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A potential thermal dynamo and its astrophysical applications

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It is shown that thermal turbulence, not unlike the standard kinetic and magnetic turbulence, can be an effective driver of a mean-field dynamo. In simple models, such as hydrodynamics and magnetohydrodynamics, both vorticity and induction equations can have strong thermal drives that resemble the α and γ effects in conventional dynamo theories; the thermal drives are likely to be dominant in systems that are endowed with subsonic, low-β turbulence. A pure thermal dynamo is quite different from the conventional dynamo in which the same kinetic/magnetic mix in the ambient turbulence can yield a different ratio of macroscopic magnetic/vortical fields. The possible implications of the similarities and differences between the thermal and non-thermal dynamos are discussed. The thermal dynamo is shown to be highly important in the stellar and planetary context, and yields results broadly consistent with other theoretical and experimental approaches. *Published by AIP Publishing.*
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I. INTRODUCTION

The problem of explaining the ubiquity of magnetic fields observed at planetary,[1] stellar,[2] and cosmic scales,[3,4] ranging from the dilute intergalactic medium (IGM) to the exceptionally dense neutron stars,[5] had led to the "evolution" of what are generically (and appropriately) called dynamo mechanisms. In this paper, we limit ourselves to the so-called mean-field (large-scale) dynamos that operate via the conversion of ambient turbulence into ordered large-scale magnetic fields; see Refs. 6 and 7 in the context of galactic and extragalactic magnetic fields. Starting with Parker in the 1950s,[8] to explain solar magnetic activity, the mean-field dynamos received a major impetus via the pioneering work of Ref. 9 on mean-field electrodynamics; an excellent account of the classical theory can be found in Refs. 3, 10, and 11. The next breakthrough arose via the use of the Eddy-Damped Quasi-Normal Markovian (EDQNM) closure by Ref. 12 to account for new contributions to the α-effect. The subsequent decades witnessed an explosion of activity, as evidenced in Refs. 13–35. A comprehensive review of dynamo theory can be found in Refs. 36–38.

There are various sub-mechanisms that contribute to the broad dynamo paradigm. Originally intended for generating (mostly long range) magnetic fields, the dynamo theories have grown to be more encompassing and can be viewed as theories for generating "Generalized Vorticity" (GV) constructed from the electromagnetic (the magnetic field) and fluid components (vorticity).[39,40] Such a unified point of view becomes necessary as one deals with physical models that are more general than the standard nonrelativistic magnetohydrodynamics (MHD).[41,42] For a full covariant formulation of the GV two-form, the reader should consult Ref. 42. Extended formulations of the dynamo mechanism have been employed in discussing the creation of flows in general. Of particular importance is their application to astrophysical jets[40,43,44] which are diverse and abundant in the cosmos.

In most of the MHD-based mean-field theories, the turbulence is kinetic, in which case the (large scale) dynamo action converts *short scale kinetic energy* into *large scale magnetic fields*.[37,45,46] For instance, the α-drive responsible for the generation of large scale magnetic fields, in the very early work of Ref. 9, arises entirely from the ensemble average of the kinetic helicity $\langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle$. In extended MHD theories, such as Hall MHD,[39,40,47–50] the short scale kinetic helicity is augmented by the (short scale) cross helicity and magnetic helicity. Together, they create long scale vortical fields with kinetic and electromagnetic components. One of the chief consequences of this model is the emergence of a unified Dynamo-Reverse Dynamo mechanism[39,40] presenting a comprehensive theory for the simultaneous generation of large-scale magnetic fields and vorticity.

The exploration of generic dynamo theories in the context of physics models more comprehensive than MHD (the model used in most extant works) is of essence since many astrophysical systems do fall outside its domain.[51] The recent exploitation of Hall MHD in astrophysics, in general, and dynamo theory, in particular, has considerably advanced the fields.[47–50,52,53] In a future publication, we will study, in detail, even more encompassing (multi-fluid) models than Hall MHD.

In this paper, however, we plan to take another step towards an overall unified picture. To the best of our knowledge, barring notable exceptions such as Refs. 54–57, most of the extant literature, even in extended dynamo theories, has not paid much attention to the possibility of dynamo action fueled by thermal energy. It is important to recognize, however, that thermal drives have been invoked very often to generate the so-called seed magnetic field needed for a dynamo to work. The Biermann battery[58] has often been

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invoked as a seed field mechanism, but we also emphasize the existence of a unique (ideal) thermal relativistic drive[4,59] that does *not* possess a nonrelativistic counterpart. To accord appropriate importance to the "thermal energy," we will explore the role of thermal turbulence and thermally driven mean-field terms in dynamo action; we will also discuss the regimes (and astrophysical systems) in which the thermal contributions could be significant, even dominant.

Since this represents our first attempt to directly investigate a thermally driven dynamo, we will limit ourselves to the simplest fluid models. In Section II, we will first summarize the principal steps of what (for want of a better word) may be called a closure scheme, and then apply it to a range of fluid theories to identify the existence, and evaluate the significance of the thermal-driven terms. In Section III, exploring ideal MHD, we establish an elegant relation between the mean magnetic field and vorticity, followed by an exploration of its stellar and planetary consequences. After summarizing our main results, we discuss the prospects for future work in Section IV. In the Appendixes, we present an alternative derivation of the thermal dynamo contributions and a discussion of two-fluid effects in the "beyond MHD" models.

II. TOWARDS A THEORY OF THE THERMAL DYNAMO

In this section, we present a hierarchy of models, each of which involves a new contribution arising from thermal physics. We commence with a detailed discussion of the underlying assumptions and the closure scheme used throughout the paper.

A. Mean-field models: Assumptions and closure scheme

We first describe an internally consistent closure scheme needed to construct a mean-field dynamo model.

A generic field Ψ is decomposed into a mean-field part $\bar{\Psi}$ and a fluctuating part ψ; terms linear in the latter vanish upon a suitable statistical or spatial averaging. The fluctuating component is further decomposed as $\psi = \psi^{(0)} + \psi^{(1)}$, where $\psi^{(0)}$ and $\psi^{(1)}$ denote the small-scale fields that exist *independently* of the mean fields and the induced small scale fluctuations, respectively. The field $\psi^{(0)}$ represents the background turbulence and is the energy reservoir that serves as a source/driver for the small scale fluctuations $\psi^{(1)}$ and, eventually, the mean field $\bar{\Psi}$ that is the desired end product of the large-scale dynamo action. Hence, we refer to $\psi^{(0)}$ as the ambient field, while the mean field $\bar{\Psi}$ and $\psi^{(1)}$ are perturbations. It is straightforward to assume $|\psi^{(1)}| \ll |\psi^{0}|$.

But the ordering between $\bar{\Psi}$ and $\psi^{(1)}$ can be tricky, especially when we are trying to breed $\bar{\Psi}$, say, from an originally non-zero but very small magnitude. However, as long as the induced small scale fluctuations $\psi^{(1)}$ remain bounded at low enough levels,[37,60] it affords great algebraic simplification to incorporate $|\psi^{(1)}| \ll |\bar{\Psi}|$ in our closure model. This genre of closure models, satisfying $|\psi^{(1)}| \ll |\bar{\Psi}|$, has been employed in Refs. 47–49, and 52, and also bears similarity with the approach of Refs. 54 and 61.

