The relativistic electro-vortical field—revisiting magneto-genesis and allied problems

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The relativistic electro-vortical field—revisiting magneto-genesis and allied problems

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Following the idea of MagnetoFluid unification [S. M. Mahajan, Phys. Rev. Lett. **90**, 035001 (2003)], a very general Electro-Vortical (EV) field is constructed to describe the dynamics of a perfect relativistic fluid. Structurally similar to the electromagnetic field $F^{\mu\nu}$, the Electro-Vortical field $\mathcal{M}^{\nu\mu}$ unifies the macroscopic forces into a single grand force that is the weighted sum of the electromagnetic and the inertial/thermal forces. The new effective force may be viewed either as a vortico-thermal generalization of the electromagnetic force or as the electromagnetic generalization of the vortico-thermal forces that a fluid element experiences in course of its evolution. Two fundamental consequences follow from this grand unification: (1) emergences of a new helicity that is conserved for arbitrary thermodynamics and (2) the entire dynamics is formally expressible as an MHD (magnetohydrodynamics) like ideal Ohm's law in which the "electric" and "magnetic" components of the EV field replace the standard electric and magnetic fields. In the light of these more and more encompassing conserved helicities, the "scope and significance" of the classical problem of magneto-genesis (need for a seed field to get a dynamo started) is reexamined. It is shown that in models more advanced than MHD, looking for exotic seed-generation mechanisms (like the baroclinic thermodynamics) should not constitute a fundamental pursuit; the totally ideal dynamics is perfectly capable of generating and sustaining magnetic fields entirely within its own devices. For a specified thermodynamics, a variety of exact and semi exact self-consistent analytical solutions for equilibrium magnetic and flow fields are derived for a single species charged fluid. The scale lengths of the fields are determined by two natural scale lengths: the skin depth and the gradient length of the thermodynamic quantities. Generally, the skin depth, being the shorter (even much shorter) than the gradient length, will characterize the kinetic-magnetic reservoir of short scale energy that will drive the dynamo as well as reverse dynamo action—the creation of large scale magnetic and flow fields. *Published by AIP Publishing*. [http://dx.doi.org/10.1063/1.4967269]

I. INTRODUCTION

This work, and consequently, this paper has been motivated by several issues deeply connected with the formulation as well as applications of the physics of relativistic (thermally and kinematically) charged fluids.[2] From the constructed formalism, the dynamics of equally important non-relativistic (NR) and charge less (neutral) systems will, naturally, follow in the appropriate limit.

I will begin with a somewhat detailed review of two of the more fundamental problem that this paper will address:

A. Vorticity-magnetic fields and helicity

The notions of vorticity and helicity (in fluid mechanics, the word helicity is believed to have been introduced by Moffatt[3]) have played a key role in advancing the development of fluid dynamics. The recognition of their topological properties, and the fact that many simple plasma (charged fluid) models, such as magnetohydrodynamics (MHD),[4] Hall MHD (HMHD)[5,6] and extended MHD (XMHD),[7] and even quantum plasmas[8,9] have similar mathematical structures, which vastly extended the scope and utility of these concepts. In fact, the exploitation of the central topological property of these relative simple systems—the conservation of helicity in ideal dynamics (non-dissipative limit)—has led to the rather powerful ideas of relaxation and self-organization in fusion and astrophysical plasmas.[10–22] Theoretically, the central implication of similar mathematical structures of these simple models (neutral fluids and minimum charged particle models (MHD, HMHD)) is that, perhaps, the dynamics of more complicated/larger physical systems may also display similar structure in terms of suitably defined composite dynamical fields that may have just as desirable attributes as helicity conservation. Such composite fields will, naturally, "unify" the fluid and the electromagnetic forces.

