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Swadesh M. Mahajan and Manasvi Lingam

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Multi-fluid systems—Multi-Beltrami relaxed states and their implications

Swadesh M. Mahajan[1,2,a)] and Manasvi Lingam[1,b)]
[1]Department of Physics and Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA
[2]Department of Physics, School of Natural Sciences, Shiv Nadar University, Uttar Pradesh 201314, India

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We consider the non-dissipative multi-fluid equations, and demonstrate how multi-Beltrami equilibria emerge as natural relaxed states of the model, representing an evolution towards the minimum energy. General properties of these states are studied, and a wide class of solutions is obtained. We specialize to the cases of double and triple Beltrami states and highlight their connections with the appropriate physical invariants, viz., the generalized helicities and the energy. In particular, we demonstrate that *different* field configurations can give rise to *distinct* or *identical* values of the invariants, depending on the nature of the roots of the multi-Beltrami equation. Moreover, we also highlight equivalences between (outwardly) unconnected models allowing us to treat them in a unified manner. Some observations regarding the nature of the solutions for certain special cases of these models are presented. Potential applications for astrophysical plasmas are also highlighted.
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I. INTRODUCTION

Ever since the pioneering work of Lord Kelvin in the 19th century, the notions of vorticity and helicity, though the word helicity was introduced later by Moffatt,[1] have played a key role in advancing the development of fluid dynamics. The recognition of their topological properties, and the fact that many simple plasma (charged fluid) models, like magnetohydrodynamics (MHD) and Hall MHD, have similar mathematical structures,[1–10] has proven to be useful in fusion and astrophysical plasmas. The exploitation of the central topological property, the conservation of helicity in "ideal dynamics" (non-dissipative limit) created one of the central developments in plasma physics—the ideas of relaxation and self-organization in plasmas.[2–8] In this paper, we will explore the possibility of relaxed states in a more encompassing model: a multi-fluid plasma.

Although this field originated with the work of Woltjer,[2] it was Taylor's research[3,6] that converted a mostly complex result into something that led to an easily solvable system with immediate predictions—the Woltjer-Taylor states of ideal MHD. The Woltjer-Taylor state is a specific example of what will be called a Beltrami state obtained by aligning a "vorticity" along its corresponding "velocity," i.e., it is found via $\nabla\times\mathbf{P}=\alpha\mathbf{P}$. The principles of self-organization and relaxation in the context of the Woltjer-Taylor Beltrami states have been successfully used in modeling fusion and astrophysical plasmas.[9,11–19] However, it must be recognized that this paradigm is not an exact one, and deviations from this principle have been observed.[20–22]

Subsequently, the same paradigm was transported to extended MHD models in Refs. 7, 8, 23, 24, who established the existence of double and triple Beltrami relaxed states. Such states are of considerable importance; they emerge via a variational principle that extremizes a given "target" invariant (typically the energy) whilst holding the helicities (and other invariants) fixed. It is common to interpret such states as the minimum energy states[6,25] of the system; it is important, however, to recognize that this is *not* always the case.[8] It has also been suggested[26–28] that the relaxation process may be viewed as evolution towards a maximum entropy state, instead of a minimum energy state. It is well known[29] that the former paradigm is of considerable importance in a wide range of fields, further cementing the importance of relaxed states.

The relaxed states, a fundamental (and abstract) expression of a plasma's ability to find its "suitable" configuration, have been applied in a variety of contexts: the single and double Beltrami states have been used in modeling fusion plasmas, such as spheromaks,[30] field-reversed configurations,[31,32] and plasma boundary layer transport.[33–35] In an astrophysical context, Beltrami states, and their associated invariants, have found usage in modeling solar flares,[36–39] solar arcades and loops,[40,41] coronal heating,[42,43] large-scale dynamos,[44–47] scale hierarchies in flows,[48] and turbulence.[49] For a highly unusual, speculative, and interesting application, which entails the exploration of the double-Beltrami system to model "classical perfect diamagnetism," we refer the reader to Ref. 50.

The preceding discussion indicates that generalizations of the Beltrami states, obtained via a variational principle for multi-fluid models, are likely to be of some use in studying astrophysical environments, such as the ones where dust plays a significant role. This investigation forms the subject of our paper. The outline of the paper is as follows. We construct, motivate, and analyze the multi-Beltrami states in Section II, as well as presenting a simple, but general, set of solutions. We discuss some of the uses and implications of the multi-Beltrami states in Sections III and IV. Finally, we conclude in Section V by presenting avenues for further work.

[a)]Electronic address: mahajan@mail.utexas.edu
[b)]Electronic address: manasvi@physics.utexas.edu

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II. THE MULTI-FLUID DYNAMICS—MULTI BELTRAMI STATES

We introduce the relevant equations for the non-dissipative multi-fluid model, and their connections to relaxed states and invariants. Next, we present the multi-Beltrami solutions for this model.

A. The evolution of generalized vorticity (GV)

We begin by writing the equation of motion obeyed by the species α (with mass m_α and charge q_α)

$$\rho_\alpha\left(\frac{\partial}{\partial t}+\mathbf{V}_\alpha\cdot\nabla\right)\mathbf{V}_\alpha=q_\alpha n_\alpha\left(\mathbf{E}+\frac{\mathbf{V}_\alpha\times\mathbf{B}}{c}\right)-\nabla p_\alpha+\rho_\alpha\nabla\Phi, \tag{1}$$

where ρ_α, $\mathbf{V}_\alpha$, and p_α represent, respectively, the density, velocity, and pressure of each species. Φ represents the gravitational potential (or any other gradient force). By using (i) $\mathbf{E}=-c^{-1}\partial\mathbf{A}/\partial t-\nabla\phi$, where $\mathbf{A}$ and ϕ are the vector and electrostatic potentials, respectively, (ii) the vector calculus identity $(\mathbf{F}\cdot\nabla)\mathbf{F}=\frac{1}{2}\nabla F^2+(\nabla\times\mathbf{F})\times\mathbf{F}$, and (iii) invoking a barotropic equation-of-state which yields $\rho_\alpha^{-1}\nabla p_\alpha=\nabla H_\alpha$, where H_α is the enthalpy, we convert (1) to

$$\frac{\partial\mathbf{P}_\alpha}{\partial t}=\mathbf{V}_\alpha\times\mathbf{\Omega}_\alpha-\nabla\psi_\alpha, \tag{2}$$

which, along with its curl,

$$\frac{\partial\mathbf{\Omega}_\alpha}{\partial t}=\nabla\times(\mathbf{V}_\alpha\times\mathbf{\Omega}_\alpha), \tag{3}$$

