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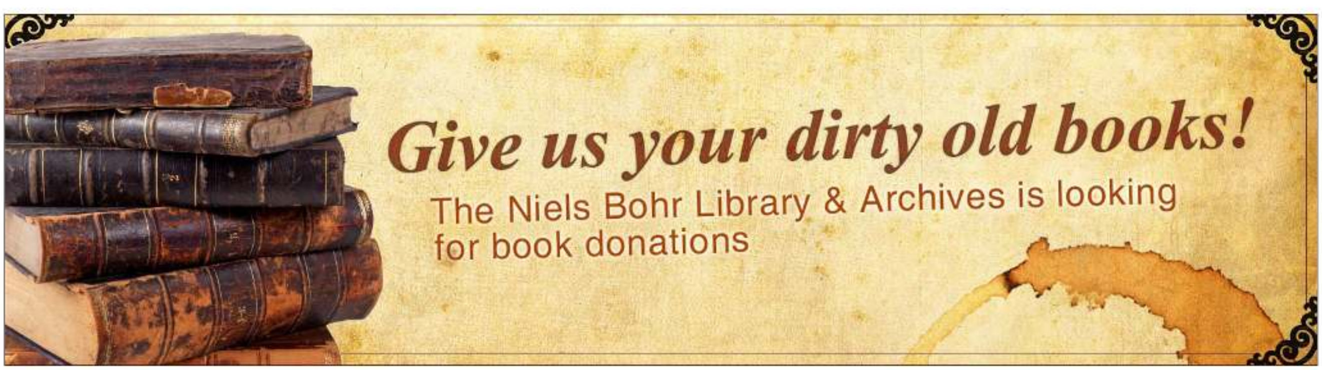

Traveling-wave solutions and the coupled Korteweg–de Vries equation

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Some coupled nonlinear equations are considered for studying traveling-wave solutions. By introducing a stream function Ψ it is shown that if one of the solutions is of the form $v\equiv v(x-ct)$, the other also must be of the form $u\equiv u(x-ct)$. In addition, the possibility of including cubic nonlinear terms has been considered and such a system, assuming that the solutions are of the traveling-wave type, has been solved.

I. INTRODUCTION

Some time ago, Ito[1] had proposed the following coupled nonlinear wave equations:

$$u_t = u_{xxx} + 6uu_x + 2vv_x, \tag{1a}$$

$$v_t = 2(uv)_x. \tag{1b}$$

An interesting characteristic of this couple is that it reduces to the familiar Korteweg–de Vries equation when $v=0$. Moreover the symmetries associated with it generate a hierarchy of coupled equations each of which is a Hamiltonian system with infinitely many constants of motion.

Recently, Kawamoto[2] has shown that of all the particular solutions obtainable from (1), the traveling-wave solutions of the type

$$u\equiv u(x-ct), \quad v\equiv v(x-ct) \tag{2}$$

(c being a constant) are necessarily cusplike in nature. It may be noted that Kawamoto had considered a more general version of (1), namely,

$$u_t + \alpha vv_x + \beta uu_x + \delta u_{xxx} = 0, \tag{3a}$$

$$v_t + \gamma(uv)_x = 0, \tag{3b}$$

where the parameters α, β, δ, and γ were kept arbitrary. The purpose of this work is twofold.

(i) First, we show that if one of the solutions of (3) is of the traveling-wave form, say $v\equiv v(x-ct)$, then the other solution also must exhibit the same form, i.e., u must also be of the form $u\equiv u(x-ct)$.

(ii) Second, even if one introduces cubic nonlinearity in (3) and modifies (3a) to make it assume the form

$$u_t + \alpha(v^3)_x + \beta(u^3)_x + \lambda(u^2)_x + \delta u_{xxx} = 0, \tag{4a}$$

$$v_t + \gamma(uv)_x = 0, \tag{4b}$$

the conclusion in (i) remains unchanged. For completeness, we also have solved (4) postulating that the functions u and v are of the traveling-wave type (2).

