# $\mathcal{CPT}-$conserving Hamiltonians and their nonlinear supersymmetrization using differential charge-operators $\mathcal{C}$

**Bijan Bagchi** and **A. Banerjee**
Department of Applied Mathematics, University of Calcutta
92 Acharya Prafulla Chandra Road, Kolkata 700 009, West Bengal, India
e-mail: bbagchi123@rediffmail.com

**Emanuela Caliceti**
Dipartimento di Matematica dell' Università and Istituto Nazionale di Fisica Nucleare, I-40127 Bologna, Italy
e-mail: caliceti@dm.unibo.it

**Francesco Cannata**
Dipartimento di Fisica dell'Università and Istituto Nazionale di Fisica Nucleare, I-40126 Bologna, Italy
e-mail: Francesco.Cannata@bo.infn.it

**Hendrik B. Geyer**
Institute of Theoretical Physics, University of Stellenbosch, Matieland 7602, South Africa
e-mail: hbg@sun.ac.za

**Christiane Quesne**
Physique Nucléaire Théorique et Physique Mathématique
Université Libre de Bruxelles, Campus de la Plaine CP229, B-1050 Brussels, Belgium
e-mail: cquesne@ulb.ac.be

and

**Miloslav Znojil**
Ústav jaderné fyziky AV ČR, 250 68 Řež, Czech Republic
e-mail: znojil@ujf.cas.cz

# Abstract

A brief overview is given of recent developments and fresh ideas at the intersection of $\mathcal{PT}-$ and/or $\mathcal{CPT}-$symmetric quantum mechanics with supersymmetric quantum mechanics (SUSY QM). Within the framework of the resulting supersymmetric version of $\mathcal{CPT}-$symmetric quantum mechanics we study the consequences of the assumption that the "charge" operator $\mathcal{C}$ is represented in a differential-operator form of the second or higher order. Besides the freedom allowed by the Hermiticity constraint for the operator $\mathcal{CP}$, encouraging results are obtained in the second-order case. In particular, the integrability of intertwining relations proves to match the closure of our nonlinear (*viz.,* polynomial) SUSY algebra. In a particular illustration, our form of $\mathcal{CPT}-$symmetric SUSY QM leads to a new class of non-Hermitian polynomial oscillators with real spectrum which turn out to be $\mathcal{PT}-$asymmetric.



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## Keywords

$\mathcal{PT}-$symmetric Hamiltonians; $\mathcal{CPT}-$symmetric quantum mechanics; supersymmetric quantum mechanics; nonlinear SUSY algebra; intertwining relations; $\mathcal{PT}-$asymmetric potentials

# 1 Introduction

The recent growth of interest in the possibility of working with non-Hermitian observables in quantum theory (cf. the concise review papers [1]) is mainly due to the influential Bender's and Boettcher's letter [2] where its authors observed that the spectrum of certain Hamiltonians $H \neq H^\dagger$ seems real and discrete and bounded below.

They conjectured that such an observation may find a firmer mathematical background and explanation in the symmetry of their models with respect to the combined action of the parity $\mathcal{P}$ and time-reversal (i.e., complex conjugation) $\mathcal{T}$. This inspiring idea has been further developed and re-formulated as proposals of the so called $\mathcal{PT}-$symmetric quantum mechanics [3], pseudo-Hermitian quantum mechanics [4] and $\mathcal{CPT}-$symmetric quantum mechanics [5]. They all deal with more or less the same class of the specific non-Hermitian models characterized, in the language of the latter reference, by another symmetry operator $\mathcal{C}$ which is very conveniently called "charge".

There exists an extensive literature on $\mathcal{PT}$-symmetric quantum mechanics [6, 7]. In particular, in a number of papers [8, 9, 10, 11], unexpected consequences of the non-Hermiticity of Hamiltonians have been noticed to emerge after its supersymmetrization a la Witten [12]. In terms of local models $H = p^2 + V(x)$ on the real line ($x \in I\!\!R$) where $V(x) = V^*(-x)$, these Hamiltonians satisfy the intertwining relation

$$H^\dagger \mathcal{P} = \mathcal{P} H \, . \tag{1}$$

Such $\mathcal{PT}$-symmetric Hamiltonians may have either complex, or real spectra. When the $\mathcal{PT}$ symmetry remains spontaneously unbroken and all the spectrum is real [2], one has elaborated the concept of quasi-Hermiticity of the Hamiltonian [13, 14]. This means that the intertwining relation (1) holds also with $\mathcal{P}$ replaced by a positive-definite operator $\Theta = \Theta^\dagger > 0$ which plays the role of a metric operator. The physical interpretation of such models is standard [15]. When the spectrum is complex, relation (1) can still be written with $\mathcal{P}$ replaced by a pseudo-metric [16, 17, 18].

We shall now generalize the previous considerations to a new type of symmetry. The framework of our constructions proposed in our recent letter [19] will incorporate Hamiltonians with both real and complex spectra. Correspondingly, we shall also deal, in general, with non-positive metric (i.e., pseudo-metric). As for the case of $\mathcal{PT}$ symmetry, the interpretation of this type of quantum mechanics can be disputable [14] and might require some innovation. However, we stress that, in our framework, we can find models which have real spectra, where, in particular, a non-Hermitian Hamiltonian is related by similarity transformations not only to a Hermitian operator, but, more specifically, to a Hermitian Schrödinger operator. Thus, for these cases, we recover the standard quantum mechanics, after similarity. Therefore, these cases are not disputable in their interpretation. From a conservative point of view, one might restrict the interest of our supersymmetric approach insofar as one takes it instrumentally as a strategy to find complex Hamiltonians with real spectrum that do not satisfy $\mathcal{PT}$ invariance (see Section 4 below).

## 1.1 SUSY intertwining relations

In the same spirit as in Ref. [19], we shall study the intertwining relations

$$\mathcal{F}H^\dagger = H\mathcal{F} \tag{2}$$

mediated by the Hermitian operator

$$\mathcal{F} = \mathcal{CP} \qquad (\mathcal{F} = \mathcal{F}^\dagger)\,, \tag{3}$$

where $\mathcal{P}$ is the parity operator, and $\mathcal{C}$ a generalized "charge" operator, assumed to be a polynomial in the differential operator $d/dx$. For any Hamiltonian $H$, Eq. (2) is equivalent to $\mathcal{CPT}$ conservation, with $\mathcal{T}$ the time reversal operator,

$$\mathcal{CPT}H = H\mathcal{CPT}\,. \tag{4}$$

In this paper we shall not discuss in detail the metric interpretation for $\mathcal{F}$, but only stress the fact that, if $\mathcal{F}$ and $H$ satisfy Eq. (2), then also $\mathcal{F}^{-1}$ (if it exists) and $H$ meet an intertwining

$$H^\dagger\mathcal{F}^{-1} = \mathcal{F}^{-1}H\,, \tag{5}$$

which means that $H$ is pseudo-Hermitian with respect to $\mathcal{F}^{-1}$, i.e., $\mathcal{F}^{-1}-$pseudo-Hermitian [4].

