Non-Hermitian Hamiltonians with real and complex eigenvalues in a Lie-algebraic framework

B. Bagchi [a,*], C. Quesne [b,†]

[a] *Department of Applied Mathematics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Calcutta 700 009, India*

[b] *Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium*

Abstract

We show that complex Lie algebras (in particular sl(2,$\mathbb{C}$)) provide us with an elegant method for studying the transition from real to complex eigenvalues of a class of non-Hermitian Hamiltonians: complexified Scarf II, generalized Pöschl-Teller, and Morse. The characterizations of these Hamiltonians under the so-called pseudo-Hermiticity are also discussed.

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Corresponding author: C. Quesne, Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium

Telephone: 32-2-6505559

Fax: 32-2-6505045

E-mail: cquesne@ulb.ac.be

*E-mail: bbagchi@cucc.ernet.in

†Directeur de recherches FNRS; E-mail: cquesne@ulb.ac.be

1 Introduction

Some years ago, it was suggested [1] that PT symmetry might be responsible for some non-Hermitian Hamiltonians to preserve the reality of their bound-state eigenvalues provided it is not spontaneously broken, in which case their complex eigenvalues should come in conjugate pairs. Following this, several non-Hermitian Hamiltonians (including the non-PT-symmetric ones [2, 3, 4]) with real or complex spectra have been analyzed using a variety of techniques, such as perturbation theory, semiclassical estimates, numerical experiments, analytical arguments, and algebraic methods. Among the latter, one may quote those connected with supersymmetrization [2, 5, 6, 7, 8, 9, 10], or some generalizations thereof [11], quasi-solvability [3, 12, 13, 14, 15, 16], and potential algebras [4, 17].

Recently, it has been shown that under some rather mild assumptions, the existence of real or complex-conjugate pairs of eigenvalues can be associated with a class of non-Hermitian Hamiltonians distinguished by either their so-called (weak) *pseudo-Hermiticity* [i.e., such that $\eta H \eta^{-1} = H^\dagger$, where η is some (Hermitian) linear automorphism] or their invariance under some antilinear operator [18, 19]. In such a context, pseudo-Hermiticity under imaginary shift of the coordinate has been identified as the explanation of the occurrence of real or complex-conjugate eigenvalues for some non-PT-symmetric Hamiltonians [20].

In the course of time, there has been a growing interest in determining the critical strengths of the interaction at which PT symmetry (or some generalization) becomes spontaneously broken, i.e., they appear *regular* complex-energy solutions, where by regular we mean eigenfunctions satisfying the asymptotic boundary conditions $\psi(\pm\infty) \to 0$, so that they are normalizable in a generalized sense [18, 20, 21, 22]. Some analytical results have been obtained both for PT-symmetric potentials [22, 23, 24, 25] and for potentials that are pseudo-Hermitian under imaginary shift of the coordinate [20].

In the present Letter, we wish to show that complex Lie algebras provide us with an easy and elegant method for studying the transition from real to complex eigenvalues, corresponding to *regular* eigenfunctions, of (PT-symmetric or non-PT-symmetric) pseudo-Hermitian and non-pseudo-Hermitian Hamiltonians.

2 Non-Hermitian Hamiltonians in an sl(2, $\mathbb{C}$) framework

The generators J_0, J_+, J_- of the complex Lie algebra sl(2,$\mathbb{C}$), characterized by the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0, \tag{1}$$

can be realized as differential operators [4]

$$J_0 = -\mathrm{i}\frac{\partial}{\partial\phi}, \qquad J_\pm = e^{\pm\mathrm{i}\phi}\left[\pm\frac{\partial}{\partial x} + \left(\mathrm{i}\frac{\partial}{\partial\phi} \mp \frac{1}{2}\right)F(x) + G(x)\right], \tag{2}$$

depending upon a real variable x and an auxiliary variable $\phi \in [0, 2\pi)$, provided the two complex-valued functions $F(x)$ and $G(x)$ in (2) satisfy coupled differential equations

$$F' = 1 - F^2, \qquad G' = -FG. \tag{3}$$

Here a prime denotes derivative with respect to spatial variable x.