A word on the role of dissipation in this paper is also mandatory. In our treatment, dissipative effects, such as the resistivity and viscosity, are not explicitly present. This, however, does *not* mean that they are unimportant—we hold it to be self-evident that they play a crucial role in dynamo theory, as noted in Ref. 37. We do not include them directly since the dissipative contributions are linear in the dynamical fields and are, hence, quite straightforward to incorporate in the mean-field analysis. Because of their absence in our simplified treatment, we do not tackle the role of the magnetic Prandtl (Pm) and Reynolds (Rm) in dynamo theory,[37,38] which remains a crucial and pressing issue.

B. Ideal hydrodynamics

Let us begin with the simplest model of a neutral fluid, obeying

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{1}$$

$$\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) = -\frac{\nabla P}{\rho}, \tag{2}$$

an appropriate P evolution equation is needed to close the model. Two commonly used choices are:

- Incompressibility: Assume $\rho = \text{const}$, reducing (1) to $\nabla \cdot \mathbf{V} = 0$. This assumption severely restricts the choice of P, which is automatically determined by taking the divergence of (2).
- Enthalpy evolution equation: The pressure is assumed to obey an adiabatic evolution, viz., $d/dt(P/\rho^{\gamma}) = 0$. We introduce the enthalpy for the barotropic fluid, which satisfies $\nabla h = \rho^{-1} \nabla P$ and evolves as per

$$\frac{\partial h}{\partial t} + \mathbf{V} \cdot \nabla h + (\gamma - 1) h \nabla \cdot \mathbf{V} = 0, \tag{3}$$

and it must also be recognized that $h = \gamma/(\gamma-1)P/\rho = \gamma/(\gamma-1)k_B T/m$, where T is the temperature and m is the total mass. In the barotropic and adiabatic case, note that $P \propto \rho^{\gamma}$ and $T \propto \rho^{\gamma-1}$, and the second equality follows from the equation of state.

Of the two approaches, in the hydrodynamic case, we shall adopt the latter. The chief reason is evident: instead of evolving the density and the pressure (or temperature) dynamics separately, we have effectively reduced them to a single equation. In other words, only one of ρ, P, and T constitutes an independent variable. Hence, our complete system of equations, in the (ideal) hydrodynamics model, comprises (2) and (3), and the continuity equation is rendered superfluous (serving just as a constraint). The governing equations are as follows:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla \left(\frac{\gamma k_B T}{m(\gamma - 1)}\right), \tag{4}$$

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T + (\gamma - 1) T \nabla \cdot \mathbf{V} = 0, \tag{5}$$

as we have used the definition of h in terms of T, since the latter is a more physically transparent variable. Taking the curl of (4) yields the equation for vorticity $\mathbf{\Omega}$

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{\Omega}) = 0. \tag{6}$$

Carrying out the decomposition $\psi = \bar{\Psi} + \psi^{(0)} + \psi^{(1)}$ with $\psi = [\mathbf{V}, \mathbf{\Omega}, T]$, and introducing the notation $\bar{\mathbf{\Omega}} = \nabla \times \bar{\mathbf{V}}$, $\boldsymbol{\omega}^{(0)} = \nabla \times \mathbf{v}^{(0)}$ and $\boldsymbol{\omega}^{(1)} = \nabla \times \mathbf{v}^{(1)}$, we derive the fundamental equations for a hydrodynamical dynamo (fluid vorticity generator). The mean-field vorticity evolves as

$$\begin{aligned}\frac{\partial \bar{\mathbf{\Omega}}}{\partial t} - \nabla \times (\bar{\mathbf{V}} \times \bar{\mathbf{\Omega}}) &= \nabla \times \langle \mathbf{v}^{(0)} \times \boldsymbol{\omega}^{(1)} + \mathbf{v}^{(1)} \times \boldsymbol{\omega}^{(0)} \rangle \\ &\equiv \nabla \times \bar{\boldsymbol{\mathcal{E}}}_{\Omega},\end{aligned} \tag{7}$$

where $\bar{\boldsymbol{\mathcal{E}}}_{\Omega}$ is an effective "electromotive force," named so in analogy with the equivalent generator of the magnetic field in standard dynamo theory. The induced fluctuations are determined by subtracting the expression for $\bar{\Psi}$ from the full ψ equation and using the Reynolds relations[11] together with our closure scheme. The induced fluctuations for our hydrodynamical model yield

$$\frac{\partial \mathbf{v}^{(1)}}{\partial t} + \bar{\mathbf{V}} \cdot \nabla \mathbf{v}^{(0)} + \mathbf{v}^{(0)} \cdot \nabla \bar{\mathbf{V}} = -\nabla \left(\frac{\gamma k_B T^{(1)}}{m(\gamma - 1)} \right), \tag{8}$$

$$\begin{aligned}&\frac{\partial T^{(1)}}{\partial t} + \bar{\mathbf{V}} \cdot \nabla T^{(0)} + \mathbf{v}^{(0)} \cdot \nabla \bar{T} \\ &+ (\gamma - 1)\left[\bar{T} \nabla \cdot \mathbf{v}^{(0)} + T^{(0)} \nabla \cdot \bar{\mathbf{V}}\right] = 0.\end{aligned} \tag{9}$$

In the latter two equations, terms proportional to $\mathbf{v}^{(1)}$ and $T^{(1)}$ have (mostly) dropped out as per the assumption that they are much smaller than *similar surviving terms* involving the mean or ambient fields. Since there is no contribution from either $\bar{T}$ or $T^{(0)}$, the only thermally driven contribution comes from $\nabla T^{(1)}$ in (8). Further details regarding the evolution equations for the small-scale fluctuations can be found in Refs. 47 and 52.

To proceed further, we use the familiar Reynolds relations, of which the most crucial assumption is that second and third order correlations are proportional to the Kronecker delta and Levi-Civita tensors, respectively. For this analytical effort, we replace the time derivatives by introducing a correlation time τ, i.e., we assume $\partial \psi^{(1)}/\partial t \approx \psi^{(1)}/\tau$.