II. TRAVELING-WAVE SOLUTIONS

We begin by writing (3a) in the form

$$u_t + ((\beta/2)u^2 + (\alpha/2)v^2 + \delta u_{xx})_x = 0.$$

This enables us to introduce an arbitrary function $\Psi = \Psi(x,t)$ [which may be called the stream function of the system (3)] such that

$$u\,dx - \left(\frac{\beta}{2}u^2 + \frac{\alpha}{2}v^2 + \delta u_{xx}\right)dt = d\Psi = \frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial t}dt.$$

On comparison, one finds

$$\frac{\partial\Psi}{\partial x} = u, \tag{5a}$$

and

$$-\frac{\partial\Psi}{\partial t} = \frac{\beta}{2}u^2 + \frac{\alpha}{2}v^2 + \delta u_{xx}. \tag{5b}$$

Substituting (5a) into (5b), v can be expressed as, let us say,

$$v^2 = -(2/\alpha)(\Psi_t + (\beta/2)\Psi_x^2 + \delta\Psi_{xxx}) \equiv (2/\alpha)\Phi(x,t). \tag{6}$$

One of the advantages of introducing a stream function is that the two coupled equations in (3) may be combined to yield a single relation in Ψ. Solving then for Ψ, one can immediately obtain u and v through the connection (5). Combining (3b), (5a), and (6) we get

$$\Phi_t + \gamma\Psi_x\Phi_x + 2\gamma\Psi_{xx}\Phi = 0, \tag{7}$$

where Φ may be represented in terms of Ψ using the definition in (6).

Let us assume that v is a function of $(x-ct)$ only. Then it may be asserted that

$$c\Phi_x + \Phi_t = 0.$$

Replacing Φ_t by Φ_x, we have from (7)

$$(c - \gamma\Psi_x)\Phi_x = 2\gamma\Psi_{xx}\Phi. \tag{8}$$

On integrating (8), Φ may be obtained in a closed form

$$\Phi = (K^2(t)/(\gamma\Psi_x - c)^2),$$

where $K(t)$ is an arbitrary function of time.

Since Φ is a function of $(x-ct)$ only, we can write without any loss of generality

$$(\gamma\Psi_x - c)^2 = K^2(t)f^2(x-ct),$$

where $f(x-ct)$ is another function of $(x-ct)$. Thus

$$\Psi_x = (1/\gamma)[c + K(t)f(x-ct)]. \tag{9}$$

In order to show that Ψ_x (and hence u) is a function of $(x-ct)$ only, we need to prove that $K(t)$ must necessarily be a constant, i.e., $K(t)$ must be independent of t.

To this end, we substitute (9) in (6) to obtain

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$$\Phi(x-ct) = -\Psi_t - (\delta/\gamma)K(t)f''(x-ct) - (\beta/2\gamma^2)[c+K(t)f(x-ct)]^2,$$

where dashes denote derivatives (partial) with respect to (w.r.t.) x only. Expanding and rearranging the right-hand side (rhs) of the above expression, one can express Ψ_t in the form

$$\Psi_t = -\Phi(x-ct) - K(t)A(x-ct) - K^2(t)B(x-ct) - (\beta c^2/2\gamma^2). \quad (10)$$

It may be noted that Ψ_t can be obtained also from (9), first, by integrating (partially) (9) w.r.t. x and then differentiating (partially) the result w.r.t. t. In this way one arrives at

$$\Psi = \frac{c}{\gamma}x + \frac{K(t)}{\gamma}g(x-ct) + \frac{\lambda(t)}{\gamma},$$

where $\lambda(t)$ is an arbitrary function of time and $g(x-ct)$ stands[3] for the quantity $\int f(x-ct)dx$. Consequently,

$$\Psi_t = \frac{\dot{K}(t)}{\gamma}g(x-ct) - \frac{cK(t)}{\gamma}h(x-ct) + \frac{\dot{\lambda}(t)}{\gamma}, \quad (11)$$

where dots represent derivatives (partial) w.r.t. t and $h(x-ct) = g_t(x-ct)$, assuming that $g_t(x-ct)$ is not a constant.