The solutions of Eq. (3) fall into the following three classes:

$$\begin{array}{lll}
\mathrm{I}: & F(x) = \tanh(x - c - \mathrm{i}\gamma), & G(x) = (b_R + \mathrm{i}b_I)\,\mathrm{sech}(x - c - \mathrm{i}\gamma), \\
\mathrm{II}: & F(x) = \coth(x - c - \mathrm{i}\gamma), & G(x) = (b_R + \mathrm{i}b_I)\,\mathrm{cosech}(x - c - \mathrm{i}\gamma), \\
\mathrm{III}: & F(x) = \pm 1, & G(x) = (b_R + \mathrm{i}b_I)e^{\mp x},
\end{array} \tag{4}$$

where c, b_R, $b_I \in \mathbb{R}$ and $-\frac{\pi}{4} \le \gamma < \frac{\pi}{4}$, thus providing us with three different realizations of sl(2,$\mathbb{C}$). For $b_I = \gamma = 0$, the latter reduce to corresponding realizations of $\mathrm{sl}(2, \mathbb{R}) \simeq \mathrm{so}(2, 1)$, for which $J_0 = J_0^\dagger$ and $J_- = J_+^\dagger$ [26].

The sl(2, $\mathbb{C}$) Casimir operator corresponding to the differential realizations of type (2) can be written as

$$\begin{aligned}
J^2 &\equiv J_0^2 \mp J_0 - J_\pm J_\mp \\
&= \frac{\partial^2}{\partial x^2} - \left(\frac{\partial^2}{\partial\phi^2} + \frac{1}{4}\right)F' + 2\mathrm{i}\frac{\partial}{\partial\phi}G' - G^2 - \frac{1}{4}.
\end{aligned} \tag{5}$$

In this work, we are going to consider the sl(2, $\mathbb{C}$) irreducible representations spanned by the states

$$|km\rangle = \Psi_{km}(x, \phi) = \psi_{km}(x)\frac{e^{\mathrm{i}m\phi}}{\sqrt{2\pi}} \tag{6}$$

with fixed k, for which

$$J_0|km\rangle = m|km\rangle, \qquad J^2|km\rangle = k(k-1)|km\rangle, \tag{7}$$

and

$$k = k_R + \mathrm{i}k_I, \qquad m = m_R + \mathrm{i}m_I, \qquad m_R = k_R + n, \qquad m_I = k_I, \tag{8}$$

where k_R, k_I, m_R, $m_I \in \mathbb{R}$ and $n \in \mathbb{N}$. The states with $m = k$ or $n = 0$ satisfy the equation $J_-|kk\rangle = 0$, while those with higher values of m (or n) can be obtained from them by repeated applications of J_+ and use of the relation $J_+|km\rangle \propto |k\, m+1\rangle$.

When the parameter m is real, i.e., $m_I = 0$, we can get rid of the auxiliary variable ϕ by extending the definition of the pseudo-norm with a multiplicative integral over ϕ from 0 to 2π. In the case m is complex, i.e., $m_I \neq 0$, a similar result can be obtained through an appropriate change of the integral over ϕ. In the former (resp. latter) case, J_0 is a Hermitian (resp. non-Hermitian) operator.

From the second relation in Eq. (7), it follows that the functions $\psi_{km}(x)$ of Eq. (6) obey the Schrödinger equation

$$-\psi''_{km} + V_m\psi_{km} = -\left(k - \tfrac{1}{2}\right)^2 \psi_{km}, \tag{9}$$

where the family of potentials V_m is defined by

$$V_m = \left(\tfrac{1}{4} - m^2\right) F' + 2mG' + G^2. \tag{10}$$

Since the irreducible representations of sl(2, $\mathbb{C}$) correspond to a given eigenvalue in Eq. (9) and the corresponding basis states to various potentials V_m, $m = k,\ k+1,\ k+2,\ \ldots,$ it is clear that sl(2, $\mathbb{C}$) is a potential algebra for the family of potentials V_m (see [26] and references quoted therein).