This observation may be useful for implementing the metric based on $\mathcal{F}^{-1}$, when $\mathcal{F}^{-1}$ has a better behavior than $\mathcal{F}$, *e.g.* with respect to boundedness. Nevertheless, in our text we also use for a $\mathcal{F}$ satisfying Eqs. (3), (2), the word "metric" operator. In fact, Eq. (2) implies that $H\mathcal{F}$ is Hermitian. As a consequence of (5), if $\mid \phi\rangle$ and $\mid \psi\rangle$ are two arbitrary vectors of the Hilbert space $L^2(\mathbf{R})$, we have

$$\int \phi^*(x)\left(\mathcal{F}^{-1}H\psi\right)(x)dx = \int \psi^*(x)\left(\mathcal{F}^{-1}H\phi\right)(x)dx\,.$$

This can be interpreted as a Hermiticity condition for $H$ provided the scalar product is defined as

$$\begin{aligned}
\langle \phi \mid \psi\rangle_{\mathcal{F}^{-1}} &= \int \phi^*(x)\left(\mathcal{F}^{-1}\psi\right)(x)\,dx\,;\\
\langle \psi \mid \phi\rangle_{\mathcal{F}^{-1}} &= \int \psi^*(x)\left(\mathcal{F}^{-1}\phi\right)(x)\,dx\,.
\end{aligned}$$

It is worthwhile to point out, however, that, in absence of additional constraints, neither $\mathcal{F}^{-1}$ nor $\mathcal{F}$ is necessarily positive definite so that, for instance, the equation $\mathcal{F} \mid \phi\rangle = 0$ might have a non-trivial solution different from $\mid \phi\rangle = 0$. At this level, $\langle\phi \mid \phi\rangle_{\mathcal{F}^{-1}}$ does not define a true norm but merely a pseudo-norm [17, 20].

It is evident that solving Eq. (2) amounts to analyzing the compatibility between $\mathcal{C}$ and $H$; in other words, $\mathcal{C}$ and $H$ are to be found contextually. Once Eq. (2) is formally solved, one can investigate its supersymmetrization [21]. By this we mean the construction of super-charges

$$Q = \begin{pmatrix} 0 & \mathcal{F} \\ 0 & 0 \end{pmatrix}, \qquad \tilde{Q} = \begin{pmatrix} 0 & 0 \\ \mathcal{F}^* & 0 \end{pmatrix} \tag{6}$$

with anti-commutator

$$K \equiv \left\{Q, \tilde{Q}\right\} = \begin{pmatrix} \mathcal{F}\mathcal{F}^* & 0 \\ 0 & \mathcal{F}^*\mathcal{F} \end{pmatrix}, \tag{7}$$

and a polynomial formulae

$$\mathcal{F}\mathcal{F}^* = \sum_{k=0}^{n} a_k H^k, \qquad \mathcal{F}^*\mathcal{F} = \sum_{k=0}^{n} a_k^*(H^*)^k \tag{8}$$

with the final goal to elucidate the conditions leading to such a type of the closure of the algebra [22].

## 1.2 Plan of the paper

In Section 2 we elaborate a particular solution to our problem inspired by the specific second-order supersymmetry (SSUSY) results of ref. [11]. Our solution of Eq. (8) will have the form

$$\mathcal{F}\mathcal{F}^* = h_1^2 - \frac{c^2}{4}, \qquad \mathcal{F}^*\mathcal{F} = h_2^2 - \frac{c^2}{4}, \tag{9}$$

where $h_1$ is naturally related to $h_2$ by Hermitian conjugation,

$$h_1 = h_2^\dagger, \tag{10}$$

if $c^2$ is real. Our explicit solution to the problem is rendered possible by a SSUSY inspired gluing constraint [21, 22]. We show that

$$\mathcal{F}h_2 = h_1\mathcal{F}, \tag{11}$$

which, because of Eq. (10), is now equivalent to Eq. (2). This amounts to

$$\mathcal{CPT}h_1 = h_1\mathcal{CPT}. \tag{12}$$

Explicit analytic examples of $\mathcal{PT}-$asymmetric models are expressed in terms of circular or hyperbolic functions.

In Section 3 we perform a detailed investigation of eq. (2) for a charge operator which is of the second order in derivatives,

$$\mathcal{C} = \frac{d^2}{dx^2} + G(x)\frac{d}{dx} + D(x)\,, \tag{13}$$

and where $G(x)$ and $D(x)$ are complex functions of the real coordinate $x$:

$$\begin{aligned} G(x) &= G_R(x) + iG_I(x)\,, \\ D(x) &= D_R(x) + iD_I(x)\,. \end{aligned}$$

We further derive the polynomial algebra of Eq. (8). In order to show explicitly that our formalism allows to generate $\mathcal{PT}-$asymmetric models with real spectrum, we discuss in Section 4 a particular polynomial oscillator model.

In Section 5 we generalize the postulate (13) and derive the general form of the charge operator $\mathcal{C}$ of any finite order in the derivative $d/dx$ such that $\mathcal{F} \equiv \mathcal{CP}$ is Hermitian. At the very end, in section 6 we give some perspectives on the impact of our results on a variety of fields where the use of similar $\mathcal{F}$ might play significant role.

# 2 SUSY gluing constraint

Starting with a second-order $\mathcal{C}$ of the form (13) we have to guarantee, first of all, the Hermiticity of $\mathcal{F} = \mathcal{CP}$ and $\mathcal{F}^{-1} = \mathcal{PC}^{-1}$. It is easy to show (see also section 5 below for an exhaustive discussion of these important conditions for polynomial charges) that the latter Hermiticity condition forces us to impose the necessary and sufficient requirements

$$D_R(x) = D_R(-x) + \frac{d}{dx}G_R(x), \qquad D_I(x) = -D_I(-x) + \frac{d}{dx}G_I(x)$$

where $G_R(x) = G_R(-x)$ is even while $G_I(x) = -G_I(-x)$ must be odd.

## 2.1 Factorization

In the subsequent step of our considerations we factorize our second-order charge operator $\mathcal{C}$ as follows,

$$\mathcal{C} = q_1 q_2\,, \quad q_1 = \frac{d}{dx} + U(x)\,, \quad q_2 = \frac{d}{dx} + W(x)\,, \tag{14}$$

where

$$U(x) + W(x) = G(x)\,, \qquad \frac{d}{dx}W(x) + U(x)W(x) = D(x)\,. \tag{15}$$

In order to simplify the problem at the start we impose the following "gluing" constraint on $q_1$ and $q_2$,

$$q_2(q_2^\dagger)^* = (q_1^\dagger)^* q_1 + c\,, \tag{16}$$

where $c$ is a complex number. By inserting Eqs. (14) into Eq. (16), we obtain

$$\left(\frac{d}{dx} + W\right)\left(-\frac{d}{dx} + W\right) = \left(-\frac{d}{dx} + U\right)\left(\frac{d}{dx} + U\right) + c\,,$$

whence

$$\frac{d}{dx}W(x) + W^2(x) = -\frac{d}{dx}U(x) + U^2(x) + c\,. \tag{17}$$

We find the following representation for $\mathcal{F}\mathcal{F}^*$ and $\mathcal{F}^*\mathcal{F}$ (Eq. (3))

$$\mathcal{F}\mathcal{F}^* = \mathcal{F}\left(\mathcal{F}^\dagger\right)^* = (q_1 q_2 \mathcal{P}) \cdot \left(\mathcal{P} q_2^\dagger q_1^\dagger\right)^* = q_1 q_2 \left(\mathcal{P}\right)^2 \left(q_2^\dagger\right)^* \left(q_1^\dagger\right)^*\,,$$

which, taking Eq. (16) into account, becomes

$$\begin{aligned}
\mathcal{F}\mathcal{F}^* &= q_1\left[\left(q_1^\dagger\right)^* q_1 + \frac{c}{2} + \frac{c}{2}\right]\left(q_1^\dagger\right)^* \\
&= \left[q_1\left(q_1^\dagger\right)^* + \frac{c}{2} + \frac{c}{2}\right] \cdot \left[q_1\left(q_1^\dagger\right)^* + \frac{c}{2} - \frac{c}{2}\right].
\end{aligned}$$

