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PT-symmetric sextic potentials

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Abstract

The family of complex PT-symmetric sextic potentials is studied to show that for various cases the system is essentially quasi-solvable and possesses real, discrete energy eigenvalues. For a particular choice of parameters, we find that under supersymmetric transformations the underlying potential picks up a reflectionless part.

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1 Introduction

Searching for non-Hermitian PT-symmetric Hamiltonians has acquired much interest in recent times (see e.g. [1, 2, 3, 4] and references quoted therein). For one thing, a rather large subclass of such Hamiltonians has been found to possess real energy eigenvalues. For another, in at least some cases, it is seen that a complex shift of the coordinate $x \in (-\infty, \infty)$ does not affect the overall normalizability of the wave functions, while at the same time retaining the real character of the energy spectrum.

The purpose of this letter is twofold:

(i) We examine the general problem of a complex sextic potential from the point of view of determining exactly a finite number of eigenvalues and eigenfunctions. A suitable ansatz scheme leads us to find discrete real energy levels under quite general conditions.

(ii) We point out that some of our results can also be arrived at by performing a complex shift of the coordinate on the reduced sextic potential consisting of even-power terms only. However, our results cover a much greater ground. In particular, we find it possible to generate an additional complex reflectionless term in the sextic potential by employing supersymmetric transformations.

2 Complex sextic potentials and their solutions

To get started, let us consider the following general representation of a sixth-degree potential

$$V(x) = \sum_{i=1}^{6} c_i x^i, \tag{1}$$

satisfying the Schrödinger equation (in units $\hbar = m = 1$)

$$\left[-\frac{1}{2} \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x), \tag{2}$$

where for $V(x)$ to be PT symmetric, c_1, c_3, $c_5 \in i\mathbb{R}$, but c_2, c_4, $c_6 \in \mathbb{R}$.

We make the ansatz that the wave function is of the form

$$\psi(x) = f(x) \exp\left(-\sum_{j=1}^{4} b_j x^j \right), \tag{3}$$

where $f(x)$ is some polynomial function of x, which, for complex potentials, is typically of the type $\sum_{m=0}^{n} \alpha_m (\mathrm{i}x)^m$. For the real analogue of (1) consisting of even power terms only, $f(x)$ is known to have a given parity.

We focus on the following choices of f:

$$\begin{aligned}
&\text{(a)} \quad f(x) = 1, \\
&\text{(b)} \quad f(x) = x + a_0, \\
&\text{(c)} \quad f(x) = x^2 + a_1 x + a_0,
\end{aligned}$$

but can generalize to higher degrees as well. For complex potentials, a_0 is imaginary in (b), whereas a_1 is imaginary, but a_0 is real in (c).

Without going into the details of calculations, which are quite straightforward, let us summarize our results.

2.1 The $f(x) = 1$ case

Here the potential parameters are found to be related to the b's as

$$\begin{aligned}
c_1 &= -3b_3 + 2b_1 b_2, \qquad c_2 = -6b_4 + 3b_1 b_3 + 2b_2^2, \qquad c_3 = 4b_1 b_4 + 6b_2 b_3, \\
c_4 &= 8b_2 b_4 + \tfrac{9}{2} b_3^2, \qquad c_5 = 12 b_3 b_4, \qquad c_6 = 8b_4^2. \qquad (4)
\end{aligned}$$

Without loss of generality, we can choose $c_6 = \frac{1}{2}$ to fix the leading coefficient of $V(x)$. It gives $b_4 = \pm\frac{1}{4}$. We take the positive sign to ensure normalizibility of the wave function, which reads

$$\psi(x) = \exp\left(-b_1 x - b_2 x^2 - b_3 x^3 - \tfrac{1}{4} x^4\right). \qquad (5)$$

The associated energy level is given by

$$E = b_2 - \tfrac{1}{2} b_1^2. \qquad (6)$$

Now b_1 and b_3 imaginary make c_1, c_3, c_5 imaginary too, so we have a complex PT-symmetric potential. Note that the energy eigenvalue in such a case is real, and the corresponding wave function is PT-symmetric.

2.2 The $f(x) = x + a_0$ case

The wave function is of the form

$$\psi(x) = (x + a_0) \exp\left(-b_1 x - b_2 x^2 - b_3 x^3 - \tfrac{1}{4}x^4\right) \tag{7}$$

for

$$\begin{aligned} c_1 &= -6b_3 + 2b_1 b_2 + a_0, \qquad c_2 = -\tfrac{5}{2} + 3b_1 b_3 + 2b_2^2, \qquad c_3 = b_1 + 6b_2 b_3, \\ c_4 &= 2b_2 + \tfrac{9}{2}b_3^2, \qquad c_5 = 3b_3, \qquad c_6 = \tfrac{1}{2}. \end{aligned} \tag{8}$$

There is also a condition on a_0,

$$a_0^3 - 3b_3 a_0^2 + 2b_2 a_0 - b_1 = 0. \tag{9}$$

The energy is given by

$$E = -\tfrac{1}{2}b_1^2 + 3b_2 - 3a_0 b_3 + a_0^2. \tag{10}$$

Let us discuss some important special cases of this scheme.