With these assumptions, the "electromotive force" $\bar{\boldsymbol{\mathcal{E}}}_{\Omega}$ may be expressed as

$$\bar{\boldsymbol{\mathcal{E}}}_{\Omega} = \mathcal{A}\bar{\mathbf{V}} + \alpha^{(\Omega)}\bar{\mathbf{\Omega}} - \beta^{(\Omega)}\nabla \times \bar{\mathbf{\Omega}} + \mathcal{C}\nabla\bar{P}, \tag{10}$$

with

$$\mathcal{A} = -\frac{2\tau}{3}\langle \boldsymbol{\omega}^{(0)} \cdot \boldsymbol{\omega}^{(0)} \rangle, \tag{11a}$$

$$\mathcal{C} = \tau\mathcal{A}, \tag{11b}$$

$$\begin{aligned}\alpha^{(\Omega)} &= \frac{\tau}{3}\langle \mathbf{v}^{(0)} \cdot \boldsymbol{\omega}^{(0)} \rangle + \frac{\tau^2}{3}\left(\frac{\gamma k_B}{m(\gamma - 1)}\right)\langle \nabla T^{(0)} \cdot \boldsymbol{\omega}^{(0)} \rangle \\ &\equiv \alpha_0^{(\Omega)} + \alpha_T^{(\Omega)},\end{aligned} \tag{11c}$$

$$\beta^{(\Omega)} = \frac{\tau}{3}\langle \mathbf{v}^{(0)} \cdot \mathbf{v}^{(0)} \rangle. \tag{11d}$$

Before subjecting (10) to a qualitative analysis, we note:

(1) Since the large scale vorticity equation (7) is driven only by $\nabla \times \bar{\boldsymbol{\mathcal{E}}}_{\Omega}$, the last term in (10) will be irrelevant.
(2) The first term, with $\mathcal{A}$ proportional to ambient enstrophy, does not appear to have an analog in mean-field MHD theories.
(3) The second and the third terms, on the other hand, mimic the α and β effects of the standard MHD theories. The nomenclature $\alpha^{(\Omega)}$ and $\beta^{(\Omega)}$ stresses the formal equivalence to their MHD counterparts under the exchange $\bar{\mathbf{\Omega}} \leftrightarrow \bar{\mathbf{B}}$.

At this stage, we emphasize a crucial point. The new term in (11c) can be re-expressed as $\alpha_T^{(\Omega)} \propto \nabla \cdot \langle T^{(0)}\boldsymbol{\omega}^{(0)} \rangle$, which *vanishes identically* for perfectly *homogeneous* and/or *isotropic* turbulence. There are several new terms that enter the mean-field equations upon relaxing these assumptions, but we include only a couple of "thermal" drives—$\alpha_T^{(\Omega)}$ and γ_T—in our analysis; the latter is defined in (17c). If the new terms are included, it is found that a host of effects open up, including the possibility of density gradients (operating on small-scales) driving dynamos, viz., large scale magnetic fields and the emergence of large-scale temperature and density gradients as well; we intend to investigate some of these effects in a forthcoming publication.

The appearance of a new term (driven by the gradient of the ambient temperature) in the α-effect is a key result of this section on ideal hydrodynamics. This term (going to zero with vanishing ambient temperature) is not reflective of some transformed "baroclinic effect" (invoked for seed generation), but belongs to the class of other dynamo-like terms driving, in this case, a vortical dynamo. We can readily compare the magnitude of $\alpha_0^{(\Omega)}$ (the pure helicity drive) and $\alpha_T^{(\Omega)}$ (the thermal-helicity drive) to define the dimensionless number

$$\epsilon_{HD} = \frac{\alpha_T^{(\Omega)}}{\alpha_0^{(\Omega)}} \sim \frac{c_s^2 \tau_c}{\mathcal{V}\ell_T}, \tag{12}$$

where we have used the fact that c_s is the characteristic isothermal sound speed, and $k_B T \sim m c_s^2$ (in dimensional variables). We note that $\mathcal{V}$ is the characteristic ambient velocity, τ_c is the correlation time scale, and ℓ_T is the scale length that characterizes the (inverse) gradient of the ambient temperature. Two natural choices for a correlation time scale come to mind:

- One can envision τ_c as analogous to a sound-crossing time implying $\tau_c \sim \ell_T/c_s$. This choice leads to $\epsilon_{HD} \sim c_s/\mathcal{V} = \mathcal{M}^{-1}$, where $\mathcal{M} = \mathcal{V}/c_s$ is the Mach number for the ambient flow.
- It is possible that a constraint similar to causality may exist at the ambient level, i.e., the length scale ℓ_T is bounded by the product of the correlation time τ_c and the ambient velocity $\mathcal{V}$. In such an event, $\tau_c \sim \ell_T/\mathcal{V}$ resulting in $\epsilon_{HD} \sim c_s^2/\mathcal{V}^2 = \mathcal{M}^{-2}$.

For either of these cases, a most interesting conclusion follows: the thermal-helicity drive ($\alpha_T^{(\Omega)}$) can become dominant for subsonic (ambient) turbulence. In most astrophysical scenarios, one observes supersonic turbulence, but environments such as the intracluster medium (ICM) and the intergalactic medium (IGM) have been proposed as sites of subsonic turbulence.[62,63] Furthermore, recent observations appear to support the existence of subsonic turbulence in protoplanetary accretion discs.[64] We emphasize that the latter is of considerable importance since (7), (10), and (11c) govern the mean-field evolution of vorticity, and it is well-known that vorticity is of critical importance in protoplanetary discs. Vorticity generation is of utmost importance as it possesses deep connections with angular momentum transport and turbulence,[65–67] thereby making it a key player in the role of planet formation.[68,69] Thus, in protoplanetary discs with subsonic turbulence, it is possible that a thermal-helicity driven dynamo may play a key role in vorticity generation and, consequently, in planetesimal formation.

C. Ideal magnetohydrodynamics

Ideal MHD is described by the *dimensional* equations

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\nabla\left(\frac{\gamma k_B T}{m(\gamma-1)}\right) + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi\rho}, \tag{13}$$

$$\frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T + (\gamma-1)T\nabla\cdot\mathbf{V} = 0, \tag{14}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{V}\times\mathbf{B}) = 0, \tag{15}$$

and the factor of ρ present in the second term on the LHS of (13) is an implicit function of T since $T \propto \rho^{\gamma-1}$. A fully self-consistent treatment of this term is a complex task since the factor of ρ is situated in the denominator and has a (fractional) power law dependence on T. For this reason, we shall mirror the approach of the hydrodynamic case and seek only a thermal drive akin to the $\alpha_T^{(\Omega)}$ term in (11c). We have already remarked in Section II B that a variety of other thermal drives can exist for our model, and that a detailed computational study of these terms will be undertaken in the near future.

The discussion could be simplified further if we restrict ourselves to the earlier stages of dynamo action where the ambient fields are much more dominant than the small and large scale fluctuations. Before proceeding further, we note that incompressibility serves as an alternative approach, enabling a more comprehensive analysis along the lines of Ref. 54. However, we must relax their assumptions of statistical homogeneity and isotropy, and we find that the $\alpha_T^{(\Omega)}$ and γ_T (discussed below) contributions can be recovered.