To the three classes of solutions of Eq. (3), given in Eq. (4), we can now associate three classes of potentials:

$$\begin{aligned}
\mathrm{I}: \quad V_m &= \left[(b_R + \mathrm{i}b_I)^2 - (m_R + \mathrm{i}m_I)^2 + \tfrac{1}{4}\right] \operatorname{sech}^2\tau \\
&\quad - 2(m_R + \mathrm{i}m_I)(b_R + \mathrm{i}b_I) \operatorname{sech}\tau \tanh\tau, \qquad \tau = x - c - \mathrm{i}\gamma, \qquad (11) \\
\mathrm{II}: \quad V_m &= \left[(b_R + \mathrm{i}b_I)^2 + (m_R + \mathrm{i}m_I)^2 - \tfrac{1}{4}\right] \operatorname{cosech}^2\tau \\
&\quad - 2(m_R + \mathrm{i}m_I)(b_R + \mathrm{i}b_I) \operatorname{cosech}\tau \coth\tau, \qquad \tau = x - c - \mathrm{i}\gamma, \qquad (12) \\
\mathrm{III}: \quad V_m &= (b_R + \mathrm{i}b_I)^2 e^{\mp 2x} \mp 2(m_R + \mathrm{i}m_I)(b_R + \mathrm{i}b_I) e^{\mp x}. \qquad (13)
\end{aligned}$$

It is worth stressing that in the generic case, such complex potentials are not invariant under PT symmetry.

Equation (9) can also be rewritten as

$$-\psi_n^{(m)\prime\prime} + V_m \psi_n^{(m)} = E_n^{(m)} \psi_n^{(m)}, \tag{14}$$

with $\psi_{km}(x) = \psi_n^{(m)}(x)$ and

$$E_n^{(m)} = -\left(m_R + \mathrm{i}m_I - n - \tfrac{1}{2}\right)^2. \tag{15}$$

Real (resp. complex) eigenvalues therefore correspond to $m_I = 0$ (resp. $m_I \neq 0$).

To be acceptable solutions of Eq. (14), the functions $\psi_n^{(m)}(x)$ have to be regular, i.e., such that $\psi_n^{(m)}(\pm\infty) \to 0$. It is straightforward to determine under which conditions there exist acceptable solutions of Eq. (14) with $n = 0$. The functions $\psi_0^{(m)}(x)$ are indeed easily obtained by solving the first-order differential equation $J_- \Psi_{mm}(x, \phi) = 0$. For the three classes of potentials (11) – (13), the results read

$$\begin{aligned} \mathrm{I}: \quad \psi_0^{(m)}(x) &\propto (\operatorname{sech}\tau)^{m_R + \mathrm{i}m_I - 1/2} \exp[(b_R + \mathrm{i}b_I)\arctan(\sinh\tau)], & (16)\\ \mathrm{II}: \quad \psi_0^{(m)}(x) &\propto (\sinh\tfrac{\tau}{2})^{b_R + \mathrm{i}b_I - m_R - \mathrm{i}m_I + 1/2} (\cosh\tfrac{\tau}{2})^{-b_R - \mathrm{i}b_I - m_R - \mathrm{i}m_I + 1/2}, & (17)\\ \mathrm{III}: \quad \psi_0^{(m)}(x) &\propto \exp[-(m_R + \mathrm{i}m_I - \tfrac{1}{2})x - (b_R + \mathrm{i}b_I)e^{-x}]. & (18) \end{aligned}$$

Such functions are regular provided $m_R > \frac{1}{2}$ and $b_R > 0$, where the second condition applies only to class III.

In the remainder of this letter, we shall illustrate the general theory developed in the present section with some selected examples.