2.2.1 $b_1 = b_3 = 0$

The condition (9) reduces to

$$a_0\left(a_0^2 + 2b_2\right) = 0. \tag{11}$$

(i) If $a_0 = 0$, then there is no imaginary term in the potential, and this corresponds to the $n = 0$, negative-parity result of ref. [5]. The energy eigenvalue is given by $E = 3b_2$ and shows a single level.

(ii) The other solution of (11), namely $a_0^2 = -2b_2$ can be studied according to whether $a_0^2 > 0$ or $a_0^2 < 0$.

If $a_0^2 > 0$, then a linear term is present in $V(x)$ with $c_1 = \pm\sqrt{2|b_2|}$, $c_2 = 2b_2^2 - \frac{5}{2}$, and $c_4 = 2b_2$. Of course $c_3 = c_5 = 0$. The energy is $E = b_2 < 0$. Thus we have two different real potentials with the same energy eigenvalue. The linear term breaks PT invariance of the potential and the wave function as well. So, in this respect, a_0 real can be viewed as an explicit symmetry breaking parameter.

On the other hand, if $a_0^2 < 0$, we get two different complex potentials, corresponding to $c_1 = \pm \mathrm{i}\sqrt{2b_2}$, with the same real energy eigenvalue:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^6 + 2b_2 x^4 + \left(2b_2^2 - \tfrac{5}{2}\right) x^2 \pm \mathrm{i}\sqrt{2b_2}\, x, && (12) \\ E &= b_2 > 0, && (13) \\ \psi(x) &= \left(x \pm \mathrm{i}\sqrt{2b_2}\right) \exp\left(-b_2 x^2 - \tfrac{1}{4}x^4\right), && (14) \end{aligned}$$

The potential (12) is PT symmetric, while the wave function (14) is odd under PT symmetry.

2.2.2 $b_1 = 0$, $b_3 \neq 0$

The solution for $a_0 = 0$ turns out to give the same conclusions as previously obtained. The second solution

$$a_0 = \frac{1}{2}\left(3b_3 \pm \sqrt{9b_3^2 - 8b_2}\right) \qquad (15)$$

yields two possibilities according as $b_3 \in \mathbb{R}$ or $b_3 \in \mathrm{i}\mathbb{R}$. If $b_3 \in \mathbb{R}$, we have $b_3^2 \geq \frac{8}{9}b_2$, implying two different real potentials with the same energy eigenvalue, except for the equality sign in (15).

If however $b_3 \in \mathrm{i}\mathbb{R}$, then a_0 must be imaginary with $b_3^2 = -|b_3|^2 \leq \frac{8}{9}b_2$. Here too we have two possibilities of obtaining two different complex potentials with the same real energy eigenvalue, except for the equality sign in (15).

2.3 The $f(x) = x^2 + a_1 x + a_0$ case

The complete set of solutions leading to more than one energy level corresponds to

$$a_1 = 2b_3, \qquad a_0 = \frac{1}{2}\left(2b_2 - b_3^2 \pm \sqrt{\left(2b_2 - 3b_3^2\right)^2 + 2}\right), \qquad (16)$$

and gives

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^6 + 3b_3 x^5 + \left(2b_2 + \tfrac{9}{2}b_3^2\right) x^4 + 2b_3\left(4b_2 - b_3^2\right) x^3 + \left[2\left(b_2^2 + 3b_2 b_3^2 - 3b_3^4\right) - \tfrac{7}{2}\right] x^2 \\ &\quad + b_3\left(4b_2^2 - 4b_2 b_3^2 - 7\right) x, && (17) \\ E_\pm &= -2b_3^2\left(b_2 - b_3^2\right)^2 + 3b_2 - b_3^2 \pm \sqrt{\left(2b_2 - 3b_3^2\right)^2 + 2}, && (18) \\ \psi_\pm &= \left[x^2 + 2b_3 x + \tfrac{1}{2}\left(2b_2 - b_3^2 \mp \sqrt{\left(2b_2 - 3b_3^2\right)^2 + 2}\right)\right] \\ &\quad \times \exp\left[-2b_3\left(b_2 - b_3^2\right) x - b_2 x^2 - b_3 x^3 - \tfrac{1}{4}x^4\right]. && (19) \end{aligned}$$

The results (16)–(19) are valid both for real and imaginary b's. In the latter case, PT symmetry is good for the potential and the wave function, whereas in the former one has a symmetry breaking. Note that in both cases $E_+ > E_-$.