Correspondingly,

$$\begin{aligned}
\mathcal{F}^*\mathcal{F} &= \left(\mathcal{F}^\dagger\right)^* \mathcal{F} \\
&= \left[\mathcal{P}\left(q_2^\dagger\right)^* q_2 \mathcal{P} - \frac{c}{2} - \frac{c}{2}\right] \cdot \left[\mathcal{P}\left(q_2^\dagger\right)^* q_2 \mathcal{P} - \frac{c}{2} + \frac{c}{2}\right].
\end{aligned}$$

Defining the Hamiltonian operators

$$\begin{aligned}
h_1 &= q_1(q_1^\dagger)^* + \frac{c}{2} \\
&= \left(\frac{d}{dx} + U\right)\left(-\frac{d}{dx} + U\right) + \frac{c}{2} \\
&= -\frac{d^2}{dx^2} + \frac{d}{dx}U(x) + U^2(x) + \frac{c}{2}\,,
\end{aligned}$$

and

$$
\begin{aligned}
h_2 &= \mathcal{P}(q_2^\dagger)^* q_2 \mathcal{P} - \frac{c}{2} \\
&= \mathcal{P}\left(-\frac{d}{dx} + W(x)\right)\left(\frac{d}{dx} + W(x)\right)\mathcal{P} - \frac{c}{2} \\
&= -\frac{d^2}{dx^2} - \frac{d}{dx}W(-x) + W^2(-x) - \frac{c}{2},
\end{aligned}
$$

equation (7) provides the following representation for $K$

$$
K = \mathcal{H}^2 - \frac{c^2}{4}, \qquad \mathcal{H} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}.
$$

Comparing with Section 3.3 below, where $\mathcal{F}\mathcal{F}^* = H^2 + \alpha H + \gamma$, and setting $h_1 = H$, we get in the present case $\alpha = 0$ and, correspondingly, $V_0 = 0$, according to Eq. (53) below, as well as $\gamma = -c^2/4$. This shows explicitly how the present model can be derived from the general results of section 3.

## 2.2 Hamiltonians

Remembering the first of Eqs. (15), Eq. (17) becomes

$$
\frac{d}{dx}G(x) + W^2(x) - (G(x) - W(x))^2 = c,
$$

*i.e.*,

$$
\frac{d}{dx}G(x) - G^2(x) + 2G(x)W(x) = c,
$$

or

$$
G(x)W(x) = \frac{1}{2}\left(G^2(x) - \frac{d}{dx}G(x) + c\right). \tag{18}
$$

We immediately deduce that

$$
\begin{aligned}
W(x) &= \frac{G^2(x) - \frac{d}{dx}G(x) + c}{2G(x)}, \\
U(x) &= G(x) - W(x) = \frac{G^2(x) + \frac{d}{dx}G(x) - c}{2G(x)}.
\end{aligned} \tag{19}
$$

Thus

$$
h_1 = -\frac{d^2}{dx^2} + V(x),
$$

with

$$
V(x) = G'(x) - \frac{(G'(x))^2}{4G^2(x)} + \frac{G''(x)}{2G(x)} + \frac{G^2(x)}{4} + \frac{c^2}{4G^2(x)}. \tag{20}
$$

From Eq. (18) we also get that at the zeros $\bar{x}$ of $G$, we must have

$$\left.\frac{dG}{dx}\right|_{x=\bar{x}} = c\,,$$

which is a constraint on $G$, too. In fact, the method would fail if $G$ had several zeros with non-identical values of the first derivative at each of them.

An important comment must be made here since even if a function does not vanish on the real axis, one can investigate its zeros in the complex $x$ plane. For instance, if

$$G(x) = G_0(x) \equiv z(x) = \frac{1 + i\sinh(\alpha x)}{2}\,, \quad \alpha \in \mathbf{R}, \tag{21}$$

it is immediate to check that $z(x_n) = 0$ at $x_n = -i\,(2n + 3/2)\pi/\alpha$, $n = 0$, $\pm 1$, .... This would mean that $dG_0(x_n)/dx = (i\alpha/2)\cosh(\alpha x_n) = 0$, thus implying that we must put $c = 0$ in this case.

In the similar spirit, we may consider the whole class of functions which depend on $x$ only via $z(x)$ of Eq. (21) in an arbitrary nonlinear manner, $G_m(x) \equiv G(z(x))$, since, as a function of $x$, $z$ is $\mathcal{PT}$-symmetric, and any real function of $z$ is $\mathcal{PT}$-symmetric, too, and is an acceptable candidate for $G$.

It becomes convenient to change variables and express the Hamiltonian, $H = -d^2/dx^2 + V(x)$, with $V(x)$ given by formula (20), as a function of $z$, by observing that

$$\begin{aligned} \frac{d}{dx} &= \frac{dz}{dx}\frac{d}{dz} = i\alpha\sqrt{z(1-z)}\frac{d}{dz}\,, \\ \frac{d^2}{dx^2} &= \left(\frac{dz}{dx}\frac{d}{dz}\right)^2 = -\alpha^2\left(\frac{1}{2} - z\right)\frac{d}{dz} - \alpha^2 z(1-z)\frac{d^2}{dz^2}\,, \end{aligned}$$

and

$$\begin{aligned} V(z) &= i\alpha\sqrt{z(1-z)}\frac{d}{dz}G + \alpha^2\frac{z(1-z)}{4G^2}\left(\frac{d}{dz}G\right)^2 - \alpha^2\frac{1-2z}{4G}\frac{d}{dz}G \\ &\quad -\alpha^2\frac{z(1-z)}{2G}\frac{d^2}{dz^2}G + \frac{G^2}{4} + \frac{c^2}{4G^2}\,. \end{aligned} \tag{22}$$

## 2.3 Consistency

We prove now an important constraint on the complex number $c = c_R + ic_I$. From the second of Eqs. (15), we have

$$\begin{aligned} D(x) &= \frac{d}{dx}W(x) + U(x)W(x) \\ &= \frac{1}{2G^2(x)}\left[\left(2G\frac{d}{dx}G - \frac{d^2}{dx^2}G\right)G - \frac{d}{dx}G\left(G^2 - \frac{d}{dx}G + c\right)\right] \\ &\quad + \frac{1}{4G^2}\left[G^4 - \left(\frac{d}{dx}G\right)^2 - c^2 + 2c\frac{d}{dx}G\right]\,, \end{aligned}$$

or

$$D(x) = \frac{1}{4G^2}\left[2G^2\frac{d}{dx}G - 2G\frac{d^2}{dx^2}G + \left(\frac{d}{dx}G\right)^2 + G^4 - c^2\right], \quad (23)$$

$$D^*(-x) = \frac{1}{4G^2}\left[-2G^2\frac{d}{dx}G - 2G\frac{d^2}{dx^2}G + \left(\frac{d}{dx}G\right)^2 + G^4 - (c^*)^2\right], \quad (24)$$

where the functions on the right-hand-sides of Eqs. (23) and (24) are all computed at $x$.