3 Complexified Scarf II potential

The potential

$$V(x) = -V_1 \operatorname{sech}^2 x - \mathrm{i}V_2 \operatorname{sech} x \tanh x, \qquad V_1 > 0, \qquad V_2 \neq 0, \tag{19}$$

which belongs to class I defined in Eq. (11), is a complexification of the real Scarf II potential [27]. It is not only invariant under PT symmetry but also P-pseudo-Hermitian. Comparison between Eqs. (11) and (19) shows that it corresponds to $c = \gamma = 0$ and

$$
\begin{aligned}
b_R^2 - b_I^2 - m_R^2 + m_I^2 + \tfrac{1}{4} &= -V_1, && (20)\\
b_R b_I - m_R m_I &= 0, && (21)\\
m_R b_R - m_I b_I &= 0, && (22)\\
2(m_R b_I + m_I b_R) &= V_2, && (23)
\end{aligned}
$$

where we may assume $b_I \neq 0$ since otherwise the sl(2, $\mathbb{C}$) generators (2) would reduce to sl(2, $\mathbb{R}$) ones.

To be able to apply the results of the previous section, the only thing we have to do is to solve Eqs. (20) – (23) in order to express the sl(2, $\mathbb{C}$) parameters b_R, b_I, m_R, m_I in terms of the potential parameters V_1, V_2. Equations (22) and (23) yield

$$m_R = \frac{V_2 b_I}{2(b_R^2 + b_I^2)}, \qquad m_I = \frac{V_2 b_R}{2(b_R^2 + b_I^2)}. \tag{24}$$

On inserting these results into Eqs. (20) and (21), we get the relations

$$
\begin{aligned}
(b_R^2 - b_I^2)\left(1 + \frac{V_2^2}{4(b_R^2 + b_I^2)^2}\right) &= -V_1 - \frac{1}{4}, && (25)\\
b_R b_I \left(1 - \frac{V_2^2}{4(b_R^2 + b_I^2)^2}\right) &= 0. && (26)
\end{aligned}
$$

The latter is satisfied if either $b_R = 0$ or $b_R \neq 0$ and $b_R^2 + b_I^2 = \frac{1}{2}|V_2|$. It now remains to solve Eq. (25) in those two possible cases.

If we choose $b_R = 0$, then Eq. (25) reduces to a quadratic equation for b_I^2, which has real positive solutions

$$b_I^2 = \tfrac{1}{4}\left(\sqrt{V_1 + \tfrac{1}{4} + V_2} + \epsilon_I \sqrt{V_1 + \tfrac{1}{4} - V_2}\right)^2, \qquad \epsilon_I = \pm 1, \tag{27}$$

provided $|V_2| \leq V_1 + \frac{1}{4}$. Equation (27) then yields for b_I the possible solutions

$$b_I = \tfrac{1}{2}\epsilon'_I \left(\sqrt{V_1 + \tfrac{1}{4} + V_2} + \epsilon_I \sqrt{V_1 + \tfrac{1}{4} - V_2}\right), \qquad \epsilon_I, \epsilon'_I = \pm 1, \tag{28}$$

while Eq. (24) leads to $m_R = V_2/(2b_I)$ and $m_I = 0$.

From the regularity condition $m_R > \frac{1}{2}$ of $\psi_0^{(m)}(x)$, given in Eq. (16), it then follows that b_I must have the same sign as V_2, which we denote by ν. Furthermore, we must choose $\epsilon'_I = +1$ or $\epsilon'_I = -\epsilon_I$ according to whether $\nu = +1$ or $\nu = -1$.

The first set of solutions of Eqs. (20) – (23), compatible with the regularity condition of $\psi_0^{(m)}(x)$, is therefore given by

$$\begin{aligned} b_R &= 0, \qquad b_I = \tfrac{1}{2}\nu\left(\sqrt{V_1 + \tfrac{1}{4} + |V_2|} - \epsilon\sqrt{V_1 + \tfrac{1}{4} - |V_2|}\right), \\ m_R &= \tfrac{1}{2}\left(\sqrt{V_1 + \tfrac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \tfrac{1}{4} - |V_2|}\right), \qquad m_I = 0, \qquad \epsilon = \pm 1, \end{aligned} \tag{29}$$

where $\epsilon = -\epsilon_I$, provided $|V_2| \leq V_1 + \frac{1}{4}$ and $\sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \frac{1}{4} - |V_2|} > 1$.

