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Algebraic bosonisation: the case of the Dirac fermion

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Abstract

On the lattice fermions paired and then composed with precise phases are the lattice bosons. We construct rules for deforming the underlying algebra that keep unchanged the boson interpretation. These deformations alter the phases of the fermion pairs. The solutions of the deformed algebra are the pair-phase excitations of the lattice bosons. We find these eigenstates of the Dirac Hamiltonian of fermions with mass. The gauge interactions are shown to excite the k-deformed modes. © 2001 Published by Elsevier Science B.V.

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1. Introduction

The background lattice, its periodic geometry, gives us the algebra for composition of the lattice bosons from fermions [1]. This happens in two regions of fermion fillings: (1) near the empty lattice (nel) pairs of fermions superposed with precise phases are the lattice bosons, and (2) near the filled lattice (nfl), i.e., close to the insulator limit, pairs of holes precisely superposed constitute the lattice bosons. The purpose of this work is to show that the underlying algebra admits of deformations [2–4] that do not change the boson [5] nature of the solutions. These deformations change the phases of the fermion pairs being superposed. The deformed phase configurations are the pair-phase excitations of the lattice bosons. We present the energy eigenspectrum of these configurations of the Dirac fermions.

The lattice background allows the construction of the algebra that has a boson interpretation in the two fermion filling regions. The algebra has elements, for instance, of the type:

$$e_{+l} = \sum c_n^\dagger c_{n+l}^\dagger, \tag{1}$$

and its conjugate

$$e_{-l} = -\sum c_n c_{n+l}, \tag{2}$$

where $c_n^\dagger$ (c_n) creates (annihilates) fermions at site n of the equispaced one-dimensional lattice with periodic boundary conditions. The size of this one-dimensional lattice is $2N$. The sums in (1), (2) are over the lattice sites denoted by the index n. Clearly,

$$\{c_n, c_m\} = \delta_{nm}, \tag{3}$$

$$\{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0. \tag{4}$$

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The elements $e_{\pm l}$, defined in (1), (2), have been shown [1] to satisfy bosonic commutation relations for the two regions of fermion filling. e_{+l} (e_{-l}) involves fermions (holes), separated l-sites apart, paired and then superposed with phases of $+1$. We want to deform the algebra by changing these phases without altering the boson interpretation of $e_{\pm l}$ in the two filling regions. We do not want to disturb the underlying lattice. The deformation parameter, called k, assume values consistent with periodic boundary conditions. They give us the k-modes of pair-phase excitations of the lattice-bosons (klb).

We solve the Dirac Hamiltonian of fermions of mass to get its klb eigenstates. For this non-interacting theory the energy of the single k-deformed boson is just equal to the sum of the energies of its fermion constituents. For large number of fermions bosonisation could offer a way of relaxing into lower energy configurations. The Pauli principle forbids fermions crowding into any state. Bosons prefer to crowd.

While the non-interacting Dirac Hamiltonian does have the klb-solutions, the k-values do not change as the bosons move. We introduce the gauge field and study its effect on the klbs. It is found that the gauge mode of number k_p changes the deformation parameter k to $k \pm k_p$. We conclude that the gauge fields are absorbed/emitted by the klbs of the interacting fermions.

The Dirac fermion of mass in one dimension is the model [6] of charge carriers in organic polymers. These systems are, however, "half-filled". We briefly discuss how to write the klbs at "half-filling", and argue that the klbs could be alternate charge carriers in these systems.

2. The k-deformations

We investigate now the general forms of pair-phase deformations of $e_{\pm l}$ that keep their bosonic nature unaltered. Recall briefly that the boson interpretation arose from the commutators:

$$[e_{-l}, e_{+l}] = 2N - 2h_0 + h_{2l}, \tag{5}$$

$$[e_{-l}, e_{+l'}] = \sum \beta^i_{ll'} h_i \ (i \neq 0;\ l \neq l'), \tag{6}$$

where the h_i are elements of the algebra defined as $h_i = \sum c_n^\dagger c_{n+i}$ + h.c. h_0 is the fermion number operator. The sum in (6) has non-zero elements for one or two values of i, and i never takes the value zero. That means h_0 is absent on the right side of (6).