In a non-relativistic fluid model (keeping the particle inertia), exploring relaxed states more general than the standard Woltjer-Taylor state[10,11,22] ($\nabla \times \mathbf{B} = \alpha \boldsymbol{B}$, $\alpha =$ constant), a composite vorticity

$$\boldsymbol{\Omega}_c = q\mathbf{B} + m\nabla \times V \qquad (1)$$

is defined. Comprising the fluid vorticity and the magnetic field, $\boldsymbol{\Omega}_c = \nabla \times \mathbf{P}_c$ ($\mathbf{P}_c = q\boldsymbol{A} + m\mathbf{V}$), is nothing but the canonical vorticity, the curl of the canonical momentum $\mathbf{P}_c$. What is important is the fact that the pure magnetic helicity

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$$h_m = \int \boldsymbol{A} \cdot \boldsymbol{B}\, d^3x, \tag{2}$$

a constant of motion in MHD, is no longer an invariant: it is replaced by a new invariant, the canonical helicity

$$h_c = \int \mathbf{P}_c \cdot \boldsymbol{\Omega}_c\, d^3x, \tag{3}$$

constructed from $\boldsymbol{\Omega}_c$. In defining $\boldsymbol{\Omega}_c$, the "vortical" and the electromagnetic fields were accorded equal status suggesting that the magnetic field may be identified as the Vorticity of the electromagnetic field. The equivalence, of course, runs both ways.

This "unifying" approach, when extended to the much more complex dynamics of a relativistic hot charged fluid, allows a very elegant formalism from which results of immense generality follow. A stand out result is that for an ideal fluid, *under well defined conditions*, we always find a suitably generalized vorticity $\boldsymbol{\omega}$ (the curl of a generalized momentum $\boldsymbol{\mathcal{P}}$) such that there does exist a relativistic helicity invariant $H_g = \int \boldsymbol{\mathcal{P}} \cdot \boldsymbol{\omega}\, d^3x$ to constrain the dynamics; "Relaxed" and Self-organized states, thus, may be accessible under such general conditions.

The big conceptual advancement, however, is underscored by the emergence of a unified tensor which I will rename as the Electro-Vortic (EV) field tensor. The basic steps in the construction of the EV field are best illustrated by a reformulation of the dynamics of a perfect isotropic fluid described by the energy momentum tensor (EMoM)[23,24]

$$T^{\mu\nu} = p\eta^{\mu\nu} + hU^\mu U^\nu, \tag{4}$$

where p, the pressure, and h, the enthalpy density, are the two thermodynamic attributes, and $U^\mu = \{U^0 = \gamma, \boldsymbol{U} = \gamma \boldsymbol{V}\}$ is the four-velocity (c = 1) of the fluid, $\boldsymbol{V}$ is the ordinary velocity, and $\gamma = (1+U^2)^2 = (1-V^2)^{-1/2}$ is the relativistic factor. In this paper, Greek indices run from 0 to 3 while the Latin indices (1–3) denote the spatial part. The contravariant ($x^\mu = [x^0, x^1, x^2, x^3]$) and covariant ($x_\mu = [x_0, x_1, x_2, x_3]$) components of a four vector are related as $x_0 = -x^0, x_i = x^i$, implying that the Minkowsky signature tensor is taken to be $\eta^{\mu\nu} = diag[-1, 1, 1, 1]$. The four velocity normalization, then, is $U^\mu U_\mu = -1$.

The dynamics of this perfect charged fluid, interacting with the EM field, is given by[1,25,26]

$$\partial_\mu T^{\mu\nu} = qnF^{\mu\nu} U_\nu, \tag{5}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and $n \equiv n_{rest}$ is the rest frame density of the fluid. It was demonstrated in Ref. 1 that Eq. (5) can be manipulated and cast into the revealing covariant form

$$U_\mu M^{\nu\mu} = -T\partial^\nu \sigma, \tag{6}$$

where $\sigma(T)$ is the entropy (temperature) of the fluid, and

$$M^{\mu\nu} = mS^{\mu\nu} + qF^{\mu\nu} = \partial^\mu \mathcal{P}^\nu - \partial^\nu \mathcal{P}^\mu, \tag{7}$$