represent, respectively, the evolution equations for the generalized vector potential (GVP), $\mathbf{P}_\alpha=\mathbf{A}_T+\frac{m_\alpha c}{q_\alpha}\mathbf{V}_\alpha$, and the GV, $\mathbf{\Omega}_\alpha=\nabla\times\mathbf{P}_\alpha=\mathbf{B}_T+\frac{m_\alpha c}{q_\alpha}\nabla\times\mathbf{V}_\alpha=\mathbf{B}_T+\frac{m_\alpha c}{q_\alpha}\omega_\alpha$. The potential, $\psi_\alpha=\frac{c}{q_\alpha}\left[\frac{m_\alpha V_\alpha^2}{2}+m_\alpha\Phi+m_\alpha H_\alpha+q_\alpha\phi\right]$, contains all the potential forces that play no direct role in the evolution of GV. The set of Eqs. (2) and (3), the latter corresponding to the standard Helmholtz vortical dynamics, constitutes a representative dynamics of a system of collisionless charged particles with a barotropic thermodynamics.[51]

The GVP ($\mathbf{P}_\alpha$), combining the kinetic and electromagnetic components of the momentum, is just the standard canonical momentum. The GV ($\mathbf{\Omega}_\alpha$), therefore, could also be called the canonical vorticity. In the definitions of GVP and GV, the suffix "T" stands for "total," as the magnetic field $\mathbf{B}_T$ could also accommodate an ambient/vacuum field. The term $\nabla\psi_\alpha$ in (2) contains contributions from the kinetic, electromagnetic, and thermodynamics components, respectively.

All species, evolving independently from each other, however, are connected through Ampère's law

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{J}=\frac{4\pi}{c}\sum\mathbf{J}_\alpha=\frac{4\pi}{c}\sum n_\alpha q_\alpha\mathbf{V}_\alpha, \tag{4}$$

since they all contribute to the electrical current. In this non-relativistic treatment, we are neglecting the displacement current.

We shall now distinguish between two different scenarios, which can be handled within the same formalism:

1. There is no ambient magnetic field and the total field is just the dynamic field, $\mathbf{B}_T=\mathbf{B}$. In this case, Eqs. (2) and (3) will serve as the starting point for further analysis.
2. There is a finite ambient magnetic field, and the total field is split as

$$\mathbf{B}_T=B_0\hat{e}_z+\mathbf{B}, \tag{5}$$

where $\mathbf{B}$ is the dynamic field, and B_0 is a guide field maintained by currents outside the plasma.

Even for this case, the dynamics can be cast exactly in the form (2)–(3) for the restricted system with (1) $\nabla\cdot\mathbf{V}_\alpha=0$, signifying incompressibility, and (2) translational symmetry, represented by $\partial/\partial z=0$. Referring the reader to Ref. 50 for details, the final equation for the evolution of the dynamical part of the generalized momentum turns out to be

$$\frac{\partial\mathbf{P}_\alpha}{\partial t}=\mathbf{V}_\alpha\times\mathbf{\Omega}_\alpha-\nabla\tilde{\psi}_\alpha, \tag{6}$$

where the velocity field has the form $\mathbf{V}_\alpha=V_\alpha^{(z)}\hat{e}_z+\hat{e}_z\times\nabla\chi_\alpha$. Notice that B_0 and χ_α are fully absorbed in the potential $\tilde{\psi}_\alpha=\psi_\alpha-B_0\chi_\alpha$, and do not affect the vortex dynamics directly.

Before concluding this section, a couple of important geometric observations are in order. Let us recall (3) along with $\nabla\cdot\mathbf{\Omega}_\alpha=0$. Hence, we can rewrite (3) as

$$\frac{\partial\mathbf{\Omega}_\alpha}{\partial t}+\mathbf{\Omega}_\alpha(\nabla\cdot\mathbf{V}_\alpha)+(\mathbf{V}_\alpha\cdot\nabla)\mathbf{\Omega}_\alpha-(\mathbf{\Omega}_\alpha\cdot\nabla)\mathbf{V}_\alpha=0. \tag{7}$$

Two different geometric interpretations are possible:

1. It can be viewed as the Lie-dragging of a vector density of weight 1, akin to the magnetic field (in ideal MHD) and the vorticity (in ideal hydrodynamics).
2. More importantly, one can rewrite (7) as

$$\left(\frac{\partial}{\partial t}+\mathfrak{L}_{\mathbf{V}_\alpha}\right)\mathbf{\Omega}_\alpha\cdot d\mathbf{S}=0, \tag{8}$$

implying that $\mathbf{\Omega}_\alpha\cdot d\mathbf{S}$ is a Lie-dragged 2-form. In turn, this implies that the flux conservation of the generalized vorticity holds true, akin to magnetic flux conservation in ideal MHD.

In both cases, the Lie-dragging is undertaken with respect to the velocity $\mathbf{V}_\alpha$, which serves as the flow vector field. The generalized vorticity at any arbitrary time is related to the generalized vorticity at $t=0$ via a relation analogous to the Cauchy vorticity formula,[52] and is given by

$$\Omega_\alpha^j=\mathcal{J}^{-1}\Omega_\alpha^i(t=0)\frac{\partial q^j}{\partial a^i}, \tag{9}$$

where $\Omega_\alpha^i(t=0)$ is the generalized vorticity at $t=0$ and $\mathbf{q}(\mathbf{a},t)\equiv\mathbf{r}$ is the Lagrangian trajectory as a function of the label $\mathbf{a}$ and t; the former is given by the initial position of the fluid particle at $t=0$. Moreover, $\mathcal{J}$ denotes the Jacobian and is equal to $\det|\frac{\partial q^j}{\partial a^i}|$.

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It is, perhaps, the right place to state a useful consequence of the Helmholtz vortical dynamics: if $\Omega^i_\alpha(t=0) = 0$, then it remains zero for all times, i.e., $\Omega^i_\alpha(t) = 0$ for arbitrary t; see for, e.g., Refs. 50 and 53. This can be seen by inspecting (9), or directly deduced from (3). Thus, we see that a theoretical basis exists for a closely associated phenomenon that was observed in the detailed Hall MHD simulations of Ref. 54; the latter are indicated to be useful in explaining the evolution of protoplanetary discs.