We find that the resulting system, due to the inclusion of magnetic fields, has a considerably higher number of terms; simplification is much more difficult. Since the primary thrust of this paper is to build a case for a thermal dynamo, we shall focus mostly on identifying and evaluating the role of a thermal drive, akin to the HD case, in generating large scale magnetic fields. Repeating the same procedure as before, we can derive an equation equivalent to (7), with $\bar{\boldsymbol{\Omega}} \to \bar{\mathbf{B}}$, for the evolution of the long scale magnetic field. The appropriate electromotive force $\bar{\boldsymbol{\mathcal{E}}} = \langle \mathbf{v}^{(0)}\times\mathbf{b}^{(1)} + \mathbf{v}^{(1)}\times\mathbf{b}^{(0)}\rangle$ yields

$$\bar{\boldsymbol{\mathcal{E}}} = \alpha\bar{\mathbf{B}} - \beta\nabla\times\bar{\mathbf{B}} + \gamma\bar{\boldsymbol{\Omega}}, \tag{16}$$

with the coefficients given by

$$\alpha = \frac{\tau}{3}\left\langle -\mathbf{v}^{(0)}\cdot\boldsymbol{\omega}^{(0)} + \frac{\mathbf{b}^{(0)}\cdot\nabla\times\mathbf{b}^{(0)}}{4\pi\rho^{(0)}}\right\rangle \equiv \alpha_V + \alpha_B, \tag{17a}$$

$$\beta = \frac{\tau}{3}\langle \mathbf{v}^{(0)}\cdot\mathbf{v}^{(0)}\rangle, \tag{17b}$$

$$\gamma = \frac{2\tau}{3}\langle \mathbf{v}^{(0)}\cdot\mathbf{b}^{(0)}\rangle + \frac{\tau^2}{3}\left(\frac{\gamma k_B}{m(\gamma-1)}\right)\langle\nabla T^{(0)}\cdot\mathbf{b}^{(0)}\rangle \equiv \gamma_0 + \gamma_T. \tag{17c}$$

It must be noted that the current helicity term in (17a) involves a factor of $\rho^{(0)}$ in the denominator, which is implicitly a function of $T^{(0)}$, given that we have adopted a barotropic (and adiabatic) equation of state. Based on our simplifying set of approximations, the α and β terms have no thermally driven contributions, and the γ term does indeed show a non-trivial temperature gradient drive γ_T, in addition to the cross helicity drive γ_0.[70] We reiterate that the turbulent electromotive force (16) also follows if we assume that the model is near-incompressible, akin to Ref. 54, but only when the homogeneity and isotropy constraints are violated (as otherwise the thermal drive considered herein vanishes).

To assess the importance of the thermal MHD drive, we define another dimensionless number

$$\epsilon_{MHD} = \frac{\gamma_T}{\gamma_0} \sim \frac{c_s^2\tau_c}{2\mathcal{V}\ell_T}, \tag{18}$$

where c_s is the sound speed, τ_c is the correlation time, $\mathcal{V}$ is the ambient velocity, and ℓ_T is the scale length corresponding to the (inverse) gradient of the ambient temperature. It is striking that (12) and (18) are identical to one another, apart from a factor of 2; the mystery disappears somewhat if we realize that both these numbers are computed from the contribution to the electromotive force that is proportional to the mean fluid vorticity. Hence, we can easily conclude that, as discussed in Section II B, γ_T is dominant when the turbulence is subsonic, i.e., it is likely to be of importance in environments such as the ICM, IGM, and protoplanetary accretion discs.

The relative magnitudes of the thermal and the standard fluid helicity drives are reflected in the following dimensionless ratio:

$$\mu_{MHD} = \frac{\gamma_T}{\alpha_V}\frac{\bar{\Omega}}{\bar{B}} \sim \left(\frac{\ell_V}{\ell_T}\right)\left(\frac{c_s^2\mathcal{V}_B}{\mathcal{V}^2}\right)\left(\frac{\tau_c}{L_V}\right)\bar{\mathcal{M}}_A, \tag{19}$$

where ℓ_V and ℓ_T are the length scales associated with the (inverse) gradients of the ambient velocity and temperature, respectively. Furthermore, c_s is the ambient sound speed, $\mathcal{V}_B$ and $\mathcal{V}$ denote the ambient Alfvén and kinetic velocities, τ_c

serves as the correlation time, and L_V is the length scale associated with the *large scale* velocity, emerges from the factor $\bar{\Omega}$ in (19). Also note that $\bar{\mathcal{M}}_A = \bar{\mathcal{V}}/\bar{\mathcal{V}}_B$ is the large scale Alfvén Mach number. At first glimpse, it may appear odd that (19) involves both large and small scale quantities. When we recognize the fact that it represents a ratio of two terms in the electromotive force, we see that both large and small scale factors enter; the former through the mean fields and the latter through the coefficients.

In order to perform a detailed analysis, we assume $\ell_V \approx \ell_T$, introduce the quantity $\zeta_V = \ell_V/L_V$, and consider three different choices for the microscopic length scale ℓ_V. To be consistent, ℓ_V must be "constructed" from the small scale quantities intrinsic to the system:

- We can work with $\ell_V/\tau_c \sim c_s$, implying that τ_c serves as the sound-crossing time for L. If so, $\mu^{(1)}_{MHD} \sim (c_s\mathcal{V}_B)/\mathcal{V}^2(\zeta_V\bar{\mathcal{M}}_A) = \mathcal{M}^{-1}\mathcal{M}_A^{-1}(\zeta_V\bar{\mathcal{M}}_A)$, where $\mathcal{M} = \mathcal{V}/c_s$ and $\mathcal{M}_A = \mathcal{V}/\mathcal{V}_B$ are, respectively, the ambient Mach and the Alfvén Mach numbers, respectively.
- A choice of $\ell_V/\tau_c \sim \mathcal{V}_B$ is possible, suggesting that τ_c behaves as an Alfvénic sound-crossing time. For this choice, we find that $\mu^{(2)}_{MHD} \sim c_s^2/\mathcal{V}^2(\zeta_V\bar{\mathcal{M}}_A) = \mathcal{M}^{-2}(\zeta_V\bar{\mathcal{M}}_A)$.
- Finally, we can suppose that a constraint akin to causality exists, and we can study the implications of the relation $\ell_V \sim \mathcal{V}\tau_c$. Upon simplification, (19) yields $\mu^{(3)}_{MHD} \sim (c_s^2\mathcal{V}_B)/\mathcal{V}^3(\zeta_V\bar{\mathcal{M}}_A) = \mathcal{M}^{-2}\mathcal{M}_A^{-1}(\zeta_V\bar{\mathcal{M}}_A)$.

In each of the three cases, we see that the value of μ is governed by the *inverse* powers of the ambient Mach numbers. Thus, we theorize that the existence of turbulence that is simultaneously subsonic *and* sub-Alfvénic in nature may ensure that γ_T in (17c) serves as the dominant factor in governing the evolution of large scale magnetic fields.

Alternatively, the different possibilities for μ, measuring the relative strength of the thermal dynamo, could be expressed in terms of the plasma beta, $\beta = c_s^2/\mathcal{V}_B^2$. We list the resultant scalings below:

(i) $\mu^{(1)}_{MHD} \sim \mathcal{M}^{-2}\beta^{-1/2}(\zeta_V\bar{\mathcal{M}}_A)$,

(ii) $\mu^{(2)}_{MHD} \sim \mathcal{M}^{-2}(\zeta_V\bar{\mathcal{M}}_A)$,

(iii) $\mu^{(3)}_{MHD} \sim \mathcal{M}^{-3}\beta^{-1/2}(\zeta_V\bar{\mathcal{M}}_A)$.