In deriving Eq. (24), use has been made of the fact that $G$ and $d^2G/dx^2$ are $\mathcal{PT}$-symmetric, while $dG/dx$ is $\mathcal{PT}$-antisymmetric, *i.e.*, $(dG/dx(-x))^* = -dG/dx(x)$. Eq. (23) is obviously consistent with the general form of $D$ as a function of $G$ given by Eqs. (48), (50), with $c^2/4 = -I_0 - D(x_0)G^2(x_0)$. Subtracting Eq. (24) from Eq. (23) side by side, we obtain

$$D(x) - D^*(-x) = \frac{d}{dx}G + \frac{(c^2)^* - c^2}{4G^2}. \quad (25)$$

Combining Eq. (25) with Eq. (33), we obtain the important result

$$(c^2)^* - c^2 = 0 \quad \rightarrow \quad \Im(c^2) = 0 \quad \rightarrow \quad c_R c_I = 0.$$

From Eq. (19) we easily obtain

$$U(x) = W^*(-x) - \frac{c_R}{G(x)},$$

whence

$$\left(\frac{d}{dx}U(x)\right)^* = -\frac{d}{dx}W(-x) + \frac{c_R}{(G^*(x))^2}\frac{d}{dx}G^*, \quad (26)$$

and

$$(U^*(x))^2 = W^2(-x) + \frac{c_R^2}{(G^*(x))^2} - 2c_R\frac{W(-x)}{G^*(x)}. \quad (27)$$

Thus

$$\begin{aligned} h_1^\dagger - h_2 &= \left(\frac{d}{dx}U(x) + U^2(x)\right)^* + \frac{d}{dx}W(-x) - W^2(-x) + c_R \\ &= c_R\left[\frac{1}{(G^2(x))^*}\left(\frac{d}{dx}G^*(x) + c_R\right) - 2\frac{W(-x)}{G^*(x)} + 1\right]. \end{aligned}$$

Using Eq. (19) to replace $W(-x)$, we obtain

$$\begin{aligned} h_1^\dagger - h_2 &= c_R\left[\frac{1}{(G^2)^*}\left(\frac{d}{dx}G^* + c_R\right) - \frac{1}{(G^2)^*}\left((G^2)^* + \frac{d}{dx}G^* + c\right) + 1\right] \\ &= c_R\frac{c_R - c}{(G^2)^*} = -i\frac{c_R c_I}{(G^2)^*} = -ic_R c_I\frac{G^2}{\mid G\mid^4}. \quad (28) \end{aligned}$$

Therefore

$$h_1^\dagger = h_2 \quad \Leftrightarrow \quad c_R c_I = 0.$$

## 2.4 Periodic potential

Let us now give an example which generalizes $\mathcal{PT}$-symmetric periodic potentials [8, 23]:

$$G(x) = e^{i\alpha x} + r\,, \quad \alpha \in \mathbf{R}\,, \qquad r \in \mathbf{R}, \;\; r \neq \pm 1.$$

In this case we have, for all $x \in \mathbf{R}$,

$$U(x) = \frac{1}{2}\left(e^{i\alpha x} + r\right) + \frac{1}{2}\left(i\alpha e^{i\alpha x} - c\right) / \left(e^{i\alpha x} + r\right)$$

$$W(x) = \frac{1}{2}\left(e^{i\alpha x} + r\right) - \frac{1}{2}\left(i\alpha e^{i\alpha x} - c\right) / \left(e^{i\alpha x} + r\right)\,.$$

Since $G$ never vanishes, we do not have any constraint on the value of $c$, in addition to the one which requires that $c$ be either real, or imaginary. The spectral analysis of the corresponding Schrödinger operators $h_1$ and $h_2$ with periodic potentials can be performed as a generalization to the non-$\mathcal{PT}$-symmetric case of the investigation done by in Ref. [24].

We now examine the invertibility of $\mathcal{C}$ and the boundedness of $\mathcal{C}^{-1}$. First notice that $\mathcal{C}$ can be written in the following form

$$\mathcal{C} = \mathcal{C}_1\mathcal{C}_2, \qquad \mathcal{C}_1 = \mathcal{C}_U + \frac{r}{2}, \qquad \mathcal{C}_2 = \mathcal{C}_U + \frac{r}{2} \tag{29}$$

where

$$\mathcal{C}_U = \frac{d}{dx} + U_1, \qquad U_1 = U - \frac{r}{2}$$

$$\mathcal{C}_W = \frac{d}{dx} + W_1, \qquad W_1 = W - \frac{r}{2}\,.$$

We will discuss the invertibility of each factor in (29) separately. As for $\mathcal{C}_1$ we first observe that the numerical range $\{z = \langle \mathcal{C}_U\psi, \psi\rangle : \psi \in H^1(I\!R)\}$ of $\mathcal{C}_U$ is contained in the strip $\{z : |Re\, z| \leq a\}$ where $a = \max_{x\in I\!R}|U_1(x)|$. Hence, if $|r| > 2a$ then $-r/2$ is in the resolvent set of $\mathcal{C}_U$ [25] and, therefore, $\mathcal{C}_1$ is invertible with bounded inverse on $L^2(\mathbf{R})$. A similar argument holds for $\mathcal{C}_W$. Thus, for sufficiently large values of $|r|$, operator $\mathcal{C}$ is invertible and $\mathcal{C}^{-1}$ is bounded on $L^2(\mathbf{R})$.

# 3 Second-order charge operator $\mathcal{C}$

## 3.1 Re-construction of the potential

We already noticed that in the second-order charge operator (13), the notation of Section 5 below implies that we have the correspondences $\gamma_2(x) = 1$, $\gamma_1(x) = G(x)$ and $\gamma_0(x) = D(x)$, so that the Hermiticity constraints on the real and imaginary parts of $\gamma_\ell(x)$, Eqs. (70) and (71), with $\omega = 2$ and $\ell = 0, 1$, immediately give

$$\begin{aligned} G_R(x) - G_R(-x) &= 0\,; \quad G_I(x) + G_I(-x) = 0; \qquad (30)\\ D_R(x) - D_R(-x) &= \frac{d}{dx}G_R(x)\,; \quad D_I(x) + D_I(-x) = \frac{d}{dx}G_I(x)\,. \qquad (31)\end{aligned}$$

As a consequence of Eq. (30), $G$ is $\mathcal{PT}$-symmetric

$$G(x) = G^*(-x)\,, \qquad (32)$$

while Eq. (31) yields

$$D(x) - D^*(-x) = \frac{d}{dx}G(x)\,. \qquad (33)$$

We assume that $\mathcal{F}$ and $H$ satisfy the intertwining condition (2) and that $H$ depends on a local complex potential, $V(x)$:

$$H = -\frac{d^2}{dx^2} + V(x)\,, \qquad (34)$$

with $V(x) = V_R(x) + iV_I(x)$. In turn, $V_R(x)$ and $V_I(x)$ are conveniently decomposed into their even and odd parts:

$$\begin{aligned} V_R(x) &= V_R^E(x) + V_R^O(x)\,,\\ V_I(x) &= V_I^E(x) + V_I^O(x)\,,\end{aligned}$$

with $V_K^E(x) = V_K^E(-x)$ and $V_K^O(x) = -V_K^O(-x)$, $(K = R, I)$. We write now condition (2) explicitly and obtain three non-trivial equations by imposing that the coefficients of $(d/dx)^2$, $d/dx$ and $(d/dx)^0$ vanish,

$$-2\left(V_R^O + iV_I^E\right) + 2\frac{d}{dx}\left(G_R + iG_I\right) = 0\,, \qquad (35)$$

$$\begin{aligned} &2\tfrac{d}{dx}\left(V_R^E + iV_I^O\right) - 2\tfrac{d}{dx}\left(V_R^O + iV_I^E\right) + 2\tfrac{d}{dx}\left(D_R + iD_I\right)\\ &+\tfrac{d^2}{dx^2}\left(G_R + iG_I\right) - 2\left(V_R^O + iV_I^E\right)\left(G_R + iG_I\right) = 0\,,\end{aligned} \qquad (36)$$

$$\begin{aligned} &\tfrac{d^2}{dx^2}\left(V_R^E + iV_I^O\right) - \tfrac{d^2}{dx^2}\left(V_R^O + iV_I^E\right) + \tfrac{d^2}{dx^2}\left(D_R + iD_I\right)\\ &+\left(G_R + iG_I\right)\tfrac{d}{dx}\left(V_R^E(x) + iV_I^O(x)\right) - \left(G_R + iG_I\right)\tfrac{d}{dx}\left(V_R^O + iV_I^E\right)\\ &-2\left(D_R + iD_I\right)\left(V_R^O + iV_I^E\right) = 0\end{aligned} \qquad (37)$$

while the coefficients of $(d/dx)^4$ and $(d/dx)^3$ are identically zero.