B. On conservation laws and variational principles

The vortex dynamics of (2) and (3) will, by a straightforward demonstration, lead to a conserved generalized helicity (GH)

$$h_\alpha = \frac{1}{8\pi}\langle \mathbf{P}_\alpha \cdot \mathbf{\Omega}_\alpha \rangle \tag{10}$$

for each α. The notation $[\int d^3x \cdots = \langle \cdots \rangle]$ will be adopted henceforth. In addition to the generalized helicities, the total energy

$$E = \left\langle \sum_\alpha \frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{B^2}{8\pi} \right\rangle \tag{11}$$

is also conserved. Thus, for an N species system, there are $N+1$ constants of motion. The conservation of generalized helicities can also be interpreted geometrically, which we shall investigate in greater detail elsewhere; earlier studies in this area include Refs. 55–58.

The exploitation of the system invariants has led to many important advances in finding accessible equilibrium states for these complex systems. This includes the famous Woltjer-Taylor states of ideal MHD,[2,3,6] the double Beltrami states of Hall MHD,[7,8] and the multi-Beltrami states of extended MHD (Ref. 23)—each of these are applicable for the non-relativistic models. We shall explore this method in further detail in Secs. III and IV.

The "constrained" variational principle is constructed based on the implicit assumption that the generalized helicities (10) are more robust against dissipation than the energy (11). Consequently, we choose the latter as the target functional and extremize it subject to the former serving as the N constraints. We demand that

$$\delta Q = \delta\left(E - \sum \frac{h_\alpha}{\mu_\alpha}\right) = 0, \tag{12}$$

where μ'_αs are the Lagrange multipliers, and $\mathbf{V}_\alpha$ and $\mathbf{A}$ are independent variables but are connected through the Ampère's law (4). Working out the variation (12), we obtain

$$\begin{aligned} &\frac{m_\alpha c}{4\pi q_\alpha}\sum_\alpha \left(\frac{\mathbf{\Omega}_\alpha}{\mu_\alpha} - \frac{4\pi}{c} n_\alpha q_\alpha \mathbf{V}_\alpha\right)\cdot \delta\mathbf{V}_\alpha \\ &\quad - \frac{1}{4\pi}\left(\nabla \times \mathbf{B} - \sum_\alpha \frac{\mathbf{\Omega}_\alpha}{\mu_\alpha}\right)\cdot \delta\mathbf{A} = 0. \end{aligned} \tag{13}$$

By equating the coefficients of $\delta\mathbf{V}_\alpha$ to zero independently, for each species α, yields the N Beltrami conditions:

$$\mathbf{\Omega}_\alpha = \mu_\alpha \frac{4\pi}{c} n_\alpha q_\alpha \mathbf{V}_\alpha = \mu_\alpha \frac{4\pi}{c}\mathbf{J}_\alpha, \tag{14}$$

which amounts to aligning the GV ($\mathbf{\Omega}_\alpha$) of each species along its corresponding velocity ($\mathbf{V}_\alpha$). Notice that in the light of (4) and (14), the coefficient of $\delta\mathbf{A}$ is automatically zero; it is just the manifestation of Ampère's law. We also observe that each of the μ's is endowed with the dimensions of length, thereby giving rise to a hierarchy of length scales in a multi-fluid system.

It is easy to verify that the multi-Beltrami states, with the condition $\mathbf{V}_\alpha \times \mathbf{\Omega}_\alpha = 0$, define an equilibrium state provided that the gradient forces, $\nabla\psi_\alpha$, are separately constrained to be zero. The latter gives rise to generalized Bernoulli conditions, necessary for closure, but they are not directly relevant to the analysis presented in this work. The equilibrium state defined by Eq. (14) and Ampère's law constitutes the minimum energy, or relaxed, states of this N-component system. We remind the reader that this equilibrium pertains only to systems that are overall charge neutral. The equilibrium has many features of interest, some of which we shall explore below.

A straightforward consequence of (14) is

$$h_\alpha \equiv \frac{1}{8\pi}\langle \mathbf{P}_\alpha \cdot \mathbf{\Omega}_\alpha\rangle = \frac{\mu_\alpha}{2c}\left\langle \left(\mathbf{A} + \frac{m_\alpha c}{q_\alpha}\mathbf{V}_\alpha\right)\cdot n_\alpha q_\alpha \mathbf{V}_\alpha \right\rangle,$$

from which we can derive

$$\begin{aligned} \sum_\alpha \frac{h_\alpha}{\mu_\alpha} &= \sum_\alpha \frac{1}{2}\langle m_\alpha n_\alpha V_\alpha^2\rangle + \sum_\alpha \frac{1}{2c}\langle \mathbf{A}\cdot\mathbf{J}_\alpha\rangle \\ &= \sum_\alpha \left\langle \frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{B^2}{8\pi}\right\rangle \equiv E. \end{aligned} \tag{15}$$

Combining Eqs. (12) and (15), we find

$$Q = E - \sum \frac{h_\alpha}{\mu_\alpha} = 0, \tag{16}$$

showing that the total variational target functional is zero for the multi-Beltrami solutions. It must be noted that the vanishing of the target functional Q, and the resulting relationship between energy and helicities, is true *only* for the multi-Beltrami states; other equilibria may not be so constrained.

For the multi-Beltrami equilibria, however, Eq. (16) provides us with a helpful relation between the Lagrange multipliers in terms of the system invariants. Hence, in the subsequent sections, it must be implicitly understood that we shall not view the energy as an independent invariant, but one that is fixed by the Lagrange multipliers and the helicities. In the special case of a single helicity invariant—the MHD Taylor states for instance—(16) reduces to the well-known relation that the Lagrange multiplier (determining the scale length of the system) is the ratio of the helicity to the total energy, viz., $\mu = \frac{h}{E}$.

C. A simple class of multi-Beltrami solutions

The N equilibrium Beltrami equations allow for general solutions from which special cases of interest can be further

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derived. Substituting the expression for the generalized vorticity into the Beltrami conditions equation (14), we find

$$\mathbf{B} + \frac{m_\alpha c}{q_\alpha}\nabla\times\mathbf{V}_\alpha = \frac{4\pi}{c}\mu_\alpha\mathbf{J}_\alpha. \tag{17}$$