is the joint Electro-Vortic field tensor. Four curls of the effective four momentum

$$\mathcal{P}^\nu = qA^\nu + fmU^\nu, \tag{8}$$

where $M^{\mu\nu}$ represents a weighted sum of the electromagnetic field $F^{\mu\nu}$ (with charge q as the weight)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{9}$$

and the inertial-thermal field $S^{\mu\nu}$ (the mass m is the weight)

$$S^{\mu\nu} = \partial^\mu (fU^\nu) - \partial^\nu (fU^\mu) \tag{10}$$

in which mfU^μ is the thermally enhanced 4-momentum of the fluid element; the enhancement factor f ($h = mnf$) is the enthalpy per unit mass.

Barring the purely thermodynamic force on the right hand side of (6), all the fluid and electromagnetic forces have combined to create the EV field which can be viewed as a weighted union of the two. This unification, of course, was crucially dependent on the fact that the underlying fluid dynamics allowed the construction of a fully antisymmetric second rate tensor $S^{\mu\nu}$ (a two form in the language of differential geometry) that contains a full description of fluid forces (inertial and thermal) just as the Faraday tensor $F^{\mu\nu}$ is the repository of all electromagnetic forces.

The Electro-Vortic field, emerging from the structural similarity of its components, is a major construct that can help us transport some of the known results, say, from hydrodynamics (fluid forces only) or from electromagnetism, into the larger dynamics. It must be clearly stated that though the construction of the EV field is a major advance in the unification of the two macroscopic forces, it is, by no means, a complete unification because the Maxwell's equations

$$\partial^\mu F^{\mu\nu} = 4\pi nqU^\nu \tag{11}$$

are not directly cognizant of $S^{\mu\nu}$; they are driven entirely by the current nqU^ν.

In Sec. II of this paper, I will go back to (6) and make attempts to propel the unification program further by investigating conditions under which the thermodynamic force can be absorbed in a more inclusive/advanced definition of the effective momentum, i.e., seek a new Π^μ such that the new Electro-Vortic tensor

$$\mathcal{M}^{\nu\nu} = \partial^\nu \Pi^\mu - \partial^\mu \Pi^\nu \tag{12}$$

obeys

$$U_\mu \mathcal{M}^{\nu\mu} = 0, \tag{13}$$

and is a fuller expression of the Electro-Vortic unification. The derivation will be followed by a working out of the consequences of (12) and comparing it with what we had learnt from analyzing Eq. (6).

It is, now, time to justify the new nomenclature—calling $\mathcal{M}^{\nu\nu}$ the Electro-Vortic Field. Interpreting this composite or the unified field as a generalization of the EM field $F^{\mu\nu}$, it is essential that we define new composite or effective electric ($\boldsymbol{\mathcal{E}}$) and magnetic fields ($\boldsymbol{\mathcal{B}}$)

$$\mathcal{M}^{0i} = \partial^0 \Pi^i - \partial^i \Pi^0 = \mathcal{E}^i, \quad \boldsymbol{\mathcal{E}} = -\partial_t \boldsymbol{\Pi} - \nabla \Pi^0, \tag{14}$$

$$\mathcal{M}^{ij} = \partial^i \Pi^j - \partial^j \Pi^i = \epsilon^{ijk} \mathcal{B}_k, \quad \boldsymbol{\mathcal{B}} = \nabla \times \boldsymbol{\Pi}. \tag{15}$$

Thus, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ are the electric and magnetic parts of the larger field—the Electro-Vortic Field. It is, then, legitimate to think of the ordinary magnetic field as the vorticity of the EM field. Equally legitimate is to view the vorticity as the "magnetic" equivalent of the fluid field. The picture will be incomplete till we realize that Eq. (14) imposes a similar equivalence on the "electric fields" of fluid and of EM origin.