For reasonable sized lattice the quantity $2N$ is some large number. Near the empty lattice the fermion number h_0 is small. The eigenvalue of h_{2l} for the single fermion is bounded by ± 2. Thus for a few fermions the value of h_{2l} is small compared to $2N$. Keeping the $2N$ term in (5) and neglecting the other two we get

$$[e_{-l}, e_{+l}] = 2N. \tag{7}$$

Therefore, the normalized objects $\frac{1}{\sqrt{2N}} e_{\pm l}$ are boson operators. For these normalized objects the r.h.s of (6) is zero. In the insulator limit the fermion states are nearly full, so that h_0 is equal to $2N$ the values of h_i $[i \neq 0]$ are small:

$$[e_{-l}, e_{+l}] = -2N, \tag{8}$$

i.e.,

$$[e_{+l}, e_{-l}] = 2N. \tag{9}$$

Thus, $\frac{1}{\sqrt{2N}} e_{-l}$ is the creation operator of the boson near the insulator region. Eq. (2) tells us these bosons are composed from hole pairs precisely superposed. Consider now the k-deformations of (1) defined as:

$$e_{+lk} = \sum e^{ikn} c_n^\dagger c_{n+l}^\dagger, \tag{10}$$

where k are obtained from periodic boundary conditions. e_{-lk} is the conjugate of (10):

$$e_{-lk} = (e_{+lk})^\dagger. \tag{11}$$

For illustration consider the six-point lattice with periodic boundary conditions

$$\begin{aligned} e_{+1k} = {} & c_1^\dagger c_2^\dagger e^{ik} + c_2^\dagger c_3^\dagger e^{2ik} + c_3^\dagger c_4^\dagger e^{3ik} \\ & + c_4^\dagger c_5^\dagger e^{4ik} + c_5^\dagger c_6^\dagger e^{5ik} + c_6^\dagger c_1^\dagger e^{6ik}. \end{aligned} \tag{12}$$

The possible values of k consistent with periodic boundary conditions are obtained by setting

$$\exp 6ik = 1. \tag{13}$$

Similarly,

$$\begin{aligned} e_{+2k} = {} & c_1^\dagger c_3^\dagger e^{ik} + c_2^\dagger c_4^\dagger e^{2ik} + c_3^\dagger c_5^\dagger e^{3ik} \\ & + c_4^\dagger c_6^\dagger e^{4ik} + c_5^\dagger c_1^\dagger e^{5ik} + c_6^\dagger c_2^\dagger e^{6ik}, \end{aligned} \tag{14}$$

$$e_{+3k} = c_1^\dagger c_4^\dagger e^{ik} + c_2^\dagger c_5^\dagger e^{2ik} + c_3^\dagger c_6^\dagger e^{3ik} + c_4^\dagger c_1^\dagger e^{4ik} + c_5^\dagger c_2^\dagger e^{5ik} + c_6^\dagger c_3^\dagger e^{6ik}, \quad (15)$$

$$e_{+4k} = c_1^\dagger c_5^\dagger e^{ik} + c_2^\dagger c_6^\dagger e^{2ik} + c_3^\dagger c_1^\dagger e^{3ik} + c_4^\dagger c_2^\dagger e^{4ik} + c_5^\dagger c_3^\dagger e^{5ik} + c_6^\dagger c_4^\dagger e^{6ik}, \quad (16)$$

etc. Notice, however, that $e_{+4k} = -e^{2ik} e_{+2k}$. Similarly, $e_{+5k} = -e^{ik} e_{+1k}$. Therefore, $e_{\pm 1k}$, $e_{\pm 2k}$ and $e_{\pm 3k}$ are the independent generators. Others are related to them upto overall phase factors. We conclude that for the lattice of size $2N$, l takes values from 1 to N.