The velocity and the current contributed by each species are related through $(m_\alpha c/q_\alpha)\nabla\times\mathbf{V}_\alpha = (4\pi/c)\lambda_\alpha^2\nabla\times\mathbf{J}_\alpha$, where $\lambda_\alpha^2 = c^2/\omega_{p\alpha}^2$ and $\omega_{p\alpha}^2 = 4\pi n_\alpha q_\alpha^2/m_\alpha$; λ_α is the appropriate skin depth corresponding to the species labelled by α. Using this relation, Eq. (17) may be written as

$$\mathbf{B} = \frac{4\pi}{c}\left[\mu_\alpha\mathbf{J}_\alpha - \lambda_\alpha^2\nabla\times\mathbf{J}_\alpha\right], \tag{18}$$

and formally manipulated into a set of linear equations

$$\left(\mu_\alpha\mathbf{B} + \lambda_\alpha^2\nabla\times\mathbf{B}\right) = \frac{4\pi}{c}\left[\mu_\alpha^2\mathbf{J}_\alpha + \lambda_\alpha^4\nabla^2\mathbf{J}_\alpha\right], \tag{19}$$

where we have used $\nabla\cdot\mathbf{J}_\alpha = 0$, which follows from the assumption of incompressibility. Naturally, the linearity of the set (19) allows a Fourier expansion

$$\mathbf{X} = \mathbf{X}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{20}$$

with each Fourier coefficient obeying

$$\mu_\alpha\mathbf{B}_k + i\lambda_\alpha^2(\mathbf{k}\times\mathbf{B}_k) = \frac{4\pi}{c}\left[\mu_\alpha^2 - \lambda_\alpha^4 k^2\right]\mathbf{J}_{\alpha k}. \tag{21}$$

Invoking the Fourier-expanded Ampère's law

$$i\mathbf{k}\times\mathbf{B}_k = \frac{4\pi}{c}\mathbf{J}_k = \frac{4\pi}{c}\sum_\alpha \mathbf{J}_{\alpha k}, \tag{22}$$

we can transform (21) into an equation determining $\mathbf{B}_k$

$$\sum_\alpha \frac{\mu_\alpha}{\mu_\alpha^2 - \lambda_\alpha^4 k^2}\mathbf{B}_k = i\left(1 - \sum_\alpha \frac{\lambda_\alpha^2}{\mu_\alpha^2 - \lambda_\alpha^4 k^2}\right)\mathbf{k}\times\mathbf{B}_k. \tag{23}$$

Thus, we conclude that the relaxed state equilibria of this highly complex multi-fluid system can be found by solving a single algebraic vector equation, namely, the single Fourier-Beltrami equation

$$\mathcal{A}\mathbf{B}_k = i\mathbf{k}\times\mathbf{B}_k, \tag{24}$$

where $\mathcal{A} \equiv \mathcal{A}(k, \mu_\alpha, \lambda_\alpha)$. Equation (24) gives rise to the following dispersion relation:

$$\mathcal{A} = \pm k = \sum_\alpha \frac{1}{\mu_\alpha \mp k\lambda_\alpha^2}, \tag{25}$$

which is obtained by observing that $\mathbf{B}$ is a complex-valued field. The fields are characterized by the properties

$$\begin{aligned}&\mathbf{k}\cdot\mathbf{B}_k = 0\\ &B_k^2 = 0 \Rightarrow B_{kR}^2 = B_{kI}^2 \text{ and } \mathbf{B}_{kR}\cdot\mathbf{B}_{kI} = 0,\end{aligned} \tag{26}$$

where the subscripts R and I denote the real and imaginary parts, respectively. The above expressions imply that $\mathbf{B}_R/B_R$ and $\mathbf{B}_I/B_I$ and $\mathbf{k}/k$ define a right-hand orthogonal triad of unit vectors; for example, $\mathbf{k} = k\hat{e}_x$ and $\mathbf{B} = (\hat{e}_y + i\hat{e}_z)B$.

Thus, it is evident that our problem is reduced to solving a polynomial in k, as seen from (25). Notice that the degree of the polynomial is, in general, $(N+1)$, where N is the number of independent species. Thus, one needs $N+1$ independent spatial scales k_j, with j ranging from 1 to $N+1$, to describe an N species plasma. The most general solution is

$$\mathbf{B} = \sum_{j=1}^{N+1}\mathbf{B}_j \mathrm{e}^{i\mathbf{k}_j\cdot\mathbf{x}} + \text{c.c.} \tag{27}$$

Since all coefficients in (25) are real—the skin depths and Lagrange multipliers—the complex roots occurring in the set $\{k_j\}$ must be conjugate pairs.

We end the section by showing two well-known elementary limits of the general dispersion formula:

- For ideal MHD Woltjer-Taylor states, we note that $\lambda_\alpha = 0$, which is equivalent to the statement that there is no effective "independent" species; this is consistent with the entirely scale-free nature of ideal MHD. Thus, we find that (25) reduces to $k = \pm\mu^{-1}$ and μ is fixed through the ratio of the (magnetic) helicity to the energy.
- For normalized Hall MHD, there is only one effective independent fluid, the ion fluid. The neglect of electron inertia results in $\lambda_e = 0$, i.e., a vanishing electron skin depth. Thus, there is only the ion skin depth λ_i which is normalized to unity by adopting suitable units. However, there are two invariants, the magnetic and the ion generalized helicities, implying the existence of two Lagrange parameters, $\mu_e = -a_0$ and $\mu_i = a_1$. The normalized two scale lengths, determined by the quadratic

$$\pm k = -\frac{1}{a_0} + \frac{1}{a_1 \mp k}, \tag{28}$$

or equivalently

$$\left(\pm k + \frac{1}{a_0}\right)(a_1 \mp k) = 1, \tag{29}$$

which forms the basis of the much studied double Beltrami equilibria.

The close connections between the multi-Beltrami states and the constrained variational principles in Section II B, constitute the *raison d'être* for envisioning the latter as the relaxed states of the system. However, a cautionary word must be added as the variational problem may not always be mathematically well-posed; see, e.g., Ref. 8 for a discussion of the same in the context of Hall MHD and double Beltrami states. Furthermore, we also wish to caution that the scenario with complex roots, viz., exponentially growing or damping solutions, must also be handled with care. There is obviously no problem for a finite domain, but growing solutions will not be permitted over large domains.

III. HALL MHD AND EQUIVALENT MODELS, DOUBLE BELTRAMI STATES AND PHYSICAL INVARIANTS

In this section, we concentrate on Hall MHD and establish its connections with other relevant models in the

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literature. Next, we use the multi-Beltrami formalism in Hall MHD to obtain some interesting relations in the context of the two helicities (and the energy).

A. Hall MHD and its relation to other models

In their simplest manifestation (in ideal MHD), the Beltrami states are described by purely sinusoidal solutions, i.e., a single Beltrami system admits only real values of $k=\pm\mu^{-1}$, since μ is real. The situation changes fundamentally when we take the next step—when we consider models ranging from Hall MHD onwards; the existence of quadratic and higher degree polynomials can give rise to complex roots, which manifest themselves as hyperbolic functions. The consequences are illustrated by the following list of applications:

- Pressure confining states become possible.[7]
- Perfectly diamagnetic states, limiting the magnetic field to an appropriate skin depth near the edge of the system, become accessible to classical systems.[50]
- States with zonal-flow structures can also emerge.