### 3.2 Integrability

The first of the above equations (35) yields

$$V_R^O = \frac{d}{dx} G_R; \qquad V_I^E = \frac{d}{dx} G_I \,. \tag{38}$$

The second equation (36)yields

$$\begin{aligned}
\frac{d}{dx}V_R^E - \frac{d}{dx}V_R^O - V_R^O G_R + V_I^E G_I + \frac{d}{dx}D_R + \frac{1}{2}\frac{d^2}{dx^2}G_R &= 0; \\
\frac{d}{dx}V_I^O - \frac{d}{dx}V_I^E - V_I^E G_R - V_R^O G_I + \frac{d}{dx}D_I + \frac{1}{2}\frac{d^2}{dx^2}G_I &= 0,
\end{aligned}$$

and is easily integrated for the other two components of the potential, $V_R^E(x)$ and $V_I^O(x)$, as functions of $G_R(x)$, $G_I(x)$, $D_R(x)$ and $D_I(x)$, by replacing $V_R^O(x)$ and $V_I^E(x)$ with their expressions (38):

$$V_R^E(x) = \frac{1}{2}\frac{d}{dx}G_R(x) + \frac{1}{2}\left(G_R(x)\right)^2 - \frac{1}{2}\left(G_I(x)\right)^2 - D_R(x) + V_0 \tag{39}$$

$$V_I^O(x) = \frac{1}{2}\frac{d}{dx}G_I(x) + G_R(x)\,G_I(x) - D_I(x)\,. \tag{40}$$

Here, $V_0$ is a real integration constant. The corresponding integration constant in the equation for $V_I^O(x)$ must be zero, because the function is odd. Both equations can be recombined as

$$V(x) = \frac{3}{2}\frac{d}{dx}G(x) + \frac{1}{2}G^2(x) - D(x) + V_0\,. \tag{41}$$

Finally, the third equation (37) allows us to express the $G_J(x)$'s, $(J = R, I)$, as functions of the $D_K(x)$'s, $(K = R, I)$, or, more conveniently, viceversa.

$$\begin{aligned}
& -\frac{1}{2}\frac{d^3}{dx^3}G_R + \frac{G_R}{2}\frac{d^2}{dx^2}G_R + \left(\frac{d}{dx}G_R\right)^2 + \left(G_R^2 - G_I^2 - 2D_R\right)\frac{d}{dx}G_R \\
& -\frac{G_I}{2}\frac{d^2 G_I}{dx^2} - \left(\frac{d}{dx}G_I\right)^2 + 2\left(D_I - G_I G_R\right)\frac{d}{dx}G_I - G_R\frac{d}{dx}D_R + G_I\frac{d}{dx}D_I \\
= \ & 0 \\
& -\frac{1}{2}\frac{d^3 G_I}{dx^3} + \frac{1}{2}G_R\frac{d^2}{dx^2}G_I + \left(-G_I^2 + G_R^2 - 2D_R\right)\frac{d}{dx}G_I + \frac{1}{2}G_I\frac{d^2}{dx^2}G_R \\
& +2\frac{d}{dx}G_R\frac{d}{dx}G_I + 2(G_R G_I - D_I)\frac{d}{dx}G_R - G_I\frac{d}{dx}D_R - G_R\frac{d}{dx}D_I \\
= \ & 0.
\end{aligned} \tag{42}$$

Eqs. (42) can be recombined in the following first-order linear equation expressing the unknown function $D(x)$ in terms of the known function $G(x)$

and its derivatives up to third order

$$\frac{1}{2}\frac{d^3}{dx^3}G - \frac{1}{2}G\frac{d^2}{dx^2}G - \left(\frac{d}{dx}G\right)^2 - G^2\frac{d}{dx}G + 2\left(\frac{d}{dx}G\right)D + G\frac{d}{dx}D = 0\,. \quad (43)$$

Eq. (43) is easily solved by direct integration. Let us define the auxiliary functions

$$g(x) \equiv 2\frac{d}{dx}G\,, \quad (44)$$

$$f(x) \equiv -\frac{1}{2}\frac{d^3}{dx^3}G + \frac{1}{2}G\frac{d^2}{dx^2}G + \left(\frac{d}{dx}G\right)^2 + G^2\frac{d}{dx}G\,, \quad (45)$$

$$\frac{1}{p(x)}\frac{d}{dx}p(x) \equiv \frac{g(x)}{G(x)}\,. \quad (46)$$

Eq. (46) is promptly integrated by use of definition (44) to

$$p(x) = \exp\left(2\int_{x_0}^{x} d\ln G(x')\right) = \frac{G^2(x)}{G^2(x_0)}\,, \quad (47)$$

where $x_0$ is an initial point where $G$ is different from zero. It is now easy to check that the general solution to Eq. (43) can be written in the form

$$p(x)D(x) = \int_{x_0}^{x} dx' \frac{p(x')f(x')}{G(x')} + p(x_0)D(x_0)\,,$$

or

$$D(x) = \frac{1}{G^2(x)}\int_{x_0}^{x} dx' G(x')f(x') + \frac{D(x_0)G^2(x_0)}{G^2(x)}\,. \quad (48)$$

The integral on the right-hand side of Eq. (48) is computed by elementary methods in the form

$$\int_{x_0}^{x} dx' G(x')f(x') = \frac{G^4(x)}{4} + \frac{G^2(x)G'(x)}{2} - \frac{G(x)G''(x)}{2} + \frac{(G'(x))^2}{4} + I_0\,, \quad (49)$$

with

$$I_0 \equiv -\frac{G^4(x_0)}{4} - \frac{G^2(x_0)G'(x_0)}{2} + \frac{G(x_0)G''(x_0)}{2} - \frac{(G'(x_0))^2}{4}\,, \quad (50)$$

where $G' \equiv dG/dx$, and so on, thus providing the most general expression of $D$ as a function of $G$ and of its derivatives.