B. Essential of the magneto genesis (MG)/seed field problem

Deeply connected with the formulation of fluid models (elementary and advanced) and the notions of vortcities and helicities is the so called problem of Magneto-Genesis (MG).

Looking for a "common" mechanism to explain the ubiquity of macroscopic magnetic fields, observed in laboratory experiments and in all varieties of astrophysical settings, has been one of the most challenging and engaging problems in theoretical physics. Spread over a large group of investigators, this effort has spawned a set of theories and simulations that are generically (and appropriately) called Dynamo Theories (DT)/Mechanisms (for a comprehensive review, see Ref. 27. The goal of the Dynamo effort is, *inter alia*, to understand the creation of magnetic fields found at scales that span the planetary,[28] the stellar,[29] and the cosmic,[30,31] and in systems ranging from the dilute intergalactic medium (IGM) to the exceptionally dense neutron stars.[32] Starting with the pioneering work of Ref. 33, explained and elaborated in Refs. 30, 34, and 35, the dynamo research has been both intense and innovative.[31,36–52]

Complementary to the dynamo theories are the reverse dynamo or unified dynamo theories,[51,52] which demonstrate that large scale flows (outflows) and magnetic fields are generated, simultaneously, in most models. The outflows show up quite prominently in several astrophysical systems.[53]

To understand much of the language and many of the concepts in Dynamo Theories (DT), one must begin with the earliest nonrelativistic MHD formulation contained in the induction equation (and its uncurled counterpart),

$$\partial_t \boldsymbol{A} - \boldsymbol{V} \times \boldsymbol{B} = -\frac{\nabla p}{n} - q\nabla\phi, \tag{16}$$

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = -\nabla \times \frac{\nabla p}{n} = \frac{\nabla n \times \nabla p}{n^2}, \tag{17}$$

where $\boldsymbol{A}(\boldsymbol{B})$ is the vector potential (the magnetic field)—all other quantities have been defined earlier. If the plasma were barotropic, i.e., $p = p(n)$, then the right hand side of (16) would become a perfect derivative and that of (17) would go to zero. The left over (17)

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) \tag{18}$$

has a seemingly trivial but most remarkable property: if $\boldsymbol{B} = 0$ at any point of time, it must always remain zero; Thus, a finite $\boldsymbol{B}$ state is not accessible (through MHD) from a state of zero magnetic field. Notice that the magnetic field can be amplified from the velocity field via (17) but only if we begin with a finite field, however, small. Thus, the dynamo action (that can take place through the second term in (17)) is possible if and only if there was a *seed* magnetic field, generated, by some additional source. This, in short, is the genesis of the Magneto-Genesis/Seed Field Problem and also of the entire industry searching for Seed-Fields.

One arrives at the same conclusion at a somewhat deeper level by noting that the MHD system (16) and (17) is endowed with a conserved quantity; the already defined magnetic helicity,

$$h_m = \int \boldsymbol{A} \cdot \boldsymbol{B}\, d^3x, \quad \frac{dh_m}{dt} = 0. \tag{19}$$

If zero initially, h_m will be zero for all times. This understanding will stand us in good steed as we deal with more and more complex dynamics.

Naturally, the very first solution had to be the jettisoning of the barotropic assumption—If the fluid had a baroclinic thermodynamics $(\nabla n \times \nabla p \neq 0)$,[54–56] then even at $\boldsymbol{B} = 0$, $\partial_t \boldsymbol{B} \neq 0$, and thus the field can, initially, grow through the baroclinic mechanism. Eventually, at sufficient magnitude, the "dynamo term" will kick in; in fact it could be dominant.

Over the years, some other seed formation mechanisms have sprung up, though the investigation of the baroclinic mechanism has been incessant.