Hence, the subsonic turbulence may not be a necessary condition to ensure that γ_T is the major driver of large scale magnetic fields. For instance, a combination of $\mathcal{M} \sim 1$ and $\beta \ll 1$ may also ensure the preponderance of γ_T. The condition $\beta \ll 1$ has been investigated in the context of a wide range of astrophysical systems, such as solar coronal holes[71,72] and black hole accretion discs.[73] Lastly, reconsidering (19), we observe that γ_T may emerge as the dominant term provided $\ell_V \gg \ell_T$, that is, if the ambient temperature (or pressure) gradients are stronger than the velocity gradients. Of course, we wish to emphasize that each of these claims is only correct *modulo* the factor $\zeta_V\bar{\mathcal{M}}_A$. For instance, it is likely that $\zeta_V \ll 1$, while $\bar{\mathcal{M}}_B$ is determined through the evolution of the vorticity and the magnetic field. Thus, our results have a greater chance of being valid when $\mathcal{M} \ll 1$ and $\beta \ll 1$ hold true simultaneously, and even this may not suffice to guarantee the dominance of the γ_T term.

Although we have compared γ_T and α_V, we note that the latter can be replaced by the beta effect (17b) and qualitatively similar results, albeit with different power laws, will follow. We present the final versions below, as the intermediate steps are quite similar. We introduce the dimensionless ratio

$$\nu_{MHD} = \frac{\gamma_T}{\beta}\frac{\bar{\Omega}}{|\nabla \times \bar{\mathbf{B}}|}, \tag{20}$$

which quantifies the ratio of the two contributions to the electromotive force arising from γ_T and β. We end up with three possibilities, depending on the choice of ℓ_T in terms of τ_c, $\mathcal{V}$, $\mathcal{V}_B$, and c_s. They are given by

(i) $\nu^{(1)}_{MHD} \sim \mathcal{M}^{-2}\beta^{-1/2}\bar{\mathcal{M}}_A$,

(ii) $\nu^{(2)}_{MHD} \sim \mathcal{M}^{-2}\bar{\mathcal{M}}_A$,

(iii) $\nu^{(3)}_{MHD} \sim \mathcal{M}^{-3}\beta^{-1/2}\bar{\mathcal{M}}_A$.

Note that the expressions for ν_{MHD} and the corresponding ones for μ_{MHD} differ only by the factor of ζ_V, which is along the expected lines (from dimensional analysis).

III. SUPPLEMENTING IDEAL MHD—DISCUSSION AND IMPLICATIONS

Here, we shall present our coupled system of equations in the ideal MHD case and discuss some of the stellar and planetary astrophysical implications of our model.

A. Closed ideal MHD system

In order to put the thermal drive in perspective, we will supplement the content of Section II by concentrating on the equations describing the mean fields: the induction equation driven by the electromotive force

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times (\bar{\mathbf{V}} \times \bar{\mathbf{B}}) + \nabla \times \bar{\boldsymbol{\mathcal{E}}}, \tag{21}$$

and the equation for the evolution of vorticity

$$\frac{\partial \bar{\mathbf{\Omega}}}{\partial t} = \nabla \times (\bar{\mathbf{V}} \times \bar{\mathbf{\Omega}}) - \nabla \times \left(\frac{\bar{\mathbf{B}} \times (\nabla \times \bar{\mathbf{B}})}{4\pi\bar{\rho}}\right) + \nabla \times \bar{\boldsymbol{\mathcal{E}}}_\Omega, \tag{22}$$

where $\bar{\boldsymbol{\mathcal{E}}}$ is given in (16) and $\bar{\boldsymbol{\mathcal{E}}}_\Omega$ has the form

$$\begin{aligned}\bar{\boldsymbol{\mathcal{E}}}_\Omega &= \alpha_V^{(\Omega)}\bar{\mathbf{\Omega}} - \beta_V^{(\Omega)}\nabla \times \bar{\mathbf{\Omega}} \\ &\quad + \alpha_B^{(\Omega)}\bar{\mathbf{B}} - \beta_B^{(\Omega)}\nabla \times \bar{\mathbf{B}} + \cdots\end{aligned} \tag{23}$$

and the coefficients in the above expression are given by

$$\begin{aligned}\alpha_V^{(\Omega)} &= \frac{\tau}{3}\langle \mathbf{v}^{(0)} \cdot \boldsymbol{\omega}^{(0)}\rangle + \frac{\tau^2}{3}\left(\frac{\gamma k_B}{m(\gamma-1)}\right)\langle \nabla T^{(0)} \cdot \boldsymbol{\omega}^{(0)}\rangle \\ &\equiv \alpha_0^{(\Omega)} + \alpha_T^{(\Omega)},\end{aligned} \tag{24a}$$

$$\beta_V^{(\Omega)} = \frac{\tau}{3}\left\langle \mathbf{v}^{(0)} \cdot \mathbf{v}^{(0)} + \frac{\mathbf{b}^{(0)} \cdot \mathbf{b}^{(0)}}{4\pi\rho^{(0)}}\right\rangle, \tag{24b}$$

$$\alpha_B^{(\Omega)} = \frac{2\tau}{3}\left\langle \frac{\boldsymbol{\omega}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)}}{4\pi\rho^{(0)}} \right\rangle, \tag{24c}$$

$$\beta_B^{(\Omega)} = \frac{2\tau}{3}\left\langle \frac{\boldsymbol{\omega}^{(0)} \cdot \mathbf{b}^{(0)}}{4\pi\rho^{(0)}} \right\rangle. \tag{24d}$$

This system may be viewed as a "complete" picture of an ideal MHD dynamo (including only the primary first-order spatial derivatives in the electromotive force). We have used "…" in (23) to indicate the additional existence of higher order spatial derivative terms; a term proportional to $\nabla \times (\nabla \times \bar{\mathbf{B}})$ is an example. Such higher order terms, with multiple derivatives of large-scale quantities, will not be analyzed in this paper. In addition, as we have stated earlier, there are other thermal drives that we do not discuss here, as our objective was to isolate the contributions that are akin to the HD case. There is also a term in (23) proportional to $\nabla\bar{T}$, but it is rendered void when the curl is taken, as seen from the last term on the RHS of (22). Moreover, it must be recognized that all factors of ρ that occur in the preceding expressions are implicitly a function of the corresponding temperature, if we assume the existence of a barotropic equation of state.

Let us draw the reader's attention to the thermal drives: $\alpha_T^{(\Omega)}$ in (24a) for $\bar{\boldsymbol{\Omega}}$, and γ_T in (17c) for $\bar{\mathbf{B}}$. Both these terms, proportional to the small-scale temperature gradients, represent the thermal-driven contributions to the mean-field equations. In Sections II B and II C, we have already explored the domain where they may be dominant—subsonic, low-β systems. Let us further go on to study an extreme approximation, where we neglect all but the thermal-driven terms to simplify our rather complex system, represented by (21) and (22). Furthermore, by ignoring the non-linear terms in (21) and (22) which come into play only at "later" times, the essence of the "very initial" (i.e., starting from infinitesimal mean fields) dynamo evolution is captured by

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \gamma_T \nabla \times \bar{\boldsymbol{\Omega}},$$
$$\frac{\partial \bar{\boldsymbol{\Omega}}}{\partial t} = \alpha_T^{(\Omega)} \nabla \times \bar{\boldsymbol{\Omega}}, \tag{25}$$

where we have supposed the homogeneity of γ_T and $\alpha_T^{(\Omega)}$. This leads us to the remarkable conclusion that

$$\bar{\mathbf{B}} = \frac{\gamma_T}{\alpha_T^{(\Omega)}} \bar{\boldsymbol{\Omega}}. \tag{26}$$

Of course, it must be understood that (25) is likely to be valid only in the highly simplified limit of (i) "early" time regime (before the non-linear dynamo terms are operational); (ii) highly subsonic and low-β systems; (iii) "small" thermal conductivity, which occurs when the Péclet number (Pe) satisfies Pe $\gg 1$.