In fact, the latter class of structures was observed in Ref. 54, where Hall MHD simulations gave rise to zonal flows, whilst ideal MHD does not (typically) do so. Thus, (25) provides an elegant and concise physical explanation of this phenomenon observed in simulations. For systems with multiple species, more complex than ideal MHD, a wider range of length scales and structures are accessible through (25).

Yet, it is not just Hall MHD that exhibits the above properties. Let us suppose that we consider an electron-positron-ion plasma, wherein there exists only one species (ions) which is much more massive than either the electrons or the positrons. Such a composition is expected to exist in pulsar and AGN winds; see, e.g., Ref. 59 for a discussion of the same. As the electrons and the positrons are effectively massless, we can set $\lambda_e=\lambda_p=0$ with the suffices e and p denoting the two species, respectively. Let us set $\mu_e=a_e$, $\mu_p=a_p$, and $\mu_i=a_1$. We shall work in normalized units where $\lambda_i=1$ just as in Hall MHD. Upon substitution into (25) and defining $-a_0^{-1}=a_e^{-1}+a_p^{-1}$, we recover (28). As a result, we see that our model is indeed "equivalent" to Hall MHD.

In fact, it is easy to show that in a model with $N-1$ massless species and one much-heavier species, one arrives at (28), rendering each of these models equivalent to Hall MHD. This equivalence occurs not just on the level of multi-Beltrami solutions but also at the level of the dynamical equations. Other examples of such models include those where the sole massive species is ionized dust—we shall examine a two-species dust model in Section IV.

B. Double Beltrami states and helicity invariants

The double Beltrami states were first proposed in the context of Hall MHD (Ref. 7) and have been used extensively since then. These states constitute, perhaps, the simplest system in which we could examine what we believe is a fundamentally interesting question: Do two field configurations have to be identical in order to have identical invariants, or, is it possible for two independent configurations could also give rise to identical invariants?

Following the (normalized) notation adopted in Ref. 7, the double Beltrami conditions reduce to

$$b^{-1}\mathbf{B}=\mathbf{V}-\nabla\times\mathbf{B};\quad \mathbf{B}+\nabla\times\mathbf{V}=a\mathbf{V}. \tag{30}$$

The system has three invariants:

$$\begin{aligned} h_1&=\frac{1}{2}\langle\mathbf{A}\cdot\mathbf{B}\rangle,\\ h_2&=\frac{1}{2}\langle(\mathbf{A}+\mathbf{V})\cdot(\mathbf{B}+\nabla\times\mathbf{V})\rangle=\frac{a}{2}\langle(\mathbf{A}+\mathbf{V})\cdot\mathbf{V}\rangle,\\ E&=\frac{1}{2}\langle B^2+V^2\rangle,\end{aligned} \tag{31}$$

and exhibits two independent solutions: $\nabla\times\mathbf{B}_\pm=\alpha_\pm\mathbf{B}_\pm$ where $\alpha_\pm=\frac{1}{2}\left[a-b^{-1}\pm\sqrt{(a+b^{-1})^2-4}\right]$ could be both real or form a complex conjugate pair.

For real roots, (30) and (31) imply

$$\begin{aligned} h_{1\pm}&=\frac{1}{2\alpha_\pm}\langle B_\pm^2\rangle,\\ h_{2\pm}&=\frac{a}{2}\left(b^{-1}+\alpha_\pm\right)\left(b^{-1}+\alpha_\pm+\alpha_\pm^{-1}\right)\langle B_\pm^2\rangle,\\ E_\pm&=\frac{1}{2}\left[1+\left(b^{-1}+\alpha_\pm\right)^2\right]\langle B_\pm^2\rangle.\end{aligned} \tag{32}$$

It must be borne in mind that only two of these three invariants are independent since $a^{-1}h_2-b^{-1}h_1=E$. Now, we seek the conditions under which $\Delta h_1=h_{1+}-h_{1-}=0$. It is possible only when $\langle B_-^2\rangle=\frac{\alpha_-}{\alpha_+}\langle B_+^2\rangle$. We use this relation and also demand that $\Delta E=E_+-E_-=0$ (note that $\Delta h_2=0$ will automatically follow). For real roots, then, $\Delta E=0$ if and only if $a+b^{-1}=2$, which implies that $\alpha_+=\alpha_-$.

In other words, for the case with real roots, we find that the physical invariants are identical only when the field configurations themselves are identical.

If the roots are a complex conjugate pair, a very different situation prevails. Two independent real magnetic fields, $\mathbf{B}_R=\frac{1}{2}\left[\mathbf{B}_\alpha+\mathbf{B}_\alpha^\star\right]$ and $\mathbf{B}_I=-\frac{i}{2}\left[\mathbf{B}_\alpha-\mathbf{B}_\alpha^\star\right]$ may be constructed from the central equation $\nabla\times\mathbf{B}_\alpha=\alpha\mathbf{B}_\alpha$, its complex conjugate $(\alpha_+=\alpha,\alpha_-=\alpha^\star)$, and (30) and (31). From these relations, we calculate

$$\langle\mathbf{A}_R\cdot\mathbf{B}_R\rangle=\langle\mathbf{A}_I\cdot\mathbf{B}_I\rangle=\frac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\alpha^\star}\right)\langle\mathbf{B}_\alpha\cdot\mathbf{B}_\alpha^\star\rangle, \tag{33}$$

$$\langle B_R^2\rangle=\langle B_I^2\rangle=\frac{1}{2}\langle\mathbf{B}_\alpha\cdot\mathbf{B}_\alpha^\star\rangle, \tag{34}$$

$$\langle V_R^2\rangle=\langle V_I^2\rangle=\frac{1}{2}\langle\mathbf{B}_\alpha\cdot\mathbf{B}_\alpha^\star\rangle, \tag{35}$$

and

$$\langle\mathbf{A}_R\cdot\mathbf{V}_R\rangle=\frac{1}{4}\left[\frac{b^{-1}+\alpha^\star}{\alpha}+\frac{b^{-1}+\alpha}{\alpha^\star}\right]\langle\mathbf{B}_\alpha\cdot\mathbf{B}_\alpha^\star\rangle=\langle\mathbf{A}_I\cdot\mathbf{V}_I\rangle, \tag{36}$$

after using the relation $(b^{-1}+\alpha)(b^{-1}+\alpha^\star)=1$.

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The results, (33)–(36), pertaining to the field configurations flowing from the complex roots, are rather revealing:

- All invariants—the magnetic helicity, the generalized helicity and the total energy—are equal for the two independent configurations denoted by $\mathbf{B}_R$ and $\mathbf{B}_I$.
- For each of these solutions, there is an equipartition in kinetic and magnetic energies. One must ascertain, however, that these configurations do imply different magnetic and flow fields.