Let us go past MHD and reexamine the question of magnetic field generation. In this subsection, we will explore the non relativistic dynamics of a charged fluid with a finite mass m,

$$\partial_t \hat{\boldsymbol{A}} - \boldsymbol{V} \times \hat{\boldsymbol{B}} = -\frac{\nabla p}{n} - \nabla\left(q\phi + mv^2/2\right), \tag{20}$$

$$\partial_t \hat{\boldsymbol{B}} - \nabla \times \left(\boldsymbol{V} \times \hat{\boldsymbol{B}}\right) = -\nabla \times \frac{\nabla p}{n} = \frac{\nabla n \times \nabla p}{n^2}, \tag{21}$$

where the generalized vector potential and the magnetic field

$$\hat{\boldsymbol{A}} = \boldsymbol{A} + m/q\boldsymbol{V},\ \hat{\boldsymbol{B}} = \boldsymbol{B} + m/q\nabla \times \boldsymbol{V} \tag{22}$$

are, respectively, proportional to the canonical momentum $\boldsymbol{P}_c$ and $\nabla \times \boldsymbol{P}_c$. In this model, the magnetic field $\boldsymbol{B}$ evolves as

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = \frac{\nabla n \times \nabla p}{n^2} + \boldsymbol{S}, \tag{23}$$

where

$$\boldsymbol{S} = -\frac{m}{q}\left[\partial_t(\nabla \times \boldsymbol{V}) - \nabla \times (\boldsymbol{V} \times (\nabla \times \boldsymbol{V}))\right] \tag{24}$$

is the source that will allow the emergence of $\boldsymbol{B}$ even from a barotropic state $(\nabla n \times \nabla p = 0)$ state with initial $\boldsymbol{B} = 0$. *The seed-generation problem, then, ceases to be a fundamental pursuit as one advances from MHD to models with more inclusive physics*. The baroclinic thermodynamics could still be important, even, dominant, but one can develop a proper (beyond MHD) dynamo starting from a zero magnetic field. In a way, the "seed" is naturally generated within the totally ideal dynamics (See Ref. 57 as an example).

However, what was true for the magnetic field $\boldsymbol{B}$ in the minimum MHD theory is, now, true for the generalized magnetic field $\hat{\boldsymbol{B}}$ (22) in the larger theory with finite particle inertia. Indeed, the evolution equation for $\boldsymbol{B}$ (17) in MHD is exactly the same as (21), the equation obeyed $\hat{\boldsymbol{B}}$. In the latter dynamics, without the Baroclinic term, one cannot create a state of finite $\hat{\boldsymbol{B}}$ starting from a state of zero $\hat{\boldsymbol{B}}$; this constraint is imposed by the constancy of h_c (that replaces h_m, the MHD invariant).

As one graduates to the more complex dynamics of a hot relativistic fluid, the expression for the Helicity invariant becomes more and more complex. But in all cases h_m is not conserved, and there is no constraint that forbids the evolution of the magnetic field for an ideal barotropic fluid. Thus, the conventional search for a baroclinic (or otherwise) seed is not necessary for Magneto-Genesis (of the ordinary magnetic field $\boldsymbol{B}$).

It will be shown in Section II that the most general Electro-Vortical field definable in this model will have an "absolutely conserved" helicity $\mathcal{H}$ (independent of thermodynamics); there will, then, be no source that could generate, starting from zero, a finite $\boldsymbol{\mathcal{B}}$, the associated grand "magnetic field."

The preceding discussion implies a paradigmatic shift in our understanding of the magneto genesis problem; this shift, hopefully, will guide us to refine our methodologies to explore the cosmic/astro magnetic fields.

In Sec. III, I will harness the formulation of Sec. II in order to solve, explicitly, several magnetic field configurations that a hot relativistic fluid can settle into. These representative analytical solutions are not only interesting by themselves but can serve as checks on more detailed numerical solutions. In Sec. IV, I will present a summary and discussion of the new results spanning the formalism as well as explicit solutions.

