Comment on "Non-Hermitian Quantum Mechanics with Minimal Length Uncertainty"*

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Abstract. We demonstrate that the recent paper by Jana and Roy entitled "Non-Hermitian quantum mechanics with minimal length uncertainty" [*SIGMA* **5** (2009), 083, 7 pages, arXiv:0908.1755] contains various misconceptions. We compare with an analysis on the same topic carried out previously in our manuscript [arXiv:0907.5354]. In particular, we show that the metric operators computed for the deformed non-Hermitian Swanson models differs in both cases and is inconsistent in the former.

Key words: non-Hermitian Hamiltonians; deformed canonical commutation relations; minimal length

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It is known for some time that the deformations of the standard canonical commutation relations between the position operator P and the momentum operator X will inevitably lead to a minimal length, that is a bound beyond which the localization of space-time events are no longer possible. In a recent manuscript [1] we investigated various limits of the q-deformationed relations

$$[X,P] = i\hbar q^{f(N)}(\alpha\delta+\beta\gamma) + \frac{i\hbar(q^2-1)}{\alpha\delta+\beta\gamma}\left(\delta\gamma X^2 + \alpha\beta\, P^2 + i\alpha\delta XP - i\beta\gamma PX\right),$$

in conjunction with the constraint $4\alpha\gamma = (q^2+1)$, with $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ and f being an arbitrary function of the number operator N. One may consider various types of Hamiltonian systems, either Hermitian or non-Hermitian, and replace the original standard canonical variables (x_0,p_0), obeying $[x_0,p_0]=i\hbar$, by (X,P). It is crucial to note that even when the undeformed Hamiltonian is Hermitian $H(x_0,p_0) = H^\dagger(x_0,p_0)$ the deformed Hamiltonian is inevitably non-Hermitian $H(X,P) \neq H^\dagger(X,P)$ as a consequence of the fact that X and/or P are no longer Hermitian. Of course one may also deform Hamiltonians, which are already non-Hermitian when undeformed $H(x_0,p_0) \neq H^\dagger(x_0,p_0)$. In both cases a proper quantum mechanical description requires the re-definition of the metric to compensate for the introduction of non-Hermitian variables and in the latter an additional change due to the fact that the Hamiltonian was non-Hermitian in the first place.

In a certain limit, as specified in [1], X and P allow for a well-known representation of the form $X=(1+\tau p_0^2)x_0$ and $P=p_0$, which in momentum space, i.e. $x_0 = i\hbar\partial_{p_0}$, corresponds to the one used by Jana and Roy [2], up to an irrelevant additional term $i\hbar\tilde{\gamma}P$. (Whenever constants with the same name but different meanings occur in [2] and [1] we dress the former with a tilde.)

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The additional term can simply be gauged away and has no physical significance. Jana and Roy have studied the non-Hermitian displaced harmonic oscillator and the Swanson model. As we have previously also investigated the latter in [1], we shall comment on the differences. The conventions in [2] are

$$H_{\rm JR}(a,a^\dagger) = \omega a^\dagger a + \lambda a^2 + \tilde{\delta}(a^\dagger)^2 + \frac{\omega}{2}$$

with $\lambda \neq \tilde{\delta} \in \mathbb{R}$ and $a = (P - i\omega X)/\sqrt{2m\hbar\omega}$, $a^\dagger = (P + i\omega X)/\sqrt{2m\hbar\omega}$, whereas in [1] we used

$$H_{\rm BF}(X,P) = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 + i\mu\{X,P\}$$

with $\mu \in \mathbb{R}$ as a starting point. Setting $\hbar = m = 1$ it is easy to see that the models coincide when $\lambda = -\tilde{\delta}$ and $\mu = \tilde{\delta} - \lambda$. The Hamiltonians exhibit a "twofold" non-Hermiticity, one resulting from the fact that when $\lambda \neq \tilde{\delta}$ even the undeformed Hamiltonian is non-Hermitian and the other resulting from the replacement of the Hermitian variables (x_0, p_0) by (X, P). The factor of the metric operator to compensate for the non-Hermiticity of X coincides in both cases, but the factor which is required due to the non-Hermitian nature of the undeformed case differs in both cases

$$\rho_{\rm BF} = e^{2\mu P^2} \qquad \text{and} \qquad \rho_{\rm JR} = (1 + \tau P^2)^{\frac{\mu}{\omega^2 \tau}}.$$

We have made the above identifications such that $H_{\rm JR}(a,a^\dagger) = H_{\rm BF}(X,P)$ and replaced the deformation parameter β used in [2] by τ employed in [1]. It is well known that when given only a non-Hermitian Hamiltonian, the metric operator can not be uniquely determined. However, as argued in [1] with the specification of the observable X, which coincides in [2] and [1], the outcome is unique and we can therefore directly compare $\rho_{\rm BF}$ and $\rho_{\rm JR}$. The limit $\tau \to 0$ reduces the deformed Hamiltonian $H_{\rm JR} = H_{\rm BF}$ to the standard Swanson Hamiltonian, such that $\rho_{\rm JR}$ and $\rho_{\rm BF}$ should acquire the form of a previously constructed metric operator. This is indeed the case for $\rho_{\rm BF}$, but not for $\rho_{\rm JR}$. In fact it is unclear how to carry out this limit for $\rho_{\rm JR}$ and we therefore conclude that the metric $\rho_{\rm JR}$ is incorrect.

References

[1] Bagchi B., Fring A., Minimal length in quantum mechanics and non-Hermitian Hamiltonian systems, arXiv:0907.5354.

[2] Jana T.K., Roy P., Non-Hermitian quantum mechanics with minimal length uncertainty, *SIGMA* **5** (2009), 083, 7 pages, arXiv:0908.1755.