On inserting these results into Eq. (15), we get two series of real eigenvalues

$$E_{n,\epsilon} = -\left[\tfrac{1}{2}\left(\sqrt{V_1 + \tfrac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \tfrac{1}{4} - |V_2|}\right) - n - \tfrac{1}{2}\right]^2, \qquad \epsilon = \pm 1. \tag{30}$$

By studying the regularity condition of the associated eigenfunctions obtained by successive applications of J_+ on $\psi_0^{(m)}(x)$, it can be shown that n is restricted to the range $n = 0, 1, 2, \ldots < \frac{1}{2}\left(\sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \frac{1}{4} - |V_2|} - 1\right)$.

If, on the contrary, we choose $b_R \neq 0$ and $b_R^2 + b_I^2 = \frac{1}{2}|V_2|$, then Eq. (25) leads to $b_R^2 - b_I^2 = -\frac{1}{2}(V_1 + \frac{1}{4})$, so that

$$b_R = \tfrac{1}{2}\epsilon_R\sqrt{|V_2| - V_1 - \tfrac{1}{4}}, \qquad b_I = \tfrac{1}{2}\epsilon_I\sqrt{|V_2| + V_1 + \tfrac{1}{4}}, \qquad \epsilon_R, \epsilon_I = \pm 1, \tag{31}$$

provided $|V_2| > V_1 + \frac{1}{4}$.

On inserting such results into Eq. (24) and imposing the regularity condition $m_R > \frac{1}{2}$, we obtain $\epsilon = \nu$. The second set of solutions of Eqs. (20) – (23), compatible with the regularity condition of $\psi_0^{(m)}(x)$, is therefore given by

$$\begin{aligned} b_R &= \tfrac{1}{2}\nu\epsilon\sqrt{|V_2| - V_1 - \tfrac{1}{4}}, \qquad b_I = \tfrac{1}{2}\nu\sqrt{|V_2| + V_1 + \tfrac{1}{4}}, \\ m_R &= \tfrac{1}{2}\sqrt{|V_2| + V_1 + \tfrac{1}{4}}, \qquad m_I = \tfrac{1}{2}\epsilon\sqrt{|V_2| - V_1 - \tfrac{1}{4}}, \qquad \epsilon = \pm 1, \end{aligned} \tag{32}$$

where we have set $\epsilon = \nu\epsilon_R$. Here we must assume $|V_2| > V_1 + \frac{1}{4}$ and $|V_2| + V_1 + \frac{1}{4} > 1$.

This set of solutions is associated with a series of complex-conjugate pairs of eigenvalues

$$E_{n,\epsilon} = -\left[\tfrac{1}{2}\left(\sqrt{|V_2| + V_1 + \tfrac{1}{4}} + \mathrm{i}\epsilon\sqrt{|V_2| - V_1 - \tfrac{1}{4}}\right) - n - \tfrac{1}{2}\right]^2, \qquad \epsilon = \pm 1, \tag{33}$$

where it can be shown that n varies in the range $n = 0, 1, 2, \ldots < \frac{1}{2}\left(\sqrt{|V_2| + V_1 + \frac{1}{4}} - 1\right)$.

We conclude that for increasing values of $|V_2|$, the two series of real eigenvalues (30) merge when $|V_2|$ reaches the value $V_1 + \frac{1}{4}$, then disappear while complex-conjugate pairs of eigenvalues (33) make their appearance, as already found elsewhere by another method [22]. Had we chosen the parametrization $V_1 = B^2 + A(A+1)$, $V_2 = -B(2A+1)$, with A and B real, as we did in Ref. [4], we would obtain that the condition $|V_2| \le V_1 + \frac{1}{4}$ is always satisfied, thus only getting the two series of real eigenvalues (30).