II. THE RELATIVISTIC ELECTRO-VORTIC FIELD

The primary aim of this section is to further extend the original magneto-fluid formalism of Ref. 1, epitomized in (6), to the construction of the EV field $\mathcal{M}^{\nu\mu}$ promised in Eqs. (12) and (13). The first step is to note that the relativistic perfect fluid of (6) is isentropic,

$$U_\nu \partial^\nu \sigma = 0, \tag{25}$$

the entropy is constant along the flow line. This property coupled with the fact $U^\mu U_\mu = -1$ allows us to write

$$\begin{aligned} -T\partial^\nu \sigma &= U^\mu U_\mu T \partial^\nu \sigma - (U_\mu \partial^\mu \sigma) T U^\nu \\ &= U_\mu [U^\mu T \partial^\nu \sigma - U^\nu T \partial^\mu \sigma]. \end{aligned} \tag{26}$$

For a restricted class of four velocity field (specified Clebsch form)

$$TU^\mu = \partial^\mu Q, \tag{27}$$

where Q is an appropriate potential function, this expression converts to

$$-T\partial^\nu \sigma = U_\mu [\partial^\nu (\sigma \partial^\mu Q) - \partial^\mu (\sigma \partial^\mu Q)], \tag{28}$$

which, when substituted in (6), leads precisely to the structure contained in (12) and (13)

$$\mathcal{M}^{\nu\nu} = \partial^\nu \Pi^\mu - \partial^\mu \Pi^\nu,$$
$$U_\mu \mathcal{M}^{\nu\mu} = 0$$

with the effective momentum

$$\Pi^\mu = qA^\mu + fmU^\mu - \sigma \partial^\mu Q = qA^\mu + m\left(f - \frac{\sigma T}{m}\right) U^\mu, \tag{29}$$

where we have used (27) to carry out the last step.

Before we proceed to work out the consequences of set (12), (13), and (29), it must be emphasized that the following treatment is not true for a general velocity field although everything prior to (27) is. Later, I will come back to discuss this topic further.

An immediate and, perhaps, the most important theoretical consequence of (12), (13), and (29) is the emergence of a new conserved (grand) helicity ($\langle\rangle = \int d^3x$)

$$\mathcal{H} = \langle \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{B}} \rangle, \quad \frac{d\mathcal{H}}{dt} = 0, \tag{30}$$

where

$$\boldsymbol{\mathcal{A}} = \frac{\boldsymbol{\Pi}}{q} = \boldsymbol{A} + \frac{m}{q}\left(f - \frac{\sigma T}{m}\right)\boldsymbol{U}, \tag{31}$$

$$\boldsymbol{\mathcal{B}} = \nabla \times \boldsymbol{\mathcal{A}} = \boldsymbol{B} + \frac{m}{q} \nabla \times \left[\left(f - \frac{\sigma T}{m}\right)\boldsymbol{U}\right], \tag{32}$$

and ($\mathcal{A}^0 = \Pi^0/q = A^0 + (m\gamma/q)(f - \sigma T/m)$),

$$\mathcal{E} = -\nabla \mathcal{A}^0 - \partial_t \boldsymbol{\mathcal{A}} \tag{33}$$

are, respectively, the new effective Electro-Vortic (vector)-potential, and the magnetic and electric components of the Electro-Vortic field.

The conservation of an appropriate helicity for systems of this type was, first, demonstrated in Ref. 1. The essential steps are:

(1) Constructing the helicity four vector

$$\mathcal{K}^\mu = \Pi_\nu \mathbb{M}^{\mu\nu}, \tag{34}$$

where ($\mathbb{M}$), defined as $\mathbb{M}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\mathcal{M}_{\alpha\beta}$, is the duel of the EV field $\mathcal{M}$.