The algebraic structure of the commutators of $e_{\pm lk}$ are analogous to the ones between $e_{\pm l}$. The deformations make the structure factors (that are the coefficients of the operators in the algebra) k-dependent. Further the operators h_i for $i > 0$ must be considered as two independent nonhermitian operators:

$$h_{+ik} = \sum e^{ikn} c_n^\dagger c_{n+i} \quad (17)$$

and

$$h_{-ik} = \sum e^{-ikn} c_{n+i}^\dagger c_n. \quad (18)$$

Just as for $e_{\pm lk}, h_{\pm ik}$ are not all independent. Some are related to others. These two operators $h_{\pm ik}$ arise with different structure-factors in the algebra. (In the limit of $ka \to 0$, the structure factors of h_{+ik} and h_{-ik} become equal.) With these nomenclature, we get:

$$[e_{-1k'}, e_{+1k}] = 2N\delta_{k'k} - \left[1 + e^{-i(k-k')}\right] h_{0(k-k')} + e^{(ik')} h_{2(k-k')} + e^{(ik)} h_{-2(k'-k)}, \quad (19)$$

as well as

$$[e_{-1k'}, e_{+2k}] = e^{ik} h_{-3(k'-k)} + e^{(-2ik')} h_{+3(k-k')} - e^{(-ik')} h_{+1(k-k')} - \exp\bigl(-i(k-k')\bigr) h_{-1(k'-k)}. \quad (20)$$

Comparing (20) with Eq. (20) of Ref. [1], we notice that the structure factors are dependent now on the deformation parameter k. The quantity $h_{0(k-k')}$ has the form:

$$h_{0(k-k')} = \sum c_n^\dagger c_n \exp i(k-k'). \quad (21)$$

For low values of filling it is small compared to $2N$ — the number of lattice points. In the insulator limit it becomes

$$h_{0(k-k')} \approx \sum \exp i(k-k').n = 2N\delta_{kk'}. \quad (22)$$

When $i \neq 0$, the other operators in (17), (18) are

$$h_{ik} = \sum c_n^\dagger c_{n+i} \exp ik.n. \quad (23)$$

These nonhermitian operators are treated as follows:

(1) transform them into Bloch space, and
(2) diagonalize.

We get then the eigenvalues of the operators h_{ik} have magnitudes bounded above by one. The structure factors [see (19) and (20)] are bounded as well. Therefore, the arguments that led us to the bosonic interpretation for $e_{\pm l}$ go through equally well for the two filling regions for $e_{\pm lk}$. We get thus,

$$\left[\frac{e_{-lk}}{\sqrt{2N}}, \frac{e_{+l'k'}}{\sqrt{2N}}\right] = \delta_{ll'}\delta_{kk'} \quad (24)$$

in the low filling region, and

$$\left[\frac{e_{+l'k'}}{\sqrt{2N}}, \frac{e_{-lk}}{\sqrt{2N}}\right] = \delta_{ll'}\delta_{kk'} \quad (25)$$

for the case when the fermion states are nearly full (nfl). We conclude, therefore, $e_{\pm lk}$ are pair-phase k-modes of the lattice bosons.