4 Complexified generalized Pöschl-Teller potential

We next consider the complexification of the generalized Pöschl-Teller potential [27], namely

$$V(x) = V_1 \operatorname{cosech}^2 \tau - V_2 \operatorname{cosech} \tau \coth \tau, \qquad \tau = x - c - \mathrm{i}\gamma, \qquad V_1 > -\tfrac{1}{4}, \qquad V_2 \ne 0. \tag{34}$$

It is easy to recognize (34) to belong to class II defined in Eq. (12). Note that the above potential is PT-symmetric as well as P-pseudo-Hermitian. Comparing with (12), we get

$$b_R^2 - b_I^2 + m_R^2 - m_I^2 - \tfrac{1}{4} = V_1, \tag{35}$$

$$b_R b_I + m_R m_I = 0, \tag{36}$$

$$2(m_R b_R - m_I b_I) = V_2, \tag{37}$$

$$m_R b_I + m_I b_R = 0. \tag{38}$$

This time there is no reason to assume that $b_I \ne 0$, since the presence of $\gamma \ne 0$ in the generators (2) ensures that we are dealing with sl(2, $\mathbb{C}$).

On successively considering the cases where $b_I = 0$ or $b_I \ne 0$ and proceeding as in the previous section, we are led to the two following sets of solutions of Eqs. (35) – (38):

$$\begin{aligned} b_R &= \tfrac{1}{2}\nu\left(\sqrt{V_1 + \tfrac{1}{4} + |V_2|} - \epsilon\sqrt{V_1 + \tfrac{1}{4} - |V_2|}\right), \qquad b_I = 0, \\ m_R &= \tfrac{1}{2}\left(\sqrt{V_1 + \tfrac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \tfrac{1}{4} - |V_2|}\right), \qquad m_I = 0, \qquad \epsilon = \pm 1, \end{aligned} \tag{39}$$

provided $|V_2| \le V_1 + \frac{1}{4}$ and $\sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \frac{1}{4} - |V_2|} > 1$, and

$$\begin{aligned} b_R &= \tfrac{1}{2}\nu\sqrt{|V_2| + V_1 + \tfrac{1}{4}}, \qquad b_I = -\tfrac{1}{2}\nu\epsilon\sqrt{|V_2| - V_1 - \tfrac{1}{4}}, \\ m_R &= \tfrac{1}{2}\sqrt{|V_2| + V_1 + \tfrac{1}{4}}, \qquad m_I = \tfrac{1}{2}\epsilon\sqrt{|V_2| - V_1 - \tfrac{1}{4}}, \qquad \epsilon = \pm 1, \end{aligned} \tag{40}$$

provided $|V_2| > V_1 + \frac{1}{4}$ and $|V_2| + V_1 + \frac{1}{4} > 1$. In both cases, ν denotes the sign of V_2.

Comparison with Eq. (15) shows that the first type solutions (39) lead to two series of real eigenvalues

$$E_{n,\epsilon} = -\left[\tfrac{1}{2}\left(\sqrt{V_1 + \tfrac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \tfrac{1}{4} - |V_2|}\right) - n - \tfrac{1}{2}\right]^2, \qquad \epsilon = \pm 1, \tag{41}$$

while the second type solutions (40) correspond to a series of complex-conjugate pairs of eigenvalues

$$E_{n,\epsilon} = -\left[\tfrac{1}{2}\left(\sqrt{|V_2| + V_1 + \tfrac{1}{4}} + \mathrm{i}\epsilon\sqrt{|V_2| - V_1 - \tfrac{1}{4}}\right) - n - \tfrac{1}{2}\right]^2, \qquad \epsilon = \pm 1. \tag{42}$$

In the former (resp. latter) case, it can be shown that n varies in the range $n = 0, 1, 2, \ldots < \frac{1}{2}\left(\sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \frac{1}{4} - |V_2|} - 1\right)$ [resp. $n = 0, 1, 2, \ldots < \frac{1}{2}\left(\sqrt{|V_2| + V_1 + \frac{1}{4}} - 1\right)$].