Concerning the case where b_3 is imaginary, we have a complex PT-symmetric two-parameter family of potentials with two distinct real energy levels. This is truly a non-trivial result and puts the spirit of quasi-solvability in the complex domain. Indeed, for the particular case of $b_3 = 0$ and $b_2 = \gamma/2$, we recover the $n = 1$, positive-parity results of the one-dimensional even-power potential $V(x) = \frac{1}{2}x^6 + \gamma x^4 + \frac{1}{2}\left(\gamma^2 + \mu\right)x^2$, where $\mu = -3 - 4n - 2r$ and r is associated with the $(-1)^r$ parity of $n + 1$ levels.

The converse also works. The results (16)–(19) can be derived from the even-power sextic potential of ref. [5] by a translation $x \to x + b_3$. If b_3 is imaginary, then this translation amounts to a complex shift, whose viability has already been pointed out in refs. [2, 3].

3 A supersymmetric viewpoint

In order to enlarge the class of PT-invariant potentials, SUSY methods have been used thoroughly [2, 4]. Here, we also outline the procedure to generate superpartners of the potentials considered in the previous section, which share their PT properties. The procedure is based on the construction of a superpotential $W(x) = -\psi'(x)/\psi(x)$, and therefore is sound in so far as the logarithmic derivative of $\psi(x)$ is well behaved on the real axis.

One can start from the wave function (5), the parameters b_1 and b_3 being assumed imaginary. Then the superpartner

$$\tilde{V}(x) = V(x) - \left(\frac{\psi''(x)}{\psi(x)} - \frac{\psi'^2(x)}{\psi^2(x)}\right) \tag{20}$$

becomes again a sextic potential, and there is no real enlargement.

More interesting is the case where one starts from the wave function (14) and constructs the partner of (12). By substituting (14) into (20), we find $\tilde{V}$ to be

$$\tilde{V}(x) = V(x) + \left(\frac{1}{x \pm \mathrm{i}\sqrt{2b_2}}\right)^2 + 2b_2 + 3x^2, \tag{21}$$

where the piece in parentheses is clearly reflectionless. The latter is indeed reminiscent of the "transparent" complex potential obtained in ref. [2], which is invariant under PT and gives a trivial S-matrix. Moreover this piece has an associated zero-energy bound state, given by $\Psi_0(x) = C\left(x \pm \mathrm{i}\sqrt{2b_2}\right)^{-1}$, where C is a constant.[1]

The form (21) also generalizes the result obtained for the harmonic oscillator in ref. [3] to the sextic case. The fact that $b_2 \neq 0$ makes the potential rather appealing in that if b_2 were vanishing, there would be a singularity on the real axis and the potential would be rendered ill defined. Here, because of a shift of the singularity, (21) remains well posed in the complex plane cut from $x = \mp\mathrm{i}\sqrt{2b_2}$ to $x = \mp\mathrm{i}\infty$, respectively.

Finally, one can consider the wave function (19) and construct the partner of (17). Again one can see that $\tilde{V}(x)$ is not trivial: for b_3 imaginary, $\psi'(x)/\psi(x)$ is well behaved on the real axis.

All results we have obtained can be generalized to the case where one shifts the variable x by a translation b, with b real. In such a case, the potentials and wave functions are reparametrized correspondingly. One may worry however about PT properties since one can generate subleading powers in x from a given power. However, one should realize that now the parity operation can be defined with respect to a mirror placed at $x = -b$, so that $x + b = x - (-b)$ goes to $-x - 2b - (-b) = -(x + b)$. When this translation b is performed, the reflectionless potential contained in (21) becomes precisely that considered in ref. [2].

4 Conclusion

To conclude, we have solved a complex PT-symmetric sextic potential in its most general form within a suitable ansatz scheme for the wave functions and shown how the associated energy levels turn out to be real. We have also demonstrated using SUSY the possibility of generating a complex reflectionless part in the potential. Although we have restricted our discussion up to the quadratic order in the coefficient of the exponential representing the wave function, it is obvious that we can build up, in an identical way, higher-order states.

[1]This transparent potential may also be viewed as the particular case $l = 1$ of the generic potential $\frac{1}{2}l(l+1)\left(x \pm \mathrm{i}\sqrt{2b_2}\right)^{-2}$ that is typical of a centrifugal barrier in a radial context (but with no singularity) and which has a zero-energy bound state $\Psi_0(x) = C\left(x \pm \mathrm{i}\sqrt{2b_2}\right)^{-l}$.

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