3. The klbs of the Dirac fermion

As an example of the above, let us illustrate the case for the Dirac fermion of mass. Recall we are on the one-dimensional lattice of size $2N$. The Hamiltonian of interest is:

$$H = H_c + H_b + H_m = i \sum \bigl(c_n^\dagger c_{n+1} - b_n^\dagger b_{n+1}\bigr) + \text{hermitian conjugate} + m \sum \bigl(c_n^\dagger b_n + b_n^\dagger c_n\bigr). \quad (26)$$

The lattice boson operators are

$$e_{+l}^c = \sum c_n^\dagger c_{n+l}, \qquad e_{-l}^c = -\sum c_n c_{n+l} \quad (27)$$

and

$$e_{+l}^b = \sum b_n^\dagger b_{n+l}^\dagger, \qquad e_{-l}^b = -\sum b_n b_{n+l}. \quad (28)$$

Further the elements:

$$d_{+l}^1 = \sum c_n^\dagger b_{n+l}^\dagger, \qquad d_{+l}^2 = \sum b_n^\dagger c_{n+l}^\dagger, \quad (29)$$

and their conjugates d^1_{-l} and d^2_{-l} are bosonic operators as well. We now define the pair-phase excitations of the lattice boson operators:

$$e^c_{+lk} = \sum c^\dagger_n c^\dagger_{n+l} e^{ikn}, \tag{30}$$

$$e^b_{+lk} = \sum b^\dagger_n b^\dagger_{n+l} e^{ikn}, \tag{31}$$

$$d^1_{+lk} = \sum c^\dagger_n b^\dagger_{n+l} e^{ikn}, \tag{32}$$

$$d^2_{+lk} = \sum b^\dagger_n c^\dagger_{n+l} e^{ikn}, \tag{33}$$

and their conjugates defined accordingly from (11). It is more convenient to work with the operators:

$$E^\pm_{\pm lk} = \frac{[e^c_{\pm lk} \pm e^b_{\pm lk}]}{\sqrt{2}}, \tag{34}$$

$$D^\pm_{\pm lk} = \frac{[d^1_{\pm lk} \pm d^2_{\pm lk}]}{\sqrt{2}}. \tag{35}$$

Working on the algebra we get the following bosonic commutators in the region of low fermion filling:

$$\left[E^\pm_{-lk}, E^\pm_{+l'k'}\right] = 2N\delta_{ll'}\delta_{kk'}\delta_{jj'}, \tag{36}$$

$$\left[D^\pm_{-lk}, D^\pm_{+l'k'}\right] = 2N\delta_{ll'}\delta_{kk'}\delta_{jj'}, \tag{37}$$

$$[D^\pm_{-lk}, E^\pm_{+l'k'}] = 0, \tag{38}$$

where j takes the values $+$ and $-$.

Near the insulator region the right-hand sides of (36) and (37) reverse their signs giving us the lattice bosons composed from precisely superposed hole pairs.

To solve for the spectra of the lattice boson excitations we calculate the commutators of $E^j_{\pm lk}$ and $D^j_{\pm lk}$ with the Hamiltonian (26) using the fermi anticommutators (3), (4). We get the following results:

$$\left[H, E^+_{+lk}\right] = i(e^{ik} - 1)E^-_{+(l+1)k} - i(e^{-ik} - 1)E^-_{+(l-1)k} + 2mD^+_{+lk}, \tag{39}$$

$$\left[H, E^-_{+lk}\right] = i(e^{ik} - 1)E^+_{+(l+1)k} - i(e^{-ik} - 1)E^+_{+(l-1)k}, \tag{40}$$

$$\left[H, D^+_{+lk}\right] = i(e^{ik} + 1)D^-_{+(l+1)k} - i(e^{-ik} + 1)D^-_{+(l-1)k} + 2mE^+_{+lk}, \tag{41}$$

$$\left[H, D^-_{+lk}\right] = i(e^{ik} + 1)D^+_{+(l+1)k} - i(e^{-ik} + 1)D^+_{+(l-1)k} \tag{42}$$

The commutators for $E^\pm_{-lk}$ and $D^\pm_{-lk}$ are obtained taking the hermitian conjugates of both sides of (39)–(42).