The relation (26) pertaining only to a pure thermal dynamo ($\equiv$*TD*) is reminiscent of a "similar" relation, derived in Refs. 39 and 40, relating the macroscopic velocity (of the flow) and the magnetic field. In Refs. 39 and 40, however, the thermal drive was neglected in analyzing the Dynamo–Reverse Dynamo ($\equiv$*DRD*) action. The DRD mechanism involved the production of large-scale flows and magnetic fields, created from a mix of short-scale kinetic and magnetic turbulence. Let us compare and contrast the TD and DRD studies:

(1) For TD, the ratio $|\bar{\mathbf{B}}|/|\bar{\boldsymbol{\Omega}}|$, estimated from (24a) and (17c), comes out to be $\gamma_T/\alpha_T^{(\Omega)} \sim b^{(0)}/\omega^{(0)}$.
(2) The equivalent ratio for the DRD that relates (in dimensionless units) the magnetic field with the flow speed yielded $|\bar{\mathbf{B}}|/|\bar{\mathbf{V}}| \sim v^{(0)}/b^{(0)}$.
(3) Notice the starkly opposite scaling of the ratio. For a pure thermal-driven dynamo, the generated macroscopic magnetic field (vorticity field) is directly proportional to the short-scale magnetic (kinetic) content of the ambient turbulence. On the other hand, the non-thermal DRD mechanism generates macroscopic magnetic (velocity) field inversely proportional to the ambient magnetic (kinetic) content.
(4) The new thermal drive introduces a potentially interesting, and qualitatively different, chapter into the book of the dynamo mechanism.

It is perhaps pertinent, at this stage, to emphasize that all thermal drives share a common feature—they are correlators of the thermal gradients (the simplest vectorial expression of available thermal free energy) with a vorticity (magnetic or kinematic). In a turbulent system, such a correlation will likely come about through a temperature-driven small-scale instability for which the perturbed amplitudes of the magnetic and velocity fields are related to ∇T. Thus, the thermal drive is a somewhat disguised form of the velocity-vorticity generator. Barring some highly symmetric systems, these small-scale perturbations will have a finite correlation with other small-scale quantities and yield finite thermal drives.

A nontrivial consequence of these different behaviors is that we have an effective diagnostic tool. From very qualitative characteristics of observed macroscopic fields, one can "gain" knowledge about the microscopic content and nature of turbulence that caused them, in addition to estimating the relative strengths of various processes that channelize turbulent energy (kinetic, magnetic, thermal) into long-scale kinetic and magnetic energies. The exact process of the extrapolation from macroscopic content to the microscopic initial state will, of course, be highly involved since all the dynamo terms will be operating simultaneously with no *a priori* specifiable ordering.

B. Implications in stellar astrophysics

Let us work out a simple example to illustrate the utility of the simple relation $\bar{B}^2 \propto \bar{\boldsymbol{\Omega}}^2$, which is implied by (26). It is well known that the total X-ray luminosity is proportional to the magnetic energy density ($\mathcal{L}_X \propto \bar{B}^2$), and the Rossby number Ro is inversely proportional to the angular velocity (Ro $\propto \bar{\boldsymbol{\Omega}}^{-1}$); see, e.g., Refs. 74 and 75 for more details. If a thermal dynamo was responsible for these large-scale fields, one would predict that, for such thermal-driven systems, $\mathcal{L}_X \propto \mathrm{Ro}^{-2}$.

Such conclusions, arrived at by very simple (though quite robust) scaling arguments, must be taken with a high degree of caution, in view of the many assumptions invoked.

The scaling may turn out to be consistent with the predictions from the classical dynamo theory,[10,11] but the issue has not yet been conclusively settled from an observational point-of-view. For relatively high values of Ro, current evidence appears to support a scaling of $\mathcal{L}_X \propto \mathrm{Ro}^{-q}$, where $2 < q < 3$.[74–76]

The simplified relation (26) also permits us to "guess" at the knowledge of the (large-scale) magnetic field profile if we know the angular velocity profile and vice-versa. We conjecture that (26) may pave the path towards a better understanding of the rotation profile observed in some regions of the Sun, where the thermal dynamo may, indeed, be dominant. An important limitation of our theory has been the elimination of stratification effects, which play a key role in such instances.[28] As our relation (26) holds true at all times, it is also likely to pinpoint the manner in which stellar spin-down is correlated with the corresponding decay of magnetic fields. A more detailed analysis of (21) and (22) is likely to enable a better understanding of this phenomenon, potentially advancing the work undertaken in Ref. 77.

C. Planetary and solar dynamos: A discussion

We note that there is an important class of astrophysical systems where thermal dynamos are important, and they have been widely studied, see, e.g., Refs. 55 and 78 for recent reviews of this subject. But, there are a couple of subtle differences between our approach, and the one adopted in most studies of planetary (and stellar) dynamos. First, we note that the latter typically focus on determining the scaling relations between magnetic fields and planetary parameters (radius and rotation rate). On the other hand, our approach is geared towards the formulation of a coupled vorticity-magnetic field model of the mean fields.

Second, we wish to note that conventional studies have been directed towards determining the magnetic field as a function of the convective flux.[55,79] The dynamo action emphasized in this paper, however, originates in ∇T (or ∇P in compressible systems). Yet, it is important to recognize that there exists a deep duality between the temperature gradient and the associated heat flux—a fact that is widely recognized in the solar dynamo literature, see, e.g., Eqs. (4) and (5) in Ref. 80. It is known that the heat flux can drive a sustained (and enhanced) temperature gradient, while the converse is also true. In mathematical terms, the heat flux is (tensorially) proportional to the specific entropy gradient, the latter of which can be written in terms of the temperature gradient.[80] Furthermore, in astrophysics, the Schwarzschild criterion quantifies the critical temperature gradient required for convection to occur,[81] which in turn can drive a convective heat flux and thereby give rise to magnetic fields, which are in accordance with the scaling laws of Refs. 55 and 79.