To show this, let us examine the Cartesian ABC solutions of the double Beltrami system. The two solutions can be constructed from

$$\mathbf{B}_\alpha = A(\hat{e}_x + i\hat{e}_y)e^{i\alpha z} + B(\hat{e}_y + i\hat{e}_z)e^{i\alpha x} + C(\hat{e}_z + i\hat{e}_x)e^{i\alpha y}, \tag{37}$$

and this leads us to

$$\mathbf{B}_R = [A\cos\alpha z - C\sin\alpha y]\hat{e}_x + [B\cos\alpha x - A\sin\alpha z]\hat{e}_y + [C\cos\alpha y - B\sin\alpha x]\hat{e}_z, \tag{38}$$

$$\mathbf{B}_I = [A\sin\alpha z + C\cos\alpha y]\hat{e}_x + [B\sin\alpha x + A\cos\alpha z]\hat{e}_y + [C\sin\alpha y + B\cos\alpha x]\hat{e}_z, \tag{39}$$

which are, indeed, distinct and independent. We have, thus, identified the complex conjugate solutions as distinct and special (double Beltrami) states that lead to exactly identical values of the physical invariants. As the roots comprise both real and imaginary components, the ABC states can exhibit exponential (and thereby unbounded) behavior in addition to the sinusoidal functions. In some domains, similar structures have been reported in Hall MHD astrophysical simulations.[54]

Finally, the very interesting nature of this class of solutions (different fields but same invariants) raises a rather profound question—how do we distinguish between these flows if not through invariants?

IV. DUST AND TWO-FLUID MODELS

We shall analyze the properties of two-fluid model in detail in this section, and point out the connections with two-species dust model. The nature of the triple Beltrami states of an electron-positron plasma, and the connections with the physical invariants of the system, are also studied.

A. An analysis of dust models and their connection to two-fluid models

To explore the multi-Beltrami flows a bit further, we consider a four-species system with positively and negatively charged dust, in addition to electrons and ions. As the masses of the latter duo are negligible in comparison, we set $\lambda_i = \lambda_e = 0$ in (25) for a four-species system. Thus, we arrive at a dispersion relation

$$\pm k = \frac{1}{\mu_{ei}} + \frac{1}{\mu_p \mp k\lambda_p^2} + \frac{1}{\mu_n \mp k\lambda_n^2}, \tag{40}$$

where the subscripts n and p denote the negatively and positively charged dust, respectively; we also have $\mu_{ei}^{-1} = \mu_i^{-1} + \mu_e^{-1}$. The dispersion relation is cubic in k since there are only two effective (dust) species. Our system is endowed with three Lagrange multipliers, and two physically specified skin depths. The three scales (the number of independent k roots) are likely to be tied to one "macroscopic" scale (the system size), and to the two "microscopic" scales (the two dust skin depths). In a possible model for a protoplanetary disc, we assume $\rho_d \sim 1\,\mathrm{g/cm^3}$, $m_d \sim 10^{-12}\,\mathrm{gm}$, and $Z_d \sim 3\times 10^3$, where $q_d = eZ_d$.[60–62] Upon solving for the dust skin depth, we find that it is on the order of 0.1 km. Hence, it is seen that the "microscopic" scale structures that can form lie within the range of planetesimal sizes. This has immediate secondary implications—the formation of zonal flow structures with finite vorticity is feasible. It is well known that such structures can "trap" particles, leading to planetesimal and planet formation.[63–65] Since the vortices are already quite large (dust skin depths) in this system, the process of trapping is likely to be efficient.

Now, we consider a system that is a near-equivalent of the two-dust species system described above. For a two-species plasma with comparable species masses, necessitating the inclusion of both skin depths, the dispersion relation, from (25), may be written as

$$\pm k = \frac{1}{\mu_1 \mp k\lambda_1^2} + \frac{1}{\mu_2 \mp k\lambda_2^2}. \tag{41}$$

We see that (41) is identical to (40) if we let $\mu_{ei}^{-1} \to 0$ in the latter. Once again, we emphasize that a species with $N-2$ massless species and two heavy species will be (almost) equivalent to the conventional two-fluid plasma. Consequently, an analysis of (40) will suffice to also cover the two-fluid model as well. Let us begin by introducing the dimensionless variables $x = k\lambda_p$, $\tilde{\mu}_{ei} = \mu_{ei}/\lambda_p$, $\tilde{\mu}_p = \mu_p/\lambda_p$, $\tilde{\mu}_n = \mu_n/\lambda_p$, and $\delta = \lambda_n^2/\lambda_p^2$. We find that (40) transforms to

$$\delta x^3 \mp (\tilde{\mu}_n + \delta\tilde{\mu}_p + \delta\tilde{\mu}_{ei}^{-1})x^2 + [\tilde{\mu}_n(\tilde{\mu}_p + \tilde{\mu}_{ei}^{-1}) + \delta\tilde{\mu}_{ei}^{-1}\tilde{\mu}_p + \delta + 1]x \mp (\tilde{\mu}_{ei}^{-1}\tilde{\mu}_p\tilde{\mu}_n + \tilde{\mu}_p + \tilde{\mu}_n) = 0. \tag{42}$$

As the above expression is a cubic, it is not easy to investigate the conditions under which one can have either three real roots or one real root (and two complex conjugate roots). To do so, one must investigate the discriminant of the above equation, which is rather cumbersome. For the special case of the two-fluid model ($\tilde{\mu}_{ei}^{-1} = 0$), the discriminant reduces to

$$\Delta = (\tilde{\mu}_n + \delta\tilde{\mu}_p)^2(\tilde{\mu}_n\tilde{\mu}_p + \delta + 1)^2 - 4\delta(\tilde{\mu}_n\tilde{\mu}_p + \delta + 1)^3 - 4(\tilde{\mu}_n + \delta\tilde{\mu}_p)^3(\tilde{\mu}_p + \tilde{\mu}_n) - 27\delta^2(\tilde{\mu}_p + \tilde{\mu}_n)^2 + 18\delta(\tilde{\mu}_n + \delta\tilde{\mu}_p)(\tilde{\mu}_n\tilde{\mu}_p + \delta + 1)(\tilde{\mu}_p + \tilde{\mu}_n), \tag{43}$$

and this is still a very cumbersome expression. However, in the two-fluid model with quasineutrality, it must be noted that $\delta = m_n/m_p$. Thus, for an electron-positron plasma, we have $\delta = 1$ but for an electron-ion plasma, we typically have