We now construct the boson Hamiltonian equivalent to (26) but written in purely bosonic variables. We require this bosonic Hamiltonian to give us results identical to the right-hand sides of (39)–(42) but with use of boson relations (36)–(38) instead of (3)–(4). This equivalent Hamiltonian H_B is:

$$\begin{aligned} H_B = {} & i(e^{ik} - 1)\sum \left(E^-_{+(l+1)k}E^+_{-lk} + E^+_{+(l+1)k}E^-_{-lk}\right) \\ & + i(e^{ik} + 1)\sum \left(D^+_{+(l+1)k}D^-_{-lk} + D^-_{+(l+1)k}D^+_{-lk}\right) \\ & - i(e^{-ik} - 1)\sum \left(E^+_{+lk}E^-_{-(l+1)k} + E^-_{+lk}E^+_{-(l+1)k}\right) \\ & - i(e^{-ik} + 1)\sum \left(D^-_{+lk}D^+_{-(l+1)k} + D^+_{+lk}D^-_{-(l+1)k}\right) \\ & + 2m\sum \left(D^+_{+lk}E^+_{-lk} + E^+_{+lk}D^+_{-lk}\right). \end{aligned} \tag{43}$$

To diagonalize H_B we go first over to the Fourier transformed objects

$$E^\pm_{\pm lk} = \frac{1}{\sqrt{L}}\sum E^\pm_\pm(qk)\exp(\mp iql) \tag{44}$$

and

$$D^\pm_{\pm lk} = \frac{1}{\sqrt{L}}\sum D^\pm_\pm(qk)\exp(\mp iql), \tag{45}$$

where L is the number of independent points on the l-lattice, i.e., $L = N$. The Hamiltonian written in this transformed variables is

$$\begin{aligned} H_B = {} & \sum E^-_+(qk)E^+_-(qk) \times \left[ie^{i(k-q)} - ie^{-iq} - ie^{-i(k-q)} + ie^{iq}\right] \\ & + \sum E^+_+(qk)E^-_-(qk) \times \left[ie^{i(k-q)} - ie^{-iq} - ie^{-i(k-q)} + ie^{iq}\right] \\ & + \sum D^+_+(qk)D^-_-(qk) \times \left[ie^{i(k-q)} + ie^{-iq} - ie^{-i(k-q)} - ie^{+iq}\right] \\ & + \sum D^-_+(qk)D^+_-(qk) \times \left[ie^{i(k-q)} + ie^{-iq} - ie^{-i(k-q)} - ie^{+iq}\right] \end{aligned}$$

$$+2m\sum\left[D_{+}^{+}(qk)E_{-}^{+}(qk)+E_{+}^{+}(qk)D_{-}^{+}(qk)\right]. \tag{46}$$

In writing this we assumed periodic boundary conditions on the lattice bosons. Define

$$A_{\pm}=\sin q\pm\sin(k-q), \tag{47}$$

so that we get the Hamiltonian in the following matrix form for diagonalisation:

$$\begin{bmatrix} 0 & -2A_{+} & 2m & 0 \\ -2A_{+} & 0 & 0 & 0 \\ 2m & 0 & 0 & 2A_{-} \\ 0 & 0 & 2A_{-} & 0 \end{bmatrix}. \tag{48}$$

On diagonalizing this we get a biquadratic equation of the following form

$$x^4-x^2\left[4A_{+}^2+4A_{-}^2+4m^2\right]+16A_{+}^2A_{-}^2=0. \tag{49}$$

The eigenvalues are the roots of this equation, namely:

$$\pm 2\left[\left(A_{+}^2+A_{-}^2+m^2\right)\pm\sqrt{\left(A_{+}^2+A_{-}^2+m^2\right)^2-4A_{+}^2A_{-}^2}\right]. \tag{50}$$

