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Nonlinear dynamics of a position-dependent mass driven Duffing-type oscillator

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Abstract

We examine some nontrivial consequences that emerge from interpreting a position-dependent mass (PDM) driven Duffing oscillator in the presence of a quartic potential. The propagation dynamics is studied numerically and sensitivity to the PDM-index is noted. Remarkable transitions from a limit cycle to chaos through period doubling and from a chaotic to a regular motion through intermediate periodic and chaotic routes are demonstrated.

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While position-dependent (effective) mass (PDM) quantum mechanical systems have repeatedly received attention in many areas of physics (such as for instance, in the problems of compositionally graded crystals [1], quantum dots [2], nuclei [3], quantum liquids [4], metal clusters [5] etc; see also [6, 7, 8, 9, 10, 11, 12, 13] for theoretical developments and references therein), interest in classical problems having a PDM is relatively recent and rapidly developing subject [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The simplest case of a PDM classical oscillator has been approached by several authors [16, 17, 18, 19, 20]. In what follows we shall demonstrate that a PDM ascribed Duffing oscillator provides an attractive possibility of defining a dynamical system that exhibits various features of bifurcations, chaos and regular motions. To be specific we will focus on the following equation of motion which emerges from a Lagrangian of the form [24]

$$L = \frac{1}{2}\left(\frac{1}{1+\xi x^2}\right)(\dot{x}^2 - \omega_0^2 x^2), \quad \xi \in \mathbb{R} \tag{1}$$

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(where an overhead dot indicates a time derivative) namely,

$$(1+\xi x^2)\ddot{x} - \xi x \dot{x}^2 + \omega_0^2 x = 0. \tag{2}$$

The above equation represents a special type of nonlinear oscillator which can be reduced to a first-order form by effecting a change in the variable x [24]. The latter equation can be immediately solved yielding periodic solutions. Hence, the absolute regularity of the motion does not depend on the specific value of ξ.

Classically when the mass is position dependent, Newton's equation of motion gets modified to

$$m(x)\ddot{x} + m'(x)\dot{x}^2 = 0 \tag{3}$$

(where the prime indicates a spatial derivative) in the absence of any external force term. When compared with (2) the following profile of the mass function comes out

$$m(x) := \frac{1}{\sqrt{1+\xi x^2}} \tag{4}$$

on ignoring the presence of the harmonic term $\omega_0^2 x$ to effect such a comparison. In (4) we have scaled the constant mass to unity. Notice that it is also possible to deal with other mass functions [15, 20] but for concreteness we focus on (4) in the rest of this paper.

In [20] an attempt was made to construct the underlying Lagrangian wherein the PDM system (3) is acted upon by a force F that could be dependent on position x, velocity $\dot{x}$ and time t i.e.

$$F(x,\dot{x},t) := \frac{dp}{dt} = m'(x)\dot{x}^2 + m(x)\ddot{x} \tag{5}$$

where $p = m(x)\dot{x}$ is the linear momentum. They have found that, in addition to the kinetic energy T, a reacting thrust $\tilde{R}$ is at play as well

$$T := \frac{1}{2}m(x)\dot{x}^2, \qquad \tilde{R}(x,\dot{x},t) := -\frac{1}{2}m'(x)\dot{x}^2. \tag{6}$$

where $\tilde{R}$ is essentially a non-inertial force. Assuming that F can be split as $F = F(x) + \tilde{R}(x,\dot{x},t)$, where $F(x)$ is controlled by a scalar potential function $F = -\frac{\partial V}{\partial x}$, the Lagrangian form of (5) obeys

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \tilde{R}, \qquad L = T - V. \tag{7}$$

Corresponding to the above L which is $L = \frac{1}{2}m(x)\dot{x}^2 - V(x)$, the Hamiltonian reads $H = \frac{p^2}{2m(x)} + V(x)$. These are of standard text-book forms[1] with $m = m(x)$.

[1] Quantum mechanically such forms are evidently non-Hermitian [25]

We stress that the non-potential force term $\tilde{R}$ has to depend on the velocities for otherwise they can be identified with the potential force $F(x)$. When written explicitly (7) looks

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{x}}) - \frac{\partial T}{\partial x} = -\frac{\partial V}{\partial x} + \tilde{R}. \tag{8}$$

As an aside, the time rate of change of T can be easily worked out as

$$\frac{dT}{dt} = \frac{d}{dt}(\frac{\partial T}{\partial \dot{x}}\dot{x}) + (\frac{\partial V}{\partial x} - \tilde{R})\dot{x} + \frac{\partial T}{\partial t} \tag{9}$$

from which, for a scleronomic system it at once follows that the time rate of energy $E(= T + V)$ is given by

$$\frac{dE}{dt} = \tilde{R}\dot{x} = -\frac{1}{2}m'\dot{x}^3 = \frac{1}{2}\frac{\xi x \dot{x}^3}{(1+\xi x^2)\sqrt{1+\xi x^2}} \tag{10}$$

(10) speaks of the power of the non-potential force. This result is new for PDM systems and of interest. If the power is negative we encounter dissipative systems.