On comparison of (10) and (11), we have therefore,

$$\Phi(x-ct) = -K(t)A(x-ct) - K^2(t)B(x-ct) - \frac{\dot{K}(t)}{\gamma}g(x-ct) + \frac{cK(t)}{\gamma}h(x-ct) - \frac{\beta c^2}{2\gamma^2} - \frac{\dot{\lambda}(t)}{\gamma}. \quad (12)$$

Since the lhs of (12) is a function of $(x-ct)$ alone while the rhs is a product of functions of time and $(x-ct)$, it follows that $K(t)$ and $\dot{\lambda}(t)$ must reduce to a constant value. Accordingly, Ψ_x (and therefore u) must be a function of $(x-ct)$ only.

III. INCLUSION OF CUBIC NONLINEAR TERMS

We now turn to Eqs. (4a) and (4b). Here instead of (10) and (11) we have

$$\Psi_t = -\Phi(x-ct) - K(t)A(x-ct) - K^2(t)B(x-ct) - K^3(t)D(x-ct) - \frac{c^2}{\gamma^2}\left(\frac{c\beta}{\gamma} + \lambda\right) \quad (10')$$

and

$$\Psi_t = \frac{\dot{K}(t)}{\gamma}g(x-ct) - \frac{cK(t)}{\gamma}h(x-ct) + \frac{\dot{\lambda}(t)}{\gamma}. \quad (11')$$

Comparing (10′) and (11′), we can write

$$\Phi(x-ct) = -K(t)A(x-ct) - K^2(t)B(x-ct) - K^3(t)D(x-ct) - \frac{\dot{K}(t)}{\gamma}g(x-ct) + \frac{cK(t)}{\gamma}h(x-ct) - \frac{c^2}{\gamma^2}\left(\frac{c\beta}{\gamma} + \lambda\right) - \frac{\dot{\lambda}(t)}{\gamma}. \quad (12')$$

Applying the arguments similar to those given before, we can claim that here too u must be a function of $(x-ct)$ only.

IV. TRAVELING-WAVE SOLUTIONS IN THE PRESENCE OF CUBIC NONLINEAR TERMS

In order to extract traveling-wave solutions of (4), let us write

$$u = u(\rho), \quad v = v(\rho),$$

where

$$\rho = (x-t).$$

Then (4a) and (4b) become

$$-u_\rho + \alpha(v^3)_\rho + \beta(u^3)_\rho + \lambda(u^2)_\rho + \delta u_{\rho\rho\rho} = 0 \quad (13a)$$

and

$$-v_\rho + \gamma(uv)_\rho = 0. \quad (13b)$$

Integrating (13b), one obtains

$$v = c_1/(\gamma u - 1), \quad (14)$$

where c_1 is an arbitrary constant of integration. Substituting (14) in (13a) and integrating twice w.r.t. ρ, it is easy to obtain

$$\frac{\delta}{2}(u_\rho)^2 = c_3 + c_2 u + \frac{1}{2}u^2 - \frac{\lambda}{3}u^3 - \frac{\beta}{4}u^4 + \frac{1}{2}\frac{\alpha c_1^3}{\gamma}\frac{1}{(\gamma u - 1)^2}, \quad (15)$$

where c_2 and c_3 are again constants of integration. The above Eq. (15) can be expressed in a compact form as

$$\frac{z\,dz}{\sqrt{az^6 + bz^5 + cz^4 + dz^3 + ez^2 + g}} = \frac{1}{\sqrt{2}}d\rho, \quad (16)$$

where the parameters a, b, c, d, e, and g are given by

$$a = -\frac{\beta}{\delta\gamma^2},$$

$$b = -4\gamma^2\left(\frac{\beta}{\delta\gamma^4} + \frac{\lambda}{3\delta\gamma^3}\right),$$