4. Comparing eigenvalues of bosons to fermions

Having obtained single boson eigenstates of the Dirac Hamiltonian of fermions on the lattice we want now to compare the energy eigenvalues of fermions and bosons. The Hamiltonian (26) has single fermion energy eigenvalues ϵ_f. It has two branches:

$$\epsilon_f^1=+\sqrt{m^2+4\sin^2 q}, \tag{51}$$

$$\epsilon_f^2=-\sqrt{m^2+4\sin^2 q}. \tag{52}$$

These we obtain by writing (26) in Bloch space followed by simple diagonalisation. The single boson energy spectra, Eq. (50), has four possible branches. Simplifying (50), using (47), we get these four branches of boson eigenenergies ϵ_B:

$$\epsilon_B^1=+\sqrt{m^2+4\sin^2 q}+\sqrt{m^2+4\sin^2(k-q)}, \tag{53}$$

$$\epsilon_B^2=+\sqrt{m^2+4\sin^2 q}-\sqrt{m^2+4\sin^2(k-q)}, \tag{54}$$

$$\epsilon_B^3=-\sqrt{m^2+4\sin^2 q}+\sqrt{m^2+4\sin^2(k-q)}, \tag{55}$$

$$\epsilon_B^4=-\sqrt{m^2+4\sin^2 q}-\sqrt{m^2+4\sin^2(k-q)}. \tag{56}$$

At this point it is instructive to investigate the fermion constituents of our klbs. To begin let us look at the constituents of e_{+lk}^c defined in (30). The single boson variable e_{+lk}^c with proper normalisation reads:

$$e_{+lk}^c=\frac{1}{\sqrt{2N}}\sum c_n^{\dagger}c_{n+l}^{\dagger}\exp^{ikn}. \tag{57}$$

Writing in Bloch space we get

$$e_{+lk}^c=\frac{1}{\sqrt{2N}}\sum c_p^{\dagger}c_{k-p}^{\dagger}\exp^{-i(k-p)l}. \tag{58}$$

The single boson eigenstates of the Dirac Hamiltonian (26) are combinations of objects such as e_{+qk}^c (see Eq. (44)). The fermion constituents of e_{+qk}^c are obtained from:

$$e_{+qk}^c=\frac{1}{\sqrt{N}}\sum e_{lk}^c\exp^{iql}. \tag{59}$$

Note that in the klb-space the number of independent states $L=N$. Substituting the expression for e_{lk}^c, Eq. (58), in (59), we get:

$$e_{+qk}^c=\left(1/\sqrt{2}\right).c_{k-q}^{\dagger}.c_q^{\dagger}. \tag{60}$$

The fermion constituents of e_{+qk}^c, from (60), are $c_{k-q}^{\dagger}$ and $c_q^{\dagger}$.

The energies of the two fermion eigenstates (i.e., combinations of objects like e_{+qk}^c) put together from the two branches of the fermion eigenvalues (51) and (52) give us precisely the four branches of the boson eigenspectra (53)–(56). Therefore, in making these single bosons, there is no net gain or loss in energies as is expected in this non-interacting theory. The single klb eigenstate of quantum numbers q and k are made of two single fermion eigenstates of quantum numbers q and $k-q$. The energy of the single klb is just the sum of the energies of the two fermions. The Hamiltonian (26) has no interactions and the results above are consistent. Further, the four branches of the energy spectra are expected from the two branches of the fermion spectra. This tells us the k-deformed modes are necessary to make the lattice boson set [1] complete.

What happens when we consider not the single boson/fermion but large numbers of them? The boson distribution function allows a number of bosons to get into an eigenstate. For the fermions this is forbidden by the Pauli principle. Thus even though the single/fermion energy spectra do not favour bosons to fermions or vice versa, the boson distributions give lower free energy for the boson states. In that sense in the two filling regions the Dirac fermions remain susceptible towards bosonisation.

5. Gauge interactions and klb excitations

The Dirac Hamiltonian of the previous section has the klbs as its eigenstates. e_{lk} sits on the effective boson lattice at the site l. The fermion Dirac Hamiltonian, when mapped to this boson space, leads to hopping of the e_{lk} to the $e_{(l\pm1)k}$. The deformation parameter k remains unchanged in this simple hopping process. We show now the gauge interactions excite the k-deformed modes of the lattice bosons.