(2) and demonstrating that $\mathcal{K}^\mu$ is divergence free

$$\partial_\mu \mathcal{K}^\mu = 0 \tag{35}$$

with the corollary that $\langle \mathcal{K}^0 \rangle = \langle \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{B}} \rangle \equiv \mathcal{H}$ is conserved. The reader may consult[1] for details.

It must be emphasized here that the Electro-Vortical field $\mathcal{M}^{\nu\nu}$, defined by (12), (13), and (29), is, perhaps, the most general and encompassing unified field that can describe the overall dynamics of the perfect isotropic fluid characterized by the relativistic EMoM tensor of (4). Since the $\mathcal{M}^{\nu\nu}$ has completely subsumed the thermodynamic force $T\partial^{\nu}\sigma$, it is the unified expression of all the macroscopic forces that the fluid element is subjected to. Consequently, the helicity $\mathcal{H}$ is a complete invariant of the model. There are no sources and sinks of $\mathcal{H}$; in particular, there is no equivalent, for instance, of a baroclinic source; if $\mathcal{H}=0$ at any time, it will always remain so.

A. EV field and closure with Maxwell equations

The elegant and succinct formalism for the EV field, however, must be put in perspective. It does create a natural mathematical synthesis in which the fluid and electromagnetic forces are unified, but it is not a closed system. By itself, it is excellent for proving theorems of great significance, but it must be combined with Maxwell equations for a self-consistent calculation of the electromagnetic and the velocity fields. Maxwell's equations, of course, do not recognize the EV field; they evolve *only the EM field* $F^{\mu\nu}$ under the influence of the currents carried by the charged fluids.

Notice that the velocity fields and the thermodynamic quantities (p, f, σ, n, and T) are all tangled up in the formalism increasing the calculational complexity. In order to work out some analytic consequences of the EV synthesis, I will dwell on a simple closure model. I will assume that the thermodynamic quantities are some known functions of space time, really of space only, because much of what follows will refer to the equilibrium states accessible to the system. I will further deal with a single dynamic charged fluid species in a neutralizing background (in some appropriate frame) so that the nontrivial part of the steady state inhomogeneous Maxwell equations ($\partial_{\mu}F^{\nu\mu}=4\pi J^{\nu}$) is contained in the Ampere's law (having assumed that $c=1$)

$$\nabla\times\boldsymbol{B}=4\pi nq\boldsymbol{U}, \tag{36}$$

where all quantities on the right hand side were defined after (5). For simplicity in later manipulations, let $n=\hat{n}n_0$ (where n_0 is an ambient measure of density and $\hat{n}$ is the density envelope). Equation (36), then, may be written as

$$\nabla\times\boldsymbol{\Omega}=\frac{\hat{n}\boldsymbol{U}}{\lambda^2}, \tag{37}$$

where $\boldsymbol{\Omega}=(q/m)B$ is the cyclotron frequency, and $\lambda=\sqrt{4\pi n_0 q^2/m}$, a constant measure of the skin depth, denotes an intrinsic length scale of the dynamics. Equation (37) is our first relationship between $\boldsymbol{\Omega}$ and $\boldsymbol{U}$. For notational convenience, we will carry out our calculations using $\boldsymbol{\Omega}$ as a measure of the magnetic field.

Let us now rewrite the set (12), (13), and (29) in the more familiar (less formal) vector notation. The zeroth and the spatial components of Eq. (13) ($\gamma\equiv U^0$)

$$\boldsymbol{\mathcal{E}}\cdot\boldsymbol{U}=0, \tag{38}$$

$$\gamma\boldsymbol{\mathcal{E}}+\boldsymbol{U}\times\boldsymbol{\mathcal{B}}=0. \tag{39}$$

There are two immediate messages in these strikingly simple-looking equations:

(1) the vector Eq. (39) is the only independent equation since Eq. (38) is just an ineluctable consequence,
(2) In terms of the EV fields, the entire dynamics of a hot relativistic fluid has the mathematical structure of the simplest MHD model; it obeys an ideal Ohm's law. In fact if we write $\mathbf{U}=\gamma\mathbf{V}$ and cancel γ, Eq. (39) reduces to $\boldsymbol{\mathcal{E}}+\boldsymbol{V}\times\boldsymbol{\mathcal{B}}=0$, precisely the non relativistic from of Ohm's law. Of course $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ are lot more complicated than the MHD counterparts as can be seen from Eqs. (31)–(33).