For increasing values of $|V_2|$, we observe a phenomenon entirely similar to that already noted for the complexified Scarf II potential: disappearance of the real eigenvalues and simultaneous appearance of complex-conjugate ones at the threshold $|V_2| = V_1 + \frac{1}{4}$. In this case, however, only partial results were reported in the literature. In Ref. [4], we obtained the two series of real eigenvalues (41) using the parametrization $V_1 = B^2 + A(A+1)$, $V_2 = B(2A+1)$, with A and B real, so that the condition $|V_2| \le V_1 + \frac{1}{4}$ is automatically satisfied. Furthermore, Lévai and Znojil considered both the real [8] and the complex [24] eigenvalues in a parametrization $V_1 = \frac{1}{4}[2(\alpha^2 + \beta^2) - 1]$, $V_2 = \frac{1}{2}(\beta^2 - \alpha^2)$, wherein α and β are real or one of them is real and the other imaginary, respectively. Their results, however, disagree with ours in both cases.

5 Complexified Morse potential

The potential

$$V(x) = (V_{1R} + \mathrm{i}V_{1I})e^{-2x} - (V_{2R} + \mathrm{i}V_{2I})e^{-x}, \qquad V_{1R}, V_{1I}, V_{2R}, V_{2I} \in \mathbb{R}, \tag{43}$$

is the most general potential of class III for the upper sign choice in Eq. (13) and is a complexification of the standard Morse potential [27]. Comparison with Eq. (13) shows that

$$
\begin{aligned}
b_R^2 - b_I^2 &= V_{1R}, && (44)\\
2b_R b_I &= V_{1I}, && (45)\\
2(m_R b_R - m_I b_I) &= V_{2R}, && (46)\\
2(m_R b_I + m_I b_R) &= V_{2I}, && (47)
\end{aligned}
$$

where we may assume $b_I \neq 0$.

On solving Eq. (45) for b_R and inserting the result into Eq. (44), we get a quadratic equation for b_I^2, of which we only keep the real positive solutions. The results for b_R and b_I read

$$
b_R = \tfrac{1}{\sqrt{2}}\epsilon_I \nu (V_{1R} + \Delta)^{1/2}, \quad b_I = \tfrac{1}{\sqrt{2}}\epsilon_I (-V_{1R} + \Delta)^{1/2}, \quad \Delta = \sqrt{V_{1R}^2 + V_{1I}^2}, \quad \epsilon_I = \pm 1, \quad (48)
$$

where $V_{1I} \neq 0$ if $V_{1R} \geq 0$ and ν denotes the sign of V_{1I}. On introducing Eq. (48) into Eqs. (46) and (47) and solving for m_R and m_I, we then obtain

$$
\begin{aligned}
m_R &= \frac{\epsilon_I \nu}{2\sqrt{2}\Delta}\left[(V_{1R} + \Delta)^{1/2} V_{2R} + \nu(-V_{1R} + \Delta)^{1/2} V_{2I}\right], && (49)\\
m_I &= \frac{\epsilon_I \nu}{2\sqrt{2}\Delta}\left[(V_{1R} + \Delta)^{1/2} V_{2I} - \nu(-V_{1R} + \Delta)^{1/2} V_{2R}\right]. && (50)
\end{aligned}
$$

From the regularity conditions $b_R > 0$ and $m_R > \frac{1}{2}$ of $\psi_0^{(m)}(x)$, given in Eq. (18), it follows that we must choose $\epsilon_I = \nu$, $V_{1I} \neq 0$ if $V_{1R} < 0$, and

$$
(V_{1R} + \Delta)^{1/2} V_{2R} + \nu(-V_{1R} + \Delta)^{1/2} V_{2I} > \sqrt{2}\Delta. \quad (51)
$$

We conclude that $V_{1I} \neq 0$ must hold for any value of V_{1R}.

Real eigenvalues are associated with $m_I = 0$ and therefore occur whenever the condition

$$
(V_{1R} + \Delta)^{1/2} V_{2I} = \nu(-V_{1R} + \Delta)^{1/2} V_{2R} \quad (52)
$$

is satisfied. In such a case, V_{2I} can be expressed in terms of V_{1R}, V_{1I}, and V_{2R}, so that the real eigenvalues are given by

$$
E_n = -\left[\frac{V_{2R}}{\sqrt{2}|V_{1I}|}(-V_{1R} + \Delta)^{1/2} - n - \frac{1}{2}\right]^2. \quad (53)
$$

It can be shown that regular eigenfunctions correspond to $n = 0, 1, 2, \ldots < (V_{2R}/\sqrt{2}|V_{1I}|)(-V_{1R} + \Delta)^{1/2} - \frac{1}{2}$.