### 3.3 SSUSY algebra

Assuming a charge operator, $\mathcal{C}(x)$, of the form (13), we now verify that the operator

$$\mathcal{F}(x)\mathcal{F}^*(x) = \mathcal{C}(x)\mathcal{P}\mathcal{C}^*(x)\mathcal{P} = \mathcal{C}(x)\mathcal{C}^*(-x)$$

can be written as a particular case of formula (8)

$$\mathcal{F}(x)\mathcal{F}^*(x) = H^2 + \alpha H + \gamma\,,$$

where $\alpha$ and $\gamma$ are constants to be determined and $H$ is Hamiltonian (34) with $V$ given in (41). In fact, we have

$$\begin{aligned}
\mathcal{C}(x)\mathcal{C}^*(-x) &= \left(\frac{d^2}{dx^2} + G(x)\frac{d}{dx} + D(x)\right)\cdot\left(\frac{d^2}{dx^2} - G^*(-x)\frac{d}{dx} + D^*(-x)\right) \\
&= \left(\frac{d^2}{dx^2} + G(x)\frac{d}{dx} + D(x)\right)\cdot\left(\frac{d^2}{dx^2} - G(x)\frac{d}{dx} + D(x) - G'(x)\right)
\end{aligned}$$

where use has been made of relations (30), (31) stemming from Hermiticity of $C(x)$. After some algebra, the right-hand side of the above expression is brought to the form

$$\begin{aligned}
\mathcal{C}(x)\mathcal{C}^*(-x) \;=\; & \frac{d^4}{dx^4} + \Big(2D(x) - G^2(x) - 3G'(x)\Big)\frac{d^2}{dx^2} \\
& + \left(2D'(x) - 3G''(x) - 2G(x)G'(x)\right)\frac{d}{dx} \\
& + D''(x) - G'''(x) + G(x)D'(x) \\
& - G(x)G''(x) + D^2(x) - D(x)G'(x)\,,
\end{aligned} \tag{51}$$

and is to be compared with

$$\begin{aligned}
H^2 + \alpha H + \gamma &= \left(-\frac{d^2}{dx^2} + V(x)\right)^2 + \alpha\left(-\frac{d^2}{dx^2} + V(x)\right) + \gamma \\
&= \frac{d^4}{dx^4} - \left(2V(x) + \alpha\right)\frac{d^2}{dx^2} - 2V'(x)\frac{d}{dx} + V^2(x) \\
&\quad - V''(x) + \alpha V(x) + \gamma\,,
\end{aligned} \tag{52}$$

where $V(x)$ may be expressed as a function of $D(x)$ and $G(x)$ according to Eq. (41). Direct comparison of the right-hand sides of the above formulae allows us to determine the $\alpha$ constant as

$$\alpha = -2V_0\,. \tag{53}$$

The value of $\gamma$ expresses the compatibility between $\mathcal{C}$ and the polynomial algebra through the equation

$$V^2(x) - V''(x) + \alpha V(x) + \gamma =$$

$$= D''(x) - G'''(x) + G(x)D'(x) - G(x)G''(x) + D^2(x) - D(x)G'(x).$$

Here, we insert the expressions of $V(x)$ and $V''(x)$ in terms of $G(x)$, $D(x)$ and of their derivatives obtained from formula (41), and making use of Eq. (43), as well as of its general solution (48), (49), we obtain the final result

$$\gamma = V_0^2 + I_0 + D(x_0)G^2(x_0)\,, \tag{54}$$

where $I_0$ is defined in Eq. (50). This makes it possible to interpret $\gamma$ as a kind of integration constant. Thus, $\mathcal{CPT}$ invariance leads to the SSUSY polynomial algebra, Eqs. (7), (8).

# 4 Polynomial oscillators

The simplest factorization of $\mathcal{C}$ reads

$$\mathcal{C}(x) = \left(\frac{d}{dx} + \frac{G(x)}{2}\right) \cdot \left(\frac{d}{dx} + \frac{G(x)}{2}\right)\,, \tag{55}$$

so that, correspondingly,

$$D(x) = \frac{G'(x)}{2} + \frac{G^2(x)}{4}\,. \tag{56}$$

In this case, Eq. (43) yields $G'''(x) = 0$, *i.e.*,

$$G(x) = ax^2 + ibx + c \tag{57}$$

where $a, b$ and $c$ are real numbers, owing to the fact that $G(x)$ is $\mathcal{PT}$-symmetric. From Eq. (41) we obtain:

$$\begin{aligned} V(x) &= \frac{1}{4}G^2(x) + G'(x) + V_0 \\ &= \frac{1}{4}a^2x^4 - \frac{1}{4}(b^2 - 2ac)x^2 + \frac{1}{2}iabx^3 + \frac{1}{2}x(ibc + 4a) + ib + \frac{c^2}{4} + V_0. \end{aligned} \tag{58}$$

If we make the additional assumption $c = 0$, for the sake of simplicity, the polynomial algebra provides the constraint

$$\gamma = V_0^2 + \frac{b^2}{4}$$

on $\gamma$ [Eq. (54)].

## 4.1 The problem of invertibility

We will now make the spectral analysis for $H$ and study the invertibility of $\mathcal{F}$ in the case $c = 0$. Then

$$V(x) = \frac{1}{4}a^2x^4 - \frac{1}{4}b^2x^2 + \frac{1}{2}iabx^3 + 2ax + ib + V_0. \tag{59}$$

Setting $\mu^2 = \frac{a^2}{4}$ and $\nu^2 = \frac{b^2}{4}$, we obtain an expression for the Schrödinger operator $H$ of the same type as that presented in Eqs. (22), (23) of Ref. [19], namely

$$H = -\frac{d^2}{dx^2} + \mu^2x^4 - \nu^2x^2 + 2i\mu\nu x^3 + 4\mu x + 2i\nu + V_0 \tag{60}$$

and $D(H) = H^2(\mathbf{R}) \cap D(x^4)$, $\forall \mu, \nu \in \mathbf{R}$, $\mu \neq 0$. As in Ref. [19], $H$ has discrete spectrum, *i.e.*, the spectrum consists of a sequence of isolated eigenvalues with finite multiplicity.

In order to prove the reality of the spectrum of $H$, we first notice that $H$ can be rewritten as

$$H = -\frac{d^2}{dx^2} + x^2(\mu x + i\nu)^2 + 4\mu x + 2i\nu + V_0. \tag{61}$$

Let us now perform the complex translation $x \to x - \frac{i\nu}{2\mu}$. Then $H = S^{-1}H_1S$ where $S\psi(x) = \psi(x - \frac{i\nu}{2\mu})$ on a dense set of functions $\psi \in L^2(\mathbf{R})$ and

$$\begin{aligned} H_1 &= -\frac{d^2}{dx^2} + \left(x - \frac{i\nu}{2\mu}\right)^2\left(\mu x - \frac{i\nu}{2} + i\nu\right)^2 + 4\mu x - 2i\nu + 2i\nu + V_0 \\ &= -\frac{d^2}{dx^2} + \mu^2\left(x - \frac{i\nu}{2\mu}\right)^2\left(x + \frac{i\nu}{2\mu}\right)^2 + 4\mu x + V_0 \\ &= -\frac{d^2}{dx^2} + \mu^2\left(x^2 + \frac{\nu^2}{4\mu^2}\right)^2 + 4\mu x + V_0 \end{aligned} \tag{62}$$

Hence $H$ has the same spectrum of $H_1$. In turn $H_1$ is selfadjoint on $D(H_1) = D(H) = H^2(\mathbf{R}) \cap D(x^4)$, thus it has real spectrum for all $\mu, \nu, V_0 \in \mathbf{R}$, $\mu \neq 0$.

We may stress that Hamiltonian (60) is not $\mathcal{PT}-$invariant but has still a real spectrum because it is related by explicit similarity to the standard self-adjoint anharmonic oscillator. In our opinion this is an exceptional example since in general the proof of the reality of the spectra of non-Hermitian Hamiltonians cannot proceed in such a straightforward manner and, generically, the necessary maps are non-local [26]. Moreover, by our construction, the reality of the spectrum is robust insofar as its $\mathcal{CPT}-$symmetry cannot be spontaneously broken. In this sense, our example (60) may be perceived as a $\mathcal{PT}-$*asymmetric* parallel to the $\mathcal{PT}-$symmetric quartic oscillator of Buslaev and Grecchi [27].