$$c = -4\gamma^2\left(\frac{\lambda}{\delta\gamma^3} - \frac{1}{2\delta\gamma^2} + \frac{3\beta}{2\delta\gamma^4}\right),$$

$$d = -4\gamma^2\left(\frac{\beta}{\delta\gamma^4} - \frac{c_2}{\delta\gamma} - \frac{1}{\delta\gamma^2} + \frac{\lambda}{\delta\gamma^3}\right),$$

$$e = -4\gamma^2\left(\frac{\beta}{4\delta\gamma^4} + \frac{\lambda}{3\delta\gamma^3} - \frac{c_3}{\delta} - \frac{c_2}{\delta\gamma} - \frac{1}{2\delta\gamma^2}\right),$$

$$g = \frac{2\alpha c_1^3\gamma}{\delta}.$$

Knowing the precise values of the parameters, (16) may be either integrated in a closed form or evaluated numerically.

In the following, we consider the particular case when the parameters b and d vanish.[4] Equation (16) then reduces to a convenient form

$$\frac{1}{2}\left(\frac{d\tau}{d\rho}\right)^2 = a\tau^3 + c\tau^2 + e\tau + g \equiv f(\tau) \quad (17)$$

or

$$\int \frac{d\tau}{\sqrt{a\tau^3 + c\tau^2 + e\tau + g}} = \sqrt{2}\,\rho + c_4, \qquad (17')$$

where c_4 is a constant of integration.

V. NATURE OF THE SOLUTIONS

Without going into the nature of the parameters a, c, e, and g, one can make some qualitative remarks,[5] about the possible solutions of (17).

We first of all note that a real bounded solution τ is permitted when $(d\tau/d\rho)^2 \geqslant 0$. Moreover, at the vanishing points of $f(\tau)$, either $d\tau/d\rho$ should change sign or tend to zero as $\rho \to \pm\infty$. To examine this, let τ_1 be a simple zero of $f(\tau)$. Then

$$\left(\frac{d\tau}{d\rho}\right)^2 = 2f'(\tau_1)(\tau - \tau_1) + \text{higher-order terms}.$$

On integration, the above yields

$$\tau = \tau_1 + \tfrac{1}{2} f'(\tau_1)(\rho - \rho_1)^2 + \text{higher-order terms},$$

where $\rho = \rho_1$ when $\tau = \tau_1$. Thus τ has a simple minimum or maximum τ_1 at ρ_1, according as $df/d\rho$ at τ_1 is positive or negative, respectively.

If, however, τ_1 is a double zero of $f(\tau)$, then

$$\left(\frac{d\tau}{d\rho}\right)^2 = f''(\tau_1)(\tau - \tau_1)^2 + \text{higher-order terms}.$$

Obviously, the validity of such a zero is possible only when $f''(\tau_1) > 0$.

Moreover, it also follows that

$$\tau - \tau_1 \sim \text{const} \times \exp[\pm\sqrt{f''(\tau_1)}\,\rho]$$

as $\rho \to \mp\infty$ in order that τ be bounded. The crucial point to note is that τ attains the maximum value τ_1 exponentially over an infinite range ρ. This may be recognized as the typical case of a solitary wave.

In addition to the solitary waves, it is possible to find the cnoidal wave also. The solution may be expressed in terms of three distinct real zeros of $f(\tau)$, namely, τ_1, τ_2, and τ_3, as

$$\rho = \rho_3 + \int_{\tau_3}^{\tau} \frac{d\tau}{\pm\sqrt{2f(\tau)}}$$

$$= \rho_3 + \int_{\tau_3}^{\tau} \frac{d\tau}{\pm\sqrt{2(\tau - \tau_1)(\tau - \tau_2)(\tau - \tau_3)}},$$

where τ_3 is a simple minimum of $f(\tau)$ and $\tau_3 < \tau_2 < \tau_1$. Therefore,

