, \frac{1}{2}, 2) := \{\frac{1}{2} < |z| < 2\}$.*

We see, for all n, f_n is zero at each $t_{n_k} = (2k+1)\frac{\pi}{2n}$; $k = 0, \ldots, 2n-1$. Moreover $f_n'(t_{n_k}) \ne 0$. Here $a_n = 2n - 1 > n$. We have

$$g_n(e^{it}) = \frac{1}{2^{n+2}n} \cos nt(\alpha'(t) - i\beta(t))$$

$$= \frac{1}{2^{n+2}n} e^{-it} (\cos t, \sin t, -1) \cos(nt)$$

$$= \frac{1}{2^{n+2}n} e^{-it} \left(\frac{e^{it} + e^{-it}}{2}, \frac{e^{it} - e^{-it}}{2i}, -1 \right) \left(\frac{e^{int} + e^{-int}}{2} \right).$$

Each $g_n(e^{it})$ has analytic extension $g(z)$ in $z = e^{it}$. Moreover $Re \int_{\gamma_i} g_n(z)dz = 0$ for each closed curve γ_i in Ω.

We denote, $g_n = (g_{n1}, g_{n2}, g_{n3})$, for all $z \in \overline{\Omega}$, we see, for each $i = 1, 2, 3$, $|g_{ni}(z)| < \frac{1}{n}$, therefore $\|g_n(z)\| < \frac{1}{n}$. It gives $\|g_n\|_\Omega = \sup_{z \in \overline{\Omega}} \|g_n(z)\| < \frac{1}{n} \to 0$

4.2. **For general singular Björling problem.** Let λ be a curve whose trace is compact and α and β are the singular Björling data as in the definition 3.1, that have cone-like singularity on the trace of λ, so data $\{\lambda, \alpha, \beta\}$ satisfies condition of the Proposition 3.1.

Since we have for all $t \in I$ $\alpha_3'\lambda_1' + \beta_3\lambda_2' = 0$, $(\alpha_3'(t), \beta_3(t)) \ne (0, 0)$, $(\lambda_1', \lambda_2') \ne (0, 0)$, therefore we have for all t,

(4.2) $$\forall t \in I, \quad -\beta_3(t)\lambda_1'(t) + \alpha_3'(t)\lambda_2'(t) \ne 0.$$

Suppose there is a sequence of the scaling functions for the $\{\lambda, \alpha, \beta\}$, as in the definition 4.1. We define

$$\alpha_n'(t) = \alpha'(t) + f_n(t)\beta(t), \quad \beta_n(t) = \beta(t) - f_n(t)\alpha'(t)$$

For any t, both α_n' or β_n are not zero. Moreover,

$$\psi_n(\lambda(t)) = \alpha_n'(t) - i\beta_n(t) = \alpha'(t) - i\beta(t) + f_n(t)(\beta(t)) + i\alpha'(t))$$

Since for the Björling data, $\{\lambda, \alpha, \beta\}$, we have f_n and g as in the definition 4.1. We get

$$\psi_n(\lambda(t)) = \phi(\lambda(t)) - ig_n(\lambda(t)).$$

These ψ_n have analytic extension in variable $z = \lambda(t)$. For each n the triplet $\{\lambda, \alpha_n, \beta_n\}$ satisfy the condition to be a singular Björling data as in the definition 3.1.

Let t_{n_k} be as in the definition 4.1 and $\alpha'_n = (\alpha'_{n1}, \alpha'_{n2}, \alpha'_{n3})$ and $\beta_n = (\beta_{n1}, \beta_{n2}, \beta_{n3})$. At each $t_{n_k} \in I$, we have

$$\alpha'_{n3}(t_{n_k})\lambda'_1(t_{n_k}) + \beta_{n3}(t_{n_k})\lambda'_2(t_{n_k})$$
$$= \alpha'_3(t_{n_k})\lambda'_1(t_{n_k}) + \beta_3(t_{n_k})\lambda'_2(t_{n_k}) + f_n(t_{n_k})(\alpha'_3(t_{n_k})\lambda'_1(t_{n_k}) - \beta_3(t_{n_k})\lambda'_2(t_{n_k}))$$

Since for all $t \in I$ $\alpha'_3\lambda'_1 + \beta_3\lambda'_2 = 0$ and $f(t_{n_k}) = 0$ we have for all t_{n_k},

$$(4.3) \qquad\qquad \alpha'_{n3}(t_{n_k})\lambda'_1(t_{n_k}) + \beta_{n3}(t_{n_k})\lambda'_2(t_{n_k}) = 0$$