To understand the effect of the gauge interactions, we consider the Hamiltonian in two space dimensions:

$$H = i\sum c_n^\dagger c_m u_{nm} + \text{h.c.}, \tag{61}$$

where n and m are the nearest neighbours on the square lattice. u_{nm} are unimodular complex numbers that are related to the flux, ϕ, through the plaquette as:

$$e^{i\phi} = u_{12}u_{23}u_{34}u_{41}, \tag{62}$$

the sites $1, 2, 3, 4$ are the corners of the plaquette. We are interested in mapping the Hamiltonian (61) to the space of the bosons. Since our bosons are in one dimensions we choose the gauge

$$u_{n+\hat{y},n} = 1, \tag{63}$$

where the direction y is chosen perpendicular to the l-lattice. $\hat{y}$ is the unit vector along y-axis. We want to understand the effect on the motion of the bosons on the l-lattice and ignore, for now, the perpendicular [i.e., the y] direction. We choose the u along x-direction to be:

$$u_{n+\hat{x},n} = \alpha^{n_x.n_y}, \tag{64}$$

where the coordinates of the point on the lattice

$$n = (n_x, n_y), \tag{65}$$

so that the choice (63) and (64) specifies the flux through the plaquettes of the lattice. Consider the $2N$ fermionic sites on the one-dimensional strand

$$n_y = 1, \tag{66}$$

and look at their dynamics along the strand. For simplicity of illustration we choose α to be:

$$\alpha = e^{ik_p}, \tag{67}$$

$$k_p = \frac{2\pi}{2N}.p. \tag{68}$$

The Hamiltonian (61) with this choice of u along the strand $n_y = 1$ is denoted by $H(n_y = 1)$. The commutator of this Hamiltonian with e_{lk} is now calculated,

$$\begin{aligned}
&\left[H(n_y = 1), e^c_{+lk}\right] \\
&\quad = i\exp(ik)e^c_{+(l+1)(k+k_p)} \\
&\qquad - i\exp(ilk_p)e^c_{+(l+1)(k-k_p)} \\
&\qquad - i\exp(i(k_p-k))e^c_{+(l-1)(k-k_p)} \\
&\qquad + i\exp(i(l-1)k_p)e^c_{+(l-1)(k+k_p)}.
\end{aligned} \tag{69}$$

Comparing with (39) we notice (i) the limit $k_p \to 0$, i.e., the absence of gauge flux, reduces (69) to (39) [note the Hamiltonian (61) does not have the mass term], (ii) the presence of non-zero flux, $k_p \neq 0$, excites the $k \pm k_p$ deformed modes of the lattice bosons. We conclude from (69) that e_{+lk} goes to $e_{+(l\pm1)(k\pm k_p)}$ through emission/absorption of the gauge fields.

Aside from the gauge interactions that excite the klbs, there are other classical modes of these k-deformed bosons. The reasons they arise have to do with the nature of the boson space to which the fermions are mapped. The equivalent lattice of bosons has the $2N$-valued fields e_{lk} at the lattice points denoted by the site index l. The reason the field is $2N$-valued is because the deformation parameter k, owing to periodic boundary conditions, can take $2N$-values. Therefore classical excitations that have k varying from one point to another in the lattice space of the bosons are obtained with appropriate boundary conditions on the boson space.

6. Discussions

The lattice background on which the fermions reside tells us how to compose bosons of these fermions. The composition requires we add fermion pairs of precise phases. In the region near the insulator we compose them of pairs of holes precisely superposed.