$$\tau = \tau_2 - (\tau_2 - \tau_3)\,\text{cn}^2[\sqrt{\tfrac{1}{2}(\tau_1 - \tau_3)}\,(\rho - \rho_3)\,|\,m],$$

where the parameter "m" of the Jacobian elliptic function cn is given by

$$m = (\tau_2 - \tau_3)/(\tau_1 - \tau_3).$$

Noting that the period of cn for $0 \leqslant m < 1$ is given by $4\overline{K}(m)$, where

$$\overline{K}(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - m\sin^2\theta)}},$$

one may define the period of τ as

$$2\overline{K}(m)\sqrt{2/(\tau_1 - \tau_3)}\,.$$

Now if the wavelength of a nonlinear cnoidal wave be represented by $2\pi/K$, then we must have

$$K = \pi\sqrt{\tau_1 - \tau_3}/\sqrt{2} \cdot \overline{K}(m),$$

which yields the frequency (ω) of the cnoidal wave as

$$\omega = \pi c\sqrt{\tau_1 - \tau_3}/\sqrt{2} \cdot \overline{K}(m).$$

VI. LIMITING CASES OF CNOIDAL WAVES

We next consider the two important limiting cases of cnoidal waves.

Case 1: $m \to 0$. When $m \to 0$, $\tau_3 \to \tau_2$ and $c \to [-a(\tau_1 + 2\tau_2)]$. Thus

$$\tau(\rho) \to \tau_2 - (\tau_2 - \tau_3)\cos^2[\sqrt{\tfrac{1}{2}(\tau_1 - \tau_2)}\,(\rho - \rho_3)]$$

$$\to \tau_2 - A\cos[K(\rho - \rho_3)],$$

where

$$A = \tfrac{1}{2}(\tau_2 - \tau_3) \quad \text{and} \quad K = \sqrt{2(\tau_1 - \tau_2)}\,,$$

which is the solution for infinitesimal waves.

Case 2: $m \to 1$. When $m \to 1$, $\tau_2 \to \tau_1$ and $c \to [-a(2\tau_1 + \tau_3)]$. As a result

$$\tau(\rho) \to \tau_1 - (\tau_1 - \tau_3)\,\text{sech}^2[\sqrt{\tfrac{1}{2}(\tau_1 - \tau_3)}\,(\rho - \rho_3)],$$

which is similar to the Boussinesq–Rayleigh solution for the solitary wave.

VII. CONCLUDING REMARKS

In this paper we have considered some coupled nonlinear equations that involve two variables $u(x,t)$ and $v(x,t)$. We have found that if one of these is a function of $(x - ct)$, the other must exhibit the same dependence of variables. In addition, we have generalized the works of Ito and Kawamoto to include cubic nonlinearity in one of the basic equations. Assuming that the functions $u(x,t)$ and $v(x,t)$ are of the traveling-wave type, we have solved such a system and considered some interesting particular cases. For instance, inclusion of nonlinear cubic terms in the equation has been found to lead to the possibility of cnoidal waves. Some limiting cases of such waves also have been considered.

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[1]M. Ito, Phys. Lett. A **91**, 335 (1982).

[2]S. Kawamoto, J. Phys. Soc. **53**, 1203 (1984).

[3]It may be noted that the integration has been done treating t as a constant. Any arbitrary function of time that results from the constants of integration has been lumped with $\lambda(t)$.

[4]For "b" to vanish, the required condition is $\beta = -\frac{1}{2}\lambda\gamma$ whereas for "d" to vanish the condition is $\beta = \gamma(\gamma^2 c_2 + \gamma - \lambda)$. Thus, both "$b$" and "$d$" will vanish simultaneously if $\beta = -\frac{1}{2}\lambda\gamma$ and $c_2 = (2\lambda - 3\gamma)/3\gamma^2$.

[5]P. G. Drazin, *Solitons* (Cambridge U. P., London, 1983).