## 4.2 The problem of boundedness

Let us now turn to the operator $\mathcal{F} = \mathcal{CP}$. In order to prove the invertibility of $\mathcal{F}$ and the boundedness of $\mathcal{F}^{-1}$ on $L^2(\mathbf{R})$ it is enough to demonstrate the same facts for $C$. Factorization (55) implies that it will suffice to prove that $C_1 = (\frac{d}{dx} + \frac{G}{2})$ is invertible and that $C_1^{-1}$ is bounded on $L^2(\mathbf{R})$ if $G$ is given by (57). Indeed, we have

$$C_1 = \frac{d}{dx} + \frac{1}{2}ax^2 + \frac{i}{2}bx \tag{63}$$

and we now proceed as in Ref. [19]. More precisely

$$C_1 = \frac{d}{dx} + \frac{a}{2}\left(x + \frac{ib}{2a}\right)^2 + \frac{b^2}{8a} \tag{64}$$

is similar to

$$C_2 = \frac{d}{dx} + \frac{a}{2}x^2 + \frac{b^2}{8a} \tag{65}$$

via the complex translation $x \to x - \frac{ib}{2a}$. Hence $C_1$ has the same spectrum as $C_2$. In turn $C_2$ is unitarily equivalent, via the Fourier transformation, to

$$C_3 = -\frac{a}{2}\frac{d^2}{dx^2} + ix + \frac{b^2}{8a}. \tag{66}$$

Therefore $C_1$ has the same spectrum as $C_3$. Finally, we perform the unitary dilation $(U\psi)(x) = (a/2)^{1/6}\psi[(a/2)^{1/3}x]$ and obtain that $C_1$ has the same spectrum as

$$C_4 = UC_3U^{-1} = \left(\frac{a}{2}\right)^{1/3}\left[-\frac{d^2}{dx^2} + ix + \left(\frac{a}{2}\right)^{-1/3}\frac{b^2}{8a}\right]. \tag{67}$$

Now, since the Schrödinger operator $-\frac{d^2}{dx^2} + ix$ has an empty spectrum (see Ref. [28]), so does $C_1$. In particular $z = 0$ belongs to resolvent set of $C_1$, so that $C_1$ is invertible and its inverse is bounded and defined on the whole of $L^2(\mathbf{R})$.

# 5 Towards operators $\mathcal{C}$ of any finite order

We shall postulate that the charge-operator component $\mathcal{C}$ of the pseudo-metric $\mathcal{CP}$, where $\mathcal{P}$ denotes parity, is a polynomial of any finite degree $\omega = 0, 1, \ldots$ in the momentum operator $p$,

$$\mathcal{C} = \sum_{k=0}^{\omega} \gamma_k(x)\,\frac{d^k}{dx^k}, \qquad \gamma_k(x) = \gamma_k^R(x) + i\,\gamma_k^I(x)\,. \tag{68}$$

The functions $\gamma_k^R(x)$ and $\gamma_k^I(x)$ are both assumed real, and our main task here is just to guarantee, at any integer $\omega$, that the operator candidate for the metric $\mathcal{CP}$ is Hermitian.

## 5.1 The metric $\mathcal{CP}$ in differential form

From

$$\mathcal{C}^\dagger = \sum_{k=0}^{\omega} (-1)^k \sum_{\ell=0}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left[ \frac{d^{(k-\ell)}}{dx^{(k-\ell)}} \gamma_k^*(x) \right] \frac{d^\ell}{dx^\ell} = \tag{69}$$

$$= \sum_{\ell=0}^{\omega} (-1)^\ell \left\{ \sum_{m=0}^{\omega-\ell} (-1)^m \left( \begin{array}{c} \ell+m \\ \ell \end{array} \right) \left[ \gamma_{\ell+m}^{R(m)}(x) - i\, \gamma_{\ell+m}^{I(m)}(x) \right] \right\} \frac{d^\ell}{dx^\ell} ,$$

where the superscripts $(m)$ at the functions $\gamma^R$ and $\gamma^I$ indicate their $m-$tuple differentiation, one obtains that the Hermiticity condition $\mathcal{CP} = \mathcal{PC}^\dagger$ is equivalent to the $(\omega+1)-$plet of relations

$$\mathcal{P}\, \gamma_\ell\, \mathcal{P} = \gamma_\ell^R(-x) + i\, \gamma_\ell^I(-x) = \sum_{m=0}^{\omega-\ell} (-1)^m \left( \begin{array}{c} \ell+m \\ \ell \end{array} \right) \left[ \gamma_{\ell+m}^{R(m)}(x) - i\, \gamma_{\ell+m}^{I(m)}(x) \right]$$

with a trivial decoupling into its real and imaginary parts

$$\gamma_\ell^R(-x) - \gamma_\ell^R(+x) = \sum_{m=1}^{\omega-\ell} (-1)^m \left( \begin{array}{c} \ell+m \\ \ell \end{array} \right) \gamma_{\ell+m}^{R(m)}(x) \tag{70}$$

and

$$\gamma_\ell^I(-x) + \gamma_\ell^I(+x) = -\sum_{m=1}^{\omega-\ell} (-1)^m \left( \begin{array}{c} \ell+m \\ \ell \end{array} \right) \gamma_{\ell+m}^{I(m)}(x) , \tag{71}$$

respectively, with $\ell = \omega - k = 0, 1, \ldots, \omega$.

## 5.2 Functional freedom in complex coefficients $\gamma_k(x)$

At the first few $k = 0, 1, \ldots$ the above Hermiticity constraints degenerate to the comparatively elementary relations,

$$\gamma_\omega^R(x) - \gamma_\omega^R(-x) = 0, \quad k = 0,$$

$$\gamma_{\omega-1}^R(x) - \gamma_{\omega-1}^R(-x) = \left( \begin{array}{c} \omega \\ 1 \end{array} \right) \gamma_\omega^{R(1)}(x) , \qquad k = 1,$$

$$\gamma_{\omega-2}^R(x) - \gamma_{\omega-2}^R(-x) = \left( \begin{array}{c} \omega-1 \\ 1 \end{array} \right) \gamma_{\omega-1}^{R(1)}(x) - \left( \begin{array}{c} \omega \\ 2 \end{array} \right) \gamma_\omega^{R(2)}(x) , \quad k = 2,$$

etc, or, in parallel,

$$\gamma_\omega^I(x) + \gamma_\omega^I(-x) = 0, \quad k = 0,$$

$$\gamma_{\omega-1}^I(x) + \gamma_{\omega-1}^I(-x) = \left( \begin{array}{c} \omega \\ 1 \end{array} \right) \gamma_\omega^{I(1)}(x) , \qquad k = 1,$$

$$\gamma_{\omega-2}^I(x) + \gamma_{\omega-2}^I(-x) = \left( \begin{array}{c} \omega-1 \\ 1 \end{array} \right) \gamma_{\omega-1}^{I(1)}(x) - \left( \begin{array}{c} \omega \\ 2 \end{array} \right) \gamma_\omega^{I(2)}(x) , \quad k = 2,$$

etc. This means that the symmetric parts $H_\ell(x) = H_\ell(-x)$ of all $\gamma_\ell^R(x)$ are arbitrary functions while, in parallel, the antisymmetric parts $h_\ell(x) = -h_\ell(-x)$ of all $\gamma_\ell^I(x)$ are also arbitrary. We may conjecture that the remaining components $R_\ell(x) = \gamma_\ell^R(x) - H_\ell(x) = -R_\ell(-x)$ and $r_\ell(x) = \gamma_\ell^I(x) - h_\ell(x) = r_\ell(-x)$ obey the rules

$$R_\omega = 0, \quad R_{\omega-1}(x) = \frac{\omega}{2} H_\omega^{(1)}(x), \quad R_{\omega-2}(x) = \frac{\omega-1}{2} H_{\omega-1}^{(1)}(x), \quad \ldots \tag{72}$$

while

$$r_\omega = 0, \quad r_{\omega-1}(x) = \frac{\omega}{2} h_\omega^{(1)}(x), \quad r_{\omega-2}(x) = \frac{\omega-1}{2} h_{\omega-1}^{(1)}(x), \quad \ldots . \tag{73}$$

and are fully determined by the respective recurrent relations (70) and (71).