For each t, $\lambda''_1(t)\alpha'_3(t) + \lambda'_1(t)\alpha''_3 + \lambda'_2\beta'_3(t) + \lambda''_2(t)\beta_3(t) = 0$ and for each t_{n_k}, $f(t_{n_k}) = 0$, we get

$$\lambda''_1\alpha'_{n3} + \lambda'_1\alpha''_{n3} + \lambda'_2\beta'_{n3} + \lambda''_2\beta_{n3}(t_{n_k}) = f'_n(t_{n_k})\left(\beta_3(t_{n_k})\lambda'_1(t_{n_k}) - \alpha'_3(t_{n_k})\lambda'_2(t_{n_k})\right)$$

As $f'(t_{n_k}) \neq 0$ and using equation 4.2, we have

$$(4.4) \qquad\qquad \forall t_{n_k}, \quad \lambda''_1\alpha'_{n3} + \lambda'_1\alpha''_{n3} + \lambda'_2\beta'_{n3} + \lambda''_2\beta_{n3} \neq 0$$

We have $\alpha'_n(t_{n_k}) = \alpha(t_{n_k})$ and $\beta_n(t_{n_k}) = \beta(t_{n_k})$ therefore

$$(4.5) \qquad \text{when } \alpha'_n(t) \neq 0, \quad D(\alpha'_{n12}, \alpha''_{n12}) \neq 0, \quad \text{when } \beta'_n(t) \neq 0, \quad D(\beta_{n12}, \beta'_{n12}) \neq 0.$$

Therefore at each t_{n_k}, $k = 0, 1, .., a_n$, (1), (2) and (3) conditions (equations 4.3, 4.4, 4.5) of proposition 3.2 satisfies.

As λ is $1 - 1$ at these points, for each n we get $a_n + 1$ swallowtails on each X_n. Since $a_n > n$, we have increasing number of points on the trace of λ such the maxface X_n for the data $\{\lambda, \alpha_n, \beta_n\}$ has increasing numbers of swallowtails.

Let fix some point $t_0 \in I$ and we denote $X = (X^1, X^2, X^3)$, for $z \in \overline{\Omega}$, let $|\int_{\lambda(t_0)}^z dw| < L$. Then $\forall z \in \overline{\Omega}$, we for each $j = 1, 2, 3$, we get

$$|X_n^j(z) - X^j(z)| = |Re \int_{\lambda(t_0)}^z \psi_n^j(w) - \phi^j(w)dw|$$

$$= \left| Re \int_{\lambda(t_0)}^z -ig_n(w) \right|$$

$$\leq \|g_n\|_\Omega . \left| \int_{\lambda(t_0)}^z dw \right| \leq \|g_n\|_\Omega L$$

Therefore we have

$$\|X_n - X\|_\Omega \leq \|g_n\|_\Omega L$$

and therefore $X_n \to X$ on $\overline{\Omega}$

Summarizing all above, we proved the following:

Theorem 4.4. *Let X be a maxface having generalized cone-like singularity on trace of λ, with singular Björling data $\{\lambda, \alpha, \beta\}$. Moreover the data $\{\lambda, \alpha, \beta\}$ has a sequence of the scaling functions as in the definition 4.1. Then there is a sequence of maxfaces X_n defined on a neighborhood Ω of trace of λ, having an increasing number of swallowtails, and sequence of maxfaces converges (in the norm $\| . \|_\Omega$) to X, having cone-like.*

Many examples can be constructed using the theorem 4.4. In particular, we have the following example similar to the one we have already seen 3.8.

Example 4.5. *Let $\lambda(t) = e^{it}$; $0 \leq t \leq 2\pi$, we start with the data $\alpha'(t)$ and β as in the example 3.4, so that every point on the unit circle is a cone-like singularity. For this Björling data, we have already seen there exists a sequence of scaling functions as in the example 4.3 To get a sequence of maxfaces having an increasing number of swallowtails; we start with the singular Björling data on λ as*

$$\alpha_n'(t) = \alpha'(t) + f_n(t)\beta(t) = \left(\cos t + \frac{\cos nt \sin t}{n2^{n+2}}\right)(\cos t, \sin t, -1)$$
$$\beta_n(t) = \beta(t) - f_n(t)\alpha'(t) = \left(\sin t - \frac{\cos nt \cos t}{n2^{n+2}}\right)(\cos t, \sin t, -1)$$

We have $t_{n_k} = (2k+1)\frac{\pi}{2n}$, $k = 0, 1, ..., 2n-1$, and $a_n = 2n-1$, so by the theorem 4.4, every X_n has at least $2n$ swallowtails on the unit circle and that converges to the cone-like.

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