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$\delta \ll 1$. Indeed, if we take $\delta = 0$ in (42), we recover the Hall MHD quadratic dispersion relation. In the case where $\delta \ll 1$, the discriminant to linear order in δ is

$$\begin{aligned}\Delta &= \tilde{\mu}_n^2[(\tilde{\mu}_n\tilde{\mu}_p + 1)^2 - 4\tilde{\mu}_n(\tilde{\mu}_n + \tilde{\mu}_p)] \\ &\quad + \delta[-2\tilde{\mu}_n^2(\tilde{\mu}_p^2 - 10 + \tilde{\mu}_n\tilde{\mu}_p(\tilde{\mu}_p^2 - 4)) + 4(2\tilde{\mu}_n\tilde{\mu}_p - 1)].\end{aligned} \tag{44}$$

By evaluating the expression in the second line of the RHS for given values of the μ's, one can use it as a quick (albeit not fully correct) check to see whether three or one real roots emerge. Now, we shall specialize to the case where $\delta = 1$, and study the properties of the electron-positron plasma in detail; as noted earlier, these plasmas are expected to be relevant in domains, such as pulsar magnetospheres.

B. The electron-positron plasma

When we set $\delta = 1$ in (43), we find that

$$\begin{aligned}\Delta &= \tilde{\mu}_n^4\tilde{\mu}_p^2 - 4\tilde{\mu}_n^4 - 2\tilde{\mu}_n^3\tilde{\mu}_p^3 + 6\tilde{\mu}_n^3\tilde{\mu}_p + \tilde{\mu}_n^2\tilde{\mu}_p^4 - 4\tilde{\mu}_n^2\tilde{\mu}_p^2 \\ &\quad + 13\tilde{\mu}_n^2 + 6\tilde{\mu}_n\tilde{\mu}_p^3 - 22\tilde{\mu}_n\tilde{\mu}_p - 4\tilde{\mu}_p^4 + 13\tilde{\mu}_p^2 - 32,\end{aligned} \tag{45}$$

and one can use this expression to gauge the nature of the roots. For instance, if $\tilde{\mu}_n \approx \tilde{\mu}_p \ll 1$, we find that $\Delta < 0$ leading to the emergence of one real and two complex (conjugate) roots. Alternatively, consider the case where $\tilde{\mu}_n \approx \tilde{\mu}_p \gg 1$, three real roots emergence since $\Delta > 0$. Let us now investigate two interesting cases:

- We choose $\tilde{\mu}_n \ll 1$ and $\tilde{\mu}_p \gg 1$. With this choice, the largest term in (45) is likely to be $-4\tilde{\mu}_p^4$ implying that $\Delta < 0$ and leading to complex conjugate roots (and one real root). Even if we choose the converse, viz., $\tilde{\mu}_p \ll 1$ and $\tilde{\mu}_n \gg 1$, we find that the largest term in Δ is likely to be $-4\tilde{\mu}_n^4$, leading to $\Delta < 0$ once again.
- Let us operate with $\tilde{\mu}_n \approx \tilde{\mu}_p \approx 1$. We end up with $\Delta < 0$ which implies the existence of a complex conjugate pair and one real root.

If we hold the energy fixed, we see that the helicities and the μ's are related via (16) for multi-Beltrami states. Hence, if we consider the case where all the μ's are very "small" (as considered in one of the above cases), it is reasonable to suppose that the helicities must be small as well in order for the ratio h_α/μ_α to be finite and thus give rise to the fixed energy. Similarly, the existence of large μ's will necessitate large values of the helicity as per the same reasoning. Thus, we see that knowledge of the helicities and energy will enable some degree of knowledge of the μ's, which in turn enable us to determine the nature of the multi-Beltrami solutions, as described previously.

To conduct a more quantitative analysis, we introduce the auxiliary variables $\tilde{\mu} = \tilde{\mu}_p + \tilde{\mu}_n$ and $\epsilon = \tilde{\mu}_p - \tilde{\mu}_n$ for the electron-positron plasma in (42); there are two possibilities owing to the $\mp$ present, and we choose to work with the '−' case here. Using the variables defined above, we arrive at

$$\left(x - \frac{\tilde{\mu}}{2}\right)\left[x\left(x - \frac{\tilde{\mu}}{2}\right) + 2\right] = \frac{\epsilon^4}{4}x. \tag{46}$$

We shall provide some information about the nature of roots by evaluating some limiting cases.

- We set $\epsilon = 0$ in (46). This leads us to the following roots
$$\begin{aligned}x_1 &= \frac{\tilde{\mu}}{2}, \\ x_\pm &= \frac{\tilde{\mu}}{4} \pm \sqrt{\frac{\tilde{\mu}^2}{16} - 2},\end{aligned} \tag{47}$$
and this clearly indicates that three real roots are obtained when $\tilde{\mu}^2 > 32$. Similarly, for $\tilde{\mu}^2 < 32$, we find that $x_\pm$ constitutes a pair of complex conjugate roots. This result is in agreement with the more qualitative results presented earlier.
- Now, suppose that $|\epsilon/\tilde{\mu}| \ll 1$. We find that the corrections to (47) can be computed in a straightforward manner, although the exact results are not reproduced here. Once again, we find that $\tilde{\mu}^2 = 32$ remains the critical point at which the nature of the roots changes (from three real roots to one real and two complex conjugate roots).
- Next, consider the case where $\tilde{\mu} = 0$ and ϵ is finite. The roots are found to be
$$\begin{aligned}X_1 &= 0, \\ X_\pm &= \pm\sqrt{\frac{\epsilon^2}{4} - 2},\end{aligned} \tag{48}$$
and this clearly implies that $\epsilon^2 > 8$ leads to three real roots, and $\epsilon^2 < 8$ implies the existence of one real root and two complex conjugate roots given by $X_\pm$.
- We look at the scenario where $|\tilde{\mu}/\epsilon| \ll 1$. The nature of the roots does not exhibit any significant changes, with $\epsilon^2 = 8$ still serving as the critical point.