Deforming the phases of the fermi (hole) pairs do not change the boson nature of the composites. The deformations are effected by the parameter k that takes values consistent with periodic boundary conditions. The solutions of the deformed algebra are the klbs of the underlying fermions. Their eigenspectrum is presented for the Dirac fermion. For the non-interacting Dirac fermions the k-values of the klbs do not change on hopping over the lattice. When gauge interactions are introduced the k-values change with these interactions. The k changes to $k \pm k_p$ with emission/absorption of the gauge component k_p.

There are classical excitations of the klbs. The boson space is made of $2N$ component boson fields sitting on the lattice points. The $2N$ components at each point are somewhat analogous to the spins. Just as for the spins, there are classical k-excitations that change k over the lattice points of the boson space. These are obtained with appropriate boundary conditions. For the non-interacting Dirac fermions, the energy of the single boson is exactly equal to the sum of the energies of its component fermions. This is as expected. However, the free energy of the assembly of bosons, because of boson distribution, is likely to be lower compared to the fermions. The bosons can preferably populate the lower energy states. We expect this to manifest at lower temperatures.

The Dirac fermion in one dimension is a model [6] for charge carriers in organic polymers such as polyacetylene. The starting assumption in band models that these systems have $1/2$-filled bands is contradicted by experiments that show that the undoped samples are insulators. On doping these become good conductors. It turns out the interaction of the chargecarriers with the underlying lattice splits the band into two parts with a gap in between. The Brillouin zone is halved. Hence the undoped samples behave as insulators.

We argue now the lattice boson conditions apply to these cases. We need to set up the lattice boson algebra for the fermions of the lower band on the reduced symmetry lattice (halving of the Brillouin zone). Since the lower band is nearly full, we are in the insulator region and the lattice boson conditions apply. We expect precisely superposed hole pairs to be alternate transporters of charge for the doped samples.

Assume again, the lattice has $2N$ points. To create lattice bosons at half-filling we consider the elements of the form:

$$e_{+1} = c_1^\dagger c_3^\dagger + c_3^\dagger c_5^\dagger + c_5^\dagger c_7^\dagger + \cdots, \tag{70}$$

$$e_{+2} = c_1^\dagger c_5^\dagger + c_3^\dagger c_7^\dagger + c_5^\dagger c_9^\dagger + \cdots, \tag{71}$$

etc. These are defined over the odd-points of the lattice. It is easy to check they form a closed algebra. Similarly, the elements defined on the even points:

$$e_{+1} = c_2^\dagger c_4^\dagger + c_4^\dagger c_6^\dagger + c_6^\dagger c_8^\dagger + \cdots, \tag{72}$$

$$e_{+2} = c_2^\dagger c_6^\dagger + c_4^\dagger c_8^\dagger + c_6^\dagger c_{10}^\dagger + \cdots, \tag{73}$$

etc., form a closed set. For both these sets, since they go over half the lattice points, we get:

$$[e_{-l}, e_{+l}] = N - 2h_0 + h_{4l}. \tag{74}$$

For the odd/even set (70)–(71) /(72)–(73)

$$h_0 = \sum c_n^\dagger c_n, \tag{75}$$

where the sum is over the odd/even sites. Half-filling is the "insulator" limit with $h_0 = N$. Thus we have,

$$[e_{+l}, e_{-l}] = N \tag{76}$$

in the limit N is large. Hence $e_{\pm l}/\sqrt{N}$ are boson generators in the half-filled limit.

We have shown that for the non-interacting fermions the single lattice boson energy is identical to the sum of the constituent fermions. It is the statistics of the bosons that argues in favour of bosonisation.

It is well known that many of the so-called half-filled systems are in fact examples of insulators with bands separated by gaps. We expect our considerations to apply in all these cases. The hole-pair bosons near the insulator domain give us alternate modes of charge transport in the region. At this point we are investigating if these are the modes that make insulators into excellent conductors on small doping.

The generalisation of these results to higher dimensions is of interest. Our efforts in that direction indicate that the algebra, the boson interpretation, the k-deformations go over to higher dimensions. For now, these take us beyond the scope of the present work.

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