Furthermore, when condition (52) is not fulfilled but condition (51) holds, we get complex eigenvalues associated with regular eigenfunctions,

$$E_n = -\left\{\frac{1}{2\sqrt{2}\Delta}\left[(V_{1R} + \Delta)^{1/2} - \mathrm{i}\nu(-V_{1R} + \Delta)^{1/2}\right](V_{2R} + \mathrm{i}V_{2I}) - n - \frac{1}{2}\right\}^2, \qquad (54)$$

where $n = 0, 1, 2, \ldots < \frac{1}{2\sqrt{2}\Delta}\left[(V_{1R} + \Delta)^{1/2}V_{2R} + \nu(-V_{1R} + \Delta)^{1/2}V_{2I}\right] - \frac{1}{2}$.

It should be stressed that contrary to what happens for the two previous examples, here the real eigenvalues, belonging to a single series, only occur for a special value of the parameter V_{2I}, while the complex eigenvalues, which do not appear in complex-conjugate pairs (since E_n^* corresponds to $V^*(x)$), are obtained for generic values of V_{2I}.

To interprete such results, it is worth choosing the parametrization $V_{1R} = A^2 - B^2$, $V_{1I} = 2AB$, $V_{2R} = \gamma A$, $V_{2I} = \delta B$, where A, B, γ, δ are real, $A > 0$, and $B \neq 0$. The complexified Morse potential (43) can then be expressed as

$$V(x) = (A + \mathrm{i}B)^2 e^{-2x} - (2C + 1)(A + \mathrm{i}B)e^{-x}, \qquad C = \frac{(\gamma - 1)A + \mathrm{i}(\delta - 1)B}{2(A + \mathrm{i}B)}. \qquad (55)$$

Its (real or complex) eigenvalues can be written in a unified way as $E_n = -(C - n)^2$, while the regularity condition (51) amounts to $(\gamma - 1)A^2 + (\delta - 1)B^2 > 0$.

For $\delta = \gamma > 1$, and therefore $C = \frac{1}{2}(\gamma - 1) \in \mathbb{R}^+$, the potential (55) coincides with that considered in our previous work [4]. Such a potential was shown to be pseudo-Hermitian under imaginary shift of the coordinate [20]. We confirm here that it has only real eigenvalues corresponding to $n = 0, 1, 2, \ldots < C$, thus exhibiting no symmetry breaking over the whole parameter range. For the values of δ different from γ, the potential indeed fails to be pseudo-Hermitian. In such a case, C is complex as well as the eigenvalues. The eigenfunctions associated with $n = 0, 1, 2, \ldots < \mathrm{Re}\, C$ are however regular. The existence of regular eigenfunctions with complex energies for general complex potentials is a phenomenon that has been known for some time (see e.g. [28]).

6 Conclusion

In the present Letter, we have shown that complex Lie algebras (in particular sl(2, $\mathbb{C}$)) provide us with an elegant tool to easily determine both real and complex eigenvalues of non-Hermitian Hamiltonians, corresponding to regular eigenfunctions. For such a purpose, it has been essential to extend the scope of our previous work [4] to those Lie algebra irreducible representations that remain nonunitary in the real algebra limit (namely those with $k_I \neq 0$).

We have illustrated our method by deriving the real and complex eigenvalues of the PT-symmetric complexified Scarf II potential, previously determined by other means [22]. In addition, we have established similar results for the PT-symmetric generalized Pöschl-Teller potential, for which only partial results were available [4, 8, 24]. We have shown that in both cases symmetry breaking occurs for a given value of one of the potential parameters.

Finally, we have considered a more general form of the complexified Morse potential than that previously studied [4, 19, 20]. For a special value of one of its parameters, our potential reduces to the former one and becomes pseudo-Hermitian under imaginary shift of the coordinate. We have proved that here no symmetry breaking occurs, the complex eigenvalues being associated with non-pseudo-Hermitian Hamiltonians.

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