C. Physical invariants and triple Beltrami states for the electron-positron plasma

Before proceeding further, we recollect that a quasineutral electron-positron plasma exhibits the same scale lengths. We normalize the multi-Beltrami states in terms of the Alfvenic units, and the length scale by the (common) skin depth. In these dimensionless units, the trio of equations is

$$\begin{aligned}\mathbf{B} + \nabla \times \mathbf{V}_p &= a_p\mathbf{V}_p, \\ \mathbf{B} - \nabla \times \mathbf{V}_n &= a_n\mathbf{V}_n, \\ \nabla \times \mathbf{B} &= \mathbf{V}_p - \mathbf{V}_n.\end{aligned} \tag{49}$$

In the above expressions, the a's are proportional to the $\tilde{\mu}$'s employed in Section III. The three invariants are known to be

$$\begin{aligned}h_p &= \frac{1}{2}\langle(\mathbf{A} + \mathbf{V}_p)\cdot(\mathbf{B} + \nabla \times \mathbf{V}_p)\rangle = \frac{a_p}{2}\langle(\mathbf{A} + \mathbf{V}_p)\cdot\mathbf{V}_p\rangle, \\ h_n &= \frac{1}{2}\langle(\mathbf{A} - \mathbf{V}_n)\cdot(\mathbf{B} - \nabla \times \mathbf{V}_n)\rangle = \frac{a_n}{2}\langle(\mathbf{A} - \mathbf{V}_n)\cdot\mathbf{V}_n\rangle, \\ E &= \frac{1}{2}\langle V_p^2 + V_n^2 + B^2\rangle.\end{aligned} \tag{50}$$

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From (50), it is easy to establish that $h_p/a_p - h_n/a_n = E$, once again demonstrating that the three invariants are *not* linearly independent. Next, suppose that we choose $\nabla \times \mathbf{B} = \kappa \mathbf{B}$, where κ represents any of the three real roots of the cubic equation for the triple Beltrami states. We find that the two helicities h_p and h_n are given by

$$h_p = \left[\frac{a_p}{a_p - \kappa}\right]^2 \frac{\langle B^2\rangle}{2\kappa},$$
$$h_n = \left[\frac{a_n}{a_n + \kappa}\right]^2 \frac{\langle B^2\rangle}{2\kappa}, \tag{51}$$

and we employ the same analysis used in Section III B. By computing Δh_p and Δh_n for two real roots κ_1 and κ_2, we find that $\Delta h_p = \Delta h_n = 0$ occurs when $(\kappa_1 - \kappa_2)(a_p + a_n) = 0$. As before, it is evident that the trivial choice $\kappa_1 = \kappa_2$ leads to identical values of the helicities. But, we also witness the emergence of the additional condition $a_p + a_n = 0$, which was altogether absent in the double Beltrami case. If we use this expression in (49) as well as the Beltrami condition $\nabla \times \mathbf{B} = \kappa \mathbf{B}$, we find that the one of the three roots is given by $\kappa_3 = -a_n$. The other two roots exhibit the property $\kappa_1\kappa_2 = 2$, and the results are identical to the ones derived in (47) under a suitable relabelling of variables. Moreover, if we use $a_p + a_n = 0$ in (51), we find that $h_p = h_n$ for the two roots κ_1 and κ_2. Consequently, we arrive at the result $h_p/a_p = -h_n/a_n = E/2$ for this particular case.

Hence, we are led to the rather remarkable result that the special choice $a_p + a_n = 0$ can give rise to two different field configurations that admit the same values of the physical invariants. The other case, involving complex conjugate roots, is more subtle, and a comprehensive treatment is reserved for the future.

V. CONCLUSION

This paper is devoted to constructing relaxed states for multi-fluid system. We derived them by invoking a general variation principle—extremizing the total (magnetic and fluid) energy subject to the constraints of helicity conservation (which, in fact, are readily derivable from the fluid equations of motion). These equilibria, dubbed the multi-Beltrami states, were then simplified and a particular class of solutions was presented in a simple and elegant form, encapsulated by (25).

The relation (25) yielded useful information, including the existence of a wide range of length scales, determined through the skin depths, helicities, and the total energy of the system. We explored some of the implications of the multi-Beltrami system in Sections III and IV. Amongst others, we showed that the electron-positron-ion plasma could be viewed as equivalent to Hall MHD, and those models with positively and negatively dust exhibited a close relation to two-fluid model. We also analyzed the properties of the latter via the cubic discriminant by considering some limiting cases, which yielded useful information about the nature of the roots, and thus, the Beltrami states of the system. The establishment of such equivalences is of considerable importance since many of these models are endowed with a common Hamiltonian structure.[10] The work undertaken herein lends further credence to these results, suggesting that one can find suitable variable transformations that map all of these models to a common underlying noncanonical Poisson bracket.

A key result that emerged via our analysis was that double Beltrami states of Hall MHD under certain conditions (complex conjugate roots) yielded different field configurations that gave rise to identical values of the three physical invariants. In other words, we demonstrated through this simple, but revealing, example that knowledge of the physical invariants is not sufficient to determine and distinguish between different field configurations. Rather curiously, when we considered the double Beltrami solutions with two real roots, we were able to use the physical invariants as a marker to distinguish between different field configurations.

We also carried out a similar procedure for the triple Beltrami states of an electron-positron plasma, and demonstrated that a very special case (with Lagrange multipliers of equal magnitude and opposite sign) led to the physical invariants, for the two real roots κ_1 and κ_2, becoming identical. Thus, a clear hierarchy begins to emerge:

- In single Beltrami states (for ideal MHD), there is no possibility of the invariants being equal unless the field configurations are identical.
- In double Beltrami states, the physical invariants are identical even with differing field configurations, but the double Beltrami system must comprise complex conjugate roots. When the case with real roots is considered, only identical field configurations yield identical values of the invariants.
- In triple Beltrami states, the physical invariants are identical even when the system is endowed with real roots that constitute distinct field configurations (provided that a special condition is met).

In other words, we see that the inclusion of more Beltrami states widens the modes of behavior accessible to the system; it is the additional degree(s) of freedom that makes it possible for the "degeneracy" in the physical invariants to occur, even when one considers distinct field configurations.

We hypothesize that a broad spectrum of problems can be investigated within the multi-Beltrami paradigm. For instance, a multi-fluid dynamo, with the concomitant generation of flows and magnetic fields, could be constructed following the prescription in Refs. 44–47, 66, which would entail the use of these multi-Beltrami states. We could also use the multi-Beltrami states to model highly energetic eruptions, such as supernovae and gamma ray bursts. In a non-relativistic Hall MHD context, this was successfully implemented in Refs. 36–38—the transitions in the number of *real* roots available to the system led to such eruptions. The relaxed states resulting from the quenching of magneto-rotational turbulence in astrophysics[67–71] also remain unexplored via this paradigm. Although we have restricted ourselves herein to Newtonian systems, one can easily extend the formalism to incorporate relativistic[51] and

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quantum-mechanical[72–74] effects, and thereby study compact objects, AGN jets, etc.

Thus, it is clear that the multi-Beltrami solutions represent a unique means of studying a diverse range of astrophysical phenomena, and we shall investigate some of the aforementioned issues in subsequent works.

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