The purpose of this communication is to develop a mathematical framework that enables us to study the evolution of the PDM system (5) in the presence of an external periodic (non-autonomous) force with an additional damping term moving in a quartic potential. Towards this end we focus on an extended PDM equation of motion

$$m(x)\ddot{x} + m'(x)\dot{x}^2 + \omega_0^2 x + \lambda x^3 + \alpha\dot{x} = f\cos\omega t \tag{11}$$

which reduces to (3) in the absence of all the force terms.

Some remarks are in order [26, 27, 28]. For the constant mass case i.e. $m(x) = 1$, (11) goes over to a forced, damped Duffing oscillator which because of the presence of a double-well potential mimicks a magneto-elastic mechanical system. The latter is concerned with a beam placed vertically between two magnets with a fixed top end and free to swing at the bottom end. As soon as a velocity is enforced the beam begins to oscillate eventually coming to rest at an equilibrium point. However the situation changes when a periodic force is applied : stable fixed points or stable fixed angles no longer occur. In (11) λ is the governing parameter along with the periodic force $f\cos\omega t$ in the presence of a viscous drag of coupling strength α.

Taking $m(x)$ as in (4) we obtain from (11) the coupled set of equations

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= \frac{\xi x y^2}{1+\xi x^2} + \sqrt{1+\xi x^2}[f\cos z - \omega_0^2 x - \lambda x^3 - \alpha y] \\
\dot{z} &= w
\end{aligned} \tag{12}$$

The remarkable behavior of the dynamical system (12) can be understood by examining the interplay between the amplitude f of the periodic forcing term and the PDM parameter ξ. Various studies of the corresponding constant-mass case, i.e. when $\xi = 0$, have been made in the literature (see specifically [29]) and the nonlinear behavior demonstrated including the oscillation modes and the nonlinear resonances, both theoretically and experimentally. For the PDM case, we fix the parameter values to be $\omega = 1.0$, $\omega_0^2 = 0.25$, $\alpha = 0.2$ and $\lambda = 1.0$ as is standard [23]. Further, we will always assume $\xi \geq 0$. We plot in Figure 1 the phase diagrams for different values of ξ including the constant-mass case of $\xi = 0$. While in the latter situation we encounter limit-cycle oscillation, there is a drastic change in the dynamical behaviors as ξ is increased. For instance, period-four oscillations are observed for $\xi = 0.2$ which eventually give way to a chaotic behaviour both for $\xi = 0.4$ and $\xi = 0.6$ values.

Bifurcation diagram of the system (12) for x with respect to f by taking $\xi = 0.5$ are presented in Figure 2. As soon as periodic force is applied, limit cycle oscillations gather in the range $0 < f < 2$. But with the increase of the amplitude of the forcing term, period-two oscillations set in and survive briefly up to $f = 4$. Further stepping up of f produces period-four and period-eight oscillations ultimately leading to a chaotic behaviour. But such a regime is short-lived because with the f-value going beyond 7.5, a period-halving bifurcation appears yielding period-three oscillations for $f > 8$.

In Figure 3 we demonstrate the bifurcation sequence for x but with respect to ξ by taking $f = 5.0$. As is evident, the limit cycle turns to period-four oscillations progressing to period-eight and so on running into a chaotic behaviour that lasts for a while before a regular motion takes over. The latter then again yields to a chaotic dynamics and the entire motion subsequently becomes a regular one around $\xi = 1.9$. On the other hand, by carrying out a bifurcation analysis for x with respect to ξ by taking $f = 8.0$ we find from Figure 4 that the constant-mass oscillator reveals a chaotic state. But we also notice that a subtle interplay between the parameters f and ξ produces complicated dynamics from a chaotic phase to periodic oscillations back again to a chaotic character and finally settling into a regular bahaviour.

To summarize, we have demonstrated in this communication that given any value of f, if it is confronted with the PDM parameter ξ, remarkable phase transitions occur such as, for instance, from a limit cycle mode to a chaotic regime through a period doubling intermediate phase pointing to the sensitivity of the system (11) due to the interplay between f and ξ. The essential distinction between the constant mass and the variable mass case rests in the fact that the presence of the parameter ξ not only enhances the rapidity of such transitions but also initiates complicated nature of bifurcations, of course for a non-zero value of f. Conversely, we also encounter transitions such as from a chaotic phase $\rightarrow$ periodic orbit $\rightarrow$ period doubling $\rightarrow$ chaos $\rightarrow$ regular motion.