## 5.3 Proof

We see that both the sequences $R_{\omega-k}(x)$ and $r_{\omega-k}(x)$ have precisely the same structure so that just the sequence of $R_{\omega-k}(x)$ may be considered without any loss of generality. Its elements should be evaluated in the recurrent manner with respect to the growing $k$. The appropriate *Ansätze* may be written in the finite-series form where, formally, $H_{\omega+1}(x) = H_{\omega+2}(x) = \cdots = 0$ and $h_{\omega+1}(x) = h_{\omega+2}(x) = \cdots = 0$,

$$\gamma_{\omega-k}^R(x) = H_{\omega-k}(x) + \sum_{m=1}^{k} c_m \frac{(\omega-k+m)!}{(\omega-k)!} H_{\omega-k+m}^{(m)}(x)\,, \tag{74}$$

$$\gamma_{\omega-k}^I(x) = h_{\omega-k}(x) + \sum_{m=1}^{k} c_m \frac{(\omega-k+m)!}{(\omega-k)!} h_{\omega-k+m}^{(m)}(x)\,. \tag{75}$$

With an auxiliary $c_0 = 1$ these *Ansätze* describe all the $\omega-$dependence of our functions $\gamma = \gamma^R + i\gamma^I$ in closed form.

As already stated above, the first term and the subsequent sum are of an opposite parity in both these formulae since $c_{2n} = 0$ at all $n = 1, 2, \ldots$. This observation is easily proved since after the insertion of the latter two *Ansätze*, the complicated recurrences (70) are replaced by their simplified version

$$2c_1 = \frac{c_0}{1!}, \quad 2c_2 = \frac{c_1}{1!} - \frac{c_0}{2!}, \quad \ldots ,$$

*i .e.*,

$$2c_k = \sum_{m=0}^{k-1} (-1)^{k-m-1} \frac{c_m}{(k-m)!}\,. \tag{76}$$

It is worthwhile to point out that the $c_k$ coefficients with odd $k$ can be written in terms of Bernoulli numbers (see, *e.g.*, Ref. [29])

$$c_{2n-1} = \frac{2(2^{2n}-1)}{(2n)!} B_{2n} \quad (n > 0)\,. \tag{77}$$

The key idea of an explicit solution of these recurrences is that the generating function $f(x) = \sum c_k x^k$ of the coefficients $c_m$ must satisfy the functional equation $f(x) - 2 = -f(x)/e^x$ which is, in its turn, easily solvable. In this way we arrive at the solution of recurrences (76) in the following compact form,

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots = \frac{2}{1+\exp(-x)} = 1 + \tanh\frac{x}{2} = \tag{78}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{2(2^{2n}-1)}{(2n)!} B_{2n} x^{2n-1} \,.$$

Obviously, all the possible parity-violating terms in the right-hand side of our Hermiticity conditions (70) vanish. This makes the form of our polynomial charge $\mathcal{C}$ extremely flexible and confirms the consistency of its present construction.

We may conclude that the requirement of Hermiticity of the metric $\mathcal{CP}$ defines all the antisymmetric components $R_\ell(x)$ and their spatially symmetric partners $r_\ell(x)$. It does not impose any additional constraint either upon the real and spatially symmetric coefficient functions $H_\ell(x)$ or upon their purely imaginary spatially antisymmetric partners $h_\ell(x)$.

# 6 Outlook

We now sketch some possible applications of our methods to a variety of problems where quasi-Hermitian or pseudo-Hermitian operators are involved.

In the context of the Klein-Gordon description of the free motion of a spinless particle in the "usual" Hilbert space $\mathcal{H}$ the relativistic evolution is generated by the Feshbach-Villars [30] "Hamiltonian" $H_{(FV)}$ which proves non-Hermitian,

$$|\psi(t)\rangle = e^{-iH_{(FV)}(t-t_0)}|\psi(t_0)\rangle, \qquad H_{(FV)} = -\frac{1}{2}\left( \begin{array}{cc} 1-\triangle & -\triangle \\ \triangle & \triangle - 1 \end{array} \right) \tag{79}$$

(cf., e.g., p. 341 in ref. [31]). One should notice that this model works with the differential pseudo-Hermitian operator with structure which strongly resembles the usual Schrödinger operators in the simplest non-trivial, two-dimensional coupled-channel case. Thus, we may expect that the methods described in our previous study might find an immediate extension to the similar problems.

The idea may also find applications in a broader context, say, of the boson mappings in nuclear physics which were comprehensively discussed in the paper [13]. It is shown there that a consistent quantum mechanical framework, and in particular a viable variational calculation for non-hermitian Hamiltonians, can indeed be constructed after the introduction of a non-trivial metric. In the context of Holstein-Primakoff type mappings this freedom defines the link with so-called Dyson-Maleev type mappings (see [32]). In practical computations, a puzzling non-Hermiticity of observables proved more then compensated by the advantages, as has been amply demonstrated in applications of generalized Dyson-Maleev mappings (see [33] and references cited therein).

All the technical conditions imposed upon the "true physical metric" $\Theta$ in review [13] are important, especially if one tries to work within a truly infinite-dimensional Hilbert space. This has been emphasized by Kretschmer and Szymanowski [14] who showed that the use of the toy metric operators might require a careful scrutiny because these operators remain unbounded. In this context, ref. [19] as well as our present paper demonstrated persuasively that a switch to the use of the differential operators $\mathcal{C}$ might be understood as an important new idea.

All the similar observations must be perceived as individual steps of a systematic improvement of the mathematically correct understanding of the use of the differential operators in connections with many applications of the quasi-Hermitian observables which seems to range, at present, from the elementary descriptions of the localization transitions in solid state physics [34] up to many ambitious $\mathcal{PT}-$symmetric models in quantum field theory [35].

The experience gained during our study of the simple Schrödinger equations might equally well find applications on the very boundary of quantum mechanics (like, say, in cosmology [36]) or even in the domain of the classical model-building (e.g., in the magneto-dynamics of fluids [37]) and in the various physical models of different origin characterized by the simple matrix structure of their description (see a number of their most elementary samples mentioned in the short and nice review [38]) where the eigenvalues coalesce or almost coalesce in the manner which contradicts the standard and robust finite-dimensional Hermitian-matrix mathematics. Of course, all these mathematical problems and not entirely standard physical situations may impose new and challenging tasks and motivate a deeper future analysis of the questions outlined in our present paper.

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[7] cf., e.g., the collection of papers "Progress in Supersymmetric Quantum Mechanics", special issue of J. Phys. A: Math. Gen. **37** (2004), pp. 10007 - 10458 (ed. I. Aref'eva, D. J. Fernández, V. Hussin, J. Negro, L. M. Nieto and B. F. Samsonov).

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