As we have shown, in accordance with the previous literature on solar (and planetary) dynamos,[2,55,82] that the temperature gradient is the root cause of the convective flux and the thermal dynamo, it is reasonable to enquire whether any viable temperature gradient instabilities exist in plasmas. The answer is in the affirmative and has been well-known to plasma physicists since the 1960s,[83,84] although it has been rarely situated in the astrophysical context.[85] Hence, we conjecture that such temperature gradient driven instabilities might lie at the heart of thermal dynamo action, generating magnetic fields by means of convection, thereby agreeing with the predictions of scaling theories. Indeed, we anticipate that a more generalized treatment would account for both temperature and density gradient driven instabilities (treating the two as *independent* variables), but such analytical and computational explorations are left for future studies. It is also worth mentioning that the temperature gradient driven instabilities are closely connected with angular momentum transport in tokamaks[86] and may thus play a role in enhancing our understanding of stellar spin-down.[87]

We also wish to point out the importance of these thermal dynamos in astrobiology,[88] which is concerned with the potential existence of life elsewhere in the universe. Finite ∇T-driven convection results in the production of large scale magnetic fields, which are well established as crucial mediators of habitability.[89] In particular, magnetic fields are responsible for shielding a planet from the incident stellar radiation and preventing the erosion of the atmosphere.[89] Thus, it appears to be quite natural to conclude that the rapidly burgeoning field of exoplanetary science[90] would need to take into consideration thermal dynamos for evaluating which class of planets can sustain magnetic fields, and enable life to originate (abiogenesis). In this respect, we point out recent studies such as Refs. 91 and 92 which have already considered the role of magnetic fields generated in M-dwarfs, as these stars have been subjected to exhaustive analyses in the realm of habitability and astrobiology.[93,94]

IV. CONCLUSION

In the early and traditional (typical) dynamo theories, the magnetic-field evolution is considered in isolation, with the velocity field being supplied as a kinematic input. This paradigm underwent a change in the last few decades, and most current dynamo models co-evolve the magnetic and velocity fields. These models, thus, investigate the interaction and exchange of kinetic and magnetic energies. Most theories, however, are not wholly (and explicitly) cognizant of the thermal energy since the role of temperature (or pressure, quantifying the thermal contributions) in the energy exchange process is often neglected. The purpose of this paper was to extend the reach of dynamo theories by specifying the temperature as a dynamical variable so that we could explore

- whether thermal interactions appropriately couple to the magnetofluid motion,
- the regimes in which the thermal-driven terms may play a significant role, and
- the resultant consequences of the thermal terms to the dynamo dynamics, if such regimes were identified.

Within the framework of simple HD and MHD models, we demonstrated that the thermal-driven terms contributed to vorticity production, both in the magnetic field and the vorticity evolution equations. We further showed that these terms were likely to be important, possibly dominant, in systems such as protoplanetary discs and coronal holes where the ambient turbulence is subsonic and/or low-β. A qualitative exploration of "beyond MHD" effects in Appendix A added additional pathways through which thermal contributions could become increasingly important. In Appendix B, we use dimensional analysis to arrive at qualitatively similar conclusions as the ones presented in the main body of the paper.

After these initial considerations, we turned to a more exhaustive analysis of ideal MHD and presented the coupled evolution equations for the mean-field vorticity and the magnetic field. By considering the simplified limit where thermal contributions are dominant, we arrived at a remarkably simple expression, represented by (26), relating the large-scale magnetic field and the vorticity. The scaling of this ratio implied $|\bar{\mathbf{B}}|/|\bar{\mathbf{\Omega}}| \sim b^{(0)}/\omega^{(0)}$, revealing that the generated macroscopic magnetic field (vorticity field) is directly proportional to the short scale magnetic (kinetic) content of the ambient turbulence. It was shown that this result comes out to be exactly opposite to earlier results from the Dynamo-Reverse Dynamo mechanism.[39,40] We rounded off our discussion by discussing the relevance of our work in astrophysics, highlighting stellar and planetary systems in particular, where such dynamos are expected to play an important role.

It is clear, from the reasons outlined above, that the thermal dynamo may constitute a qualitatively different and profound step in the progress of dynamo theory. Further steps in the development of a fuller and more complete dynamo theory will consist of including important effects such as stratification, multi-fluid effects (for the most part), dissipation, and relativity. We also plan to apply the results of this work, and earlier papers, to astrophysical systems, in greater detail.

We could, for instance, use Eq. (26) and its counterpart in Refs. 39 and 40 as probes to expose the nature and content of the underlying microscopic turbulence. We also intend to invoke them as a starting point for more detailed investigations of the relationship between the macroscopic angular velocity and the magnetic field. Such relations in the context of stars and planets have been widely studied and debated owing to the existence of remarkable correlations between these two quantities, in addition to their importance in governing stellar spin-down and evolution. Thermal dynamos, in the solar and planetary context, will also have important ramifications in the field of astrobiology. They may indicate which class of planets and host stars is likely to be most accessible, with all other factors being held fixed, to sustained dynamo action, thereby increasing the chances of abiogenesis.

Thus, we believe that it is truly important for future simulations and analyses to consistently take into account the effects of thermal turbulence, as the ensuing implications are likely to be far-reaching.

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APPENDIX A: DISCUSSION OF THE "BEYOND MHD" MODELS

We believe that we have made a preliminary case for a predominantly thermal driven dynamo in the simplest HD and MHD models. In this Appendix, we will attempt to extend this understanding, at a very conceptual level, to a wide array of models more complex than MHD. Our discussion will be concerned more with 2-fluid effects (rather than thermal ones) as this is another area that has been mostly neglected by the dynamo community, with some recent exceptions.

We focus primarily on Hall MHD, the simplest and most widely studied of the lot, and then indicate how many of its properties carry over to even broader and complex models. The equations of Hall MHD are very similar to ideal MHD, and the Equations (13) and (14) stay the same, while (15) is modified by replacing $\mathbf{V}$ with $\mathbf{V}_e = \mathbf{V} - \lambda_H \nabla \times \mathbf{B}$ in the *incompressible* limit, where $\lambda_H = \lambda_i/L$ is the dimensionless Hall parameter and λ_i is the (dimensional) ion skin depth. The ensuing equations, for the mean fields and the fluctuations, are rendered even more complex, and we shall not report on the results here.

Instead, we merely observe that the γ-effect is still present, but the coefficient is no longer identical to its ideal MHD counterpart, the latter of which is given by (17c). However, we do find that γ_T in (17c) stays the same as before, and γ_0 can be approximated by replacing $\mathbf{v}^{(0)}$ by the ambient electron velocity $\mathbf{v}_e^{(0)} = \mathbf{v}^{(0)} - \epsilon\nabla \times \mathbf{b}^{(0)}$. Hence, much of the analysis undertaken in Section II C is also applicable herein; in addition, most conventional plasmas satisfy the criterion $\lambda_H \ll 1$. Nevertheless, we wish to point out an important, and mostly neglected, feature of Hall MHD. The governing equations can be rewritten[41] to yield

$$\frac{\partial \mathbf{\Omega}_a}{\partial t} - \nabla \times (\mathbf{V}_a \times \mathbf{\Omega}_a) = 0, \tag{A1}$$

where $\mathbf{\Omega}_a'$s and $\mathbf{V}_a$'s serve as effective vorticities and associated velocities, respectively, expressed as

$$\begin{aligned} \mathbf{\Omega}_1 = \mathbf{B}, \quad \mathbf{V}_1 = \mathbf{V} - \lambda_H \nabla \times \mathbf{B}, \\ \mathbf{\Omega}_2 = \mathbf{B} + \lambda_H \nabla \times \mathbf{V}, \quad \mathbf{V}_2 = \mathbf{V}. \end{aligned} \tag{A2}$$

The similarity between (6) and (A1) is manifest—Hall MHD engenders the simultaneous production of "generalized vorticity" and magnetic fields. Hence, we can import the results from Sections II B and II C on the conceptual level; a more comprehensive analysis is obviously warranted, which will be tackled in subsequent investigations. The presence of neutrals or dust, however, can change the dynamics significantly. Bypassing the details, we present a qualitative overview of the relevant features.