Acknowledgements

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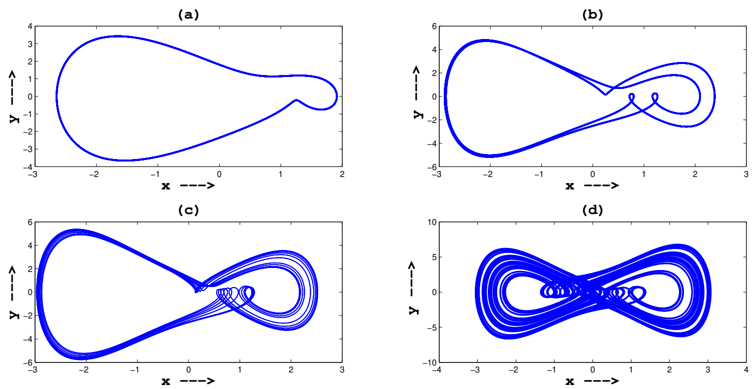

Figure 1: Phase diagram of equation (12) with $f = 5.0,\ \omega = 1.0,\ \omega_0^2 = 0.25,\ \alpha = 0.2,\ \lambda = 1.0$ for (a) constant mass ($\xi = 0$)(b) PDM with $\xi = 0.2$ (c) PDM with $\xi = 0.4$ (d) PDM with $\xi = 0.6$

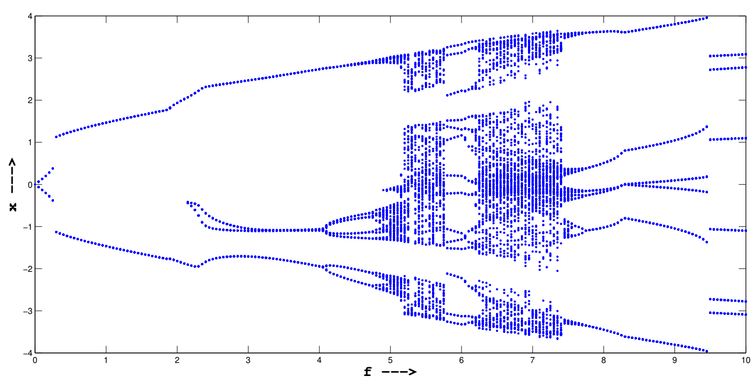

Figure 2: Bifurcation diagram of equation (12) with respect to f for $\xi = 0.5,\ \omega = 1.0,\ \omega_0^2 = 0.25,\ \alpha = 0.2,\ \lambda = 1.0$

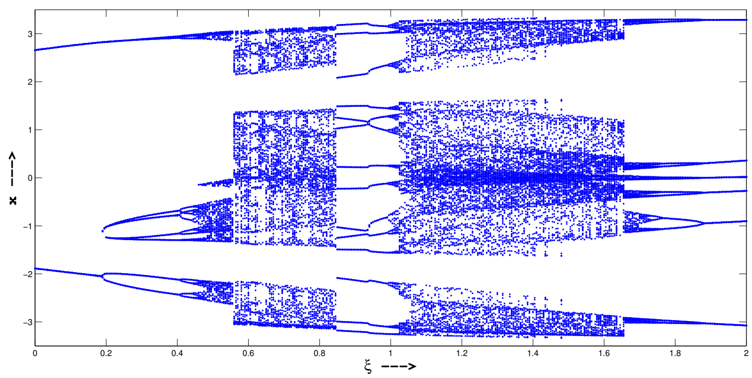

Figure 3: Bifurcation diagram of equation (12) with respect to ξ for $f = 5.0,\ \omega = 1.0,\ \omega_0^2 = 0.25,\ \alpha = 0.2,\ \lambda = 1.0$

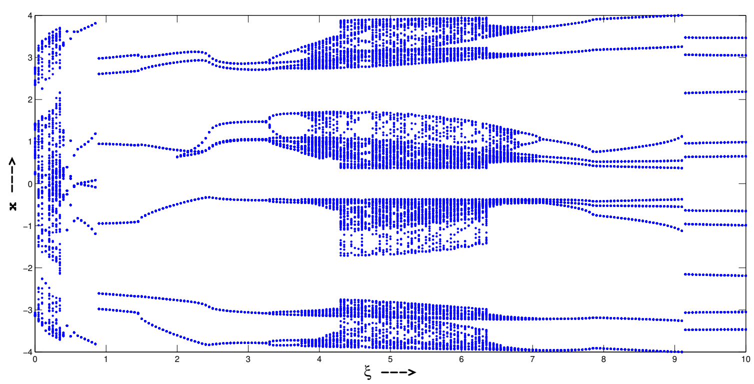

Figure 4: Bifurcation diagram of equation (12) with respect to ξ for $f = 8.0,\ \omega = 1.0,\ \omega_0^2 = 0.25,\ \alpha = 0.2,\ \lambda = 1.0$

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