Let us consider the effect of neutrals first. It is known that a three species system, with neutrals as the third species, can be duly manipulated to obtain a structure akin to that of Hall MHD.[95] As a result, one must replace λ_i with $\lambda_i\sqrt{\rho_n/\rho_i}$, and since $\sqrt{\rho_n/\rho_i} \gg 1$ in protoplanetary discs, a modified (ion) skin depth of approximately 0.4 AU[96] is obtained. As a result, the Hall parameter can be large, even of order unity, and this in turn can significantly reduce the value of $\mathbf{v}_e^{(0)}$. In (19), the quantity $\mathcal{V}$ quantifies the ambient velocity, and in the three species model, one must (effectively) replace it with the ambient electron velocity $\mathcal{V}_e$. Since the latter is much reduced owing to the "large" value of λ_H, we find that $\mu \gg 1$ can be satisfied via a different pathway—a decrease in the ambient velocity owing to the existence of a non-negligible Hall effect.

Next, let us consider the effects of dust via a three species model wherein we have inertialess ions and electrons and charged (either positive or negative) dust. It was shown in Ref. 97 that this system is formally equivalent to that of Hall MHD, but the Hall parameter now becomes the dust parameter $\lambda_D = \Lambda_d/L$, where Λ_d serves as the dust skin depth. In general, it is well known that the dust skin depth is much greater than its (fully) ionic counterpart, and this serves to increase λ_H and drive down the ambient velocity, as the latter is determined through $\mathbf{v}_d^{(0)} = \mathbf{v}^{(0)} - \lambda_D \nabla \times \mathbf{b}^{(0)}$. As a result, it is possible to achieve $\mu \gg 1$ in (19), establishing preeminence of the thermal-driven dynamo term. In summary, we conclude that neutrals and dust, through different mechanisms, could reduce the ambient velocity ensuring that the required condition $\mu \gg 1$, considered in (19), is more easily achievable.

Lastly, we emphasize that the dual vorticity structure of (A1) is near-ubiquitous, which allows us to highlight the kinship of a vast array of models with the vorticity equation of ideal hydrodynamics. These models include relativistic plasmas,[42] quantum plasmas with spin-1/2 effects,[98,99] multi-species,[97,100] and extended MHD[101,102] plasmas. Owing to this commonality, we believe that most of the qualitative results obtained in context of the thermal-driven dynamo are also equally applicable to these complex models.

APPENDIX B: A DIMENSIONAL ANALYSIS APPROACH TO THERMAL DRIVES

First, let us begin by supposing that we consider an incompressible magnetofluid system where the magnetic field $\mathbf{B}$, the velocity (vorticity) $\mathbf{V}(\boldsymbol{\omega})$, and the temperature T are the sole variables. In order to construct coefficients involving T and a different variable, we are required to construct scalar quantities that are quadratic in the fields. The first possibility is to make it a product of two scalars, such as $T(\nabla \cdot \mathbf{B})$ which vanishes. The other is to make it the scalar product of two vectors. One vector that arises immediately is ∇T and the other can be chosen to be $\mathbf{B}$ or $\boldsymbol{\omega}$. Thus, we would end up with $\nabla T \cdot \mathbf{B}$ or $\nabla T \cdot \boldsymbol{\omega}$, and these are the precise thermal drives identified in the paper. With the addition of compressibility, and simultaneously treating ρ as an independent dynamical variable, one could formulate a greater variety of thermal and density drives.

Next, we shall consider a different approach where we are given minimal information about the exact closure scheme, and instead, we shall construct scaling relations based on dimensional analysis. Our starting point is the principle of dimensional homogeneity,[103,104] which requires every dimensionless number of import to be a function of other dimensionless numbers. For a given model, the time and length scales are the chief factors of interest. One can replace the time by the velocity scale; for e.g., the sound crossing time could be formulated from the sound speed c_s and the length scale. Thus, we shall focus on length and velocity scales, especially the latter.

Any dimensionless ratio of the electromotive force coefficients, denoted by Δ, must involve the large scale (mean) and the small scale (ambient) fields. In terms of the velocities, the natural dimensionless numbers are the Mach number $\mathcal{M} = \mathcal{V}/c_s$ and the Alfvén Mach number $\mathcal{M}_A = \mathcal{V}/\mathcal{V}_B$. These numbers are equally applicable for the mean and ambient fields; we denote the former with an overbar to distinguish them from the latter. Thus, we may expect, by the principle of dimensional homogeneity, that $\Delta = \mathcal{F}(\mathcal{M}, \mathcal{M}_A, \bar{\mathcal{M}}, \bar{\mathcal{M}}_A)$ holds true, where $\mathcal{F}$ is an arbitrary function.

A crucial aspect of dimensional analysis is to identify what are the parameters of interest, both in the input and the output. Suppose that we wish to evaluate the ratio of two electromotive force contributions, where the mean fields comprise only of the vorticity and the magnetic field (and *not* the temperature). In such an instance, we would not expect any dependence on $\bar{\mathcal{M}}$. In other words, we would have $\Delta = \mathcal{F}(\mathcal{M}, \mathcal{M}_A, \bar{\mathcal{M}}_A)$ for this *specific* case. Next, let us suppose that we introduce the length scales. We shall make the (grossly) simplifying assumption that all ambient fields have the same characteristic length scale, and that all the mean fields exhibit a similar property. In such an instance, one could introduce a factor $\zeta = \ell/L$, where ℓ is the ambient length scale and L is the mean field length scale. Of course, in most real-world situations, we cannot expect the separation between these two length scales to be clearly resolvable or defined.

With the above set of assumptions, the ratio of any two electromotive force coefficients is restricted to the following form:

$$\Delta = \mathcal{F}(\mathcal{M}, \mathcal{M}_A, \zeta, \bar{\mathcal{M}}_A). \tag{B1}$$

Moreover, we are free to introduce the scaling hypothesis[105] which leads to a multivariable power law function, i.e., we end up with $\Delta = \mathcal{M}^{p_1} \mathcal{M}_A^{p_2} \zeta^{p_3} \bar{\mathcal{M}}_A^{p_4}$, where $p_i \in \mathbb{R}$. Upon comparing this with the dimensionless ratios (12), (18), (20), and (19) and the concomitant discussions, we find that dimensional analysis yields results that are fully consistent with our mean field model. Of course, a chief deficiency of the former approach is that we cannot determine the power laws explicitly, but it does serve as a consistency check to verify that the dimensionless ratios computed herein have incorporated all of the relevant parameters. If one wishes to obtain values of the p's, they must be determined either by fitting against numerical simulations, or by imposing additional physical hypotheses, since dimensional analysis cannot, by itself, take us any further.

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