Since I will be attempting to solve the steady-state, charge neutral ($\partial_t=0, A^0=0$) fluid-Maxwell system in Section III, it will be helpful to spell out the relevant expressions,

$$(q/m)\boldsymbol{\mathcal{E}}=-\nabla(\hat{f}\gamma), \tag{40}$$

$$(q/m)\boldsymbol{\mathcal{B}}=\boldsymbol{\Omega}+T\nabla\hat{f}\times\mathbf{U}, \tag{41}$$

where (27) has been explicitly used. Notice that the inhomogeneous thermodynamics appears through the gradients of $\hat{f}=(f/T-\sigma/m)$, a somewhat involved combination of enthalpy, entropy, and temperature.

For a given spatial dependence of the thermodynamics variables, Eqs. (37) and (39) (in conjunction with (40) and (41)) constitute a closed set of equations to solve for the physical fields: $\boldsymbol{U}$, the vector part of the four velocity, and $\boldsymbol{\Omega}$, the measure of the magnetic field.

III. REVISITING MAGNETO-GENESIS—VARIETY OF MAGNETOFLUID STATES

It was pointed out in Sec. I B that in most models beyond MHD, there was no constraint that prevented the evolution of a magnetic field from a state of zero field even if the prevailing thermodynamics were barotropic, in fact, even homogeneous. Therefore, the state with $\boldsymbol{B}=0$ is not at all special in these advanced models. Consequently, the high priority given to the quest for searching seed—creation mechanisms—baroclinic, relativistic, and general relativistic (or a combination thereof) was quite unnecessary; there was no primeval magneto-genesis *a la seed creation* needed for realistic plasma systems to develop and sustain magnetic fields. Of course, the structure and nature of the accessible magnetic states will be determined just as much by the thermal properties of the plasma, as for instance, by the kinematic parts often invoked in the dynamo theories. It is no wonder that in a self-consistent equilibrium model, the thermal and kinematic forces are so intertwined that it is their combined rather than individual manifestation that emerges as the full determinant of the magnetic state. Most of all, one must appreciate that self-consistent states will have related magnetic and flow-fields, and thus all states of interest will be Magneto-Fluid states.

Before the formulation of the new Electro-Vortical field $\mathcal{M}^{\mu\nu}$(and its concomitant absolutely conserved helicity), there was, in addition to calculating the magnetic/velocity fields, another interesting set of calculations one could do: one could, for instance, explore the seed-generation problem, that is, the seed for the appropriate generalized vorticity (of which the magnetic field is just a part) since there existed sources and sinks within the dynamical model of a relativistic perfect fluid. For the fully unified EV field of (12), (13), and (29), however, such a pursuit is meaningless since there are no sources that could lead to a state of finite $\boldsymbol{\mathcal{B}}$ from an initial state of zero $\boldsymbol{\mathcal{B}}$.

I will, therefore, concentrate on simply exploring the magnetic/flow configurations supported by the equilibrium system of Eqs. (37) and (39)–(41). Successive configurations will have an increasing content as well as complexity.

A. A super "superconducting" state

Equation (39) is trivially satisfied if the EV electric (EV_e) and EV magnetic (EV_m) fields were, separately, zero

$$\mathcal{E} = 0 = \nabla(\gamma \hat{f}), \tag{42}$$

and

$$\boldsymbol{\mathcal{B}} = 0 = \boldsymbol{\Omega} + T\nabla\hat{f} \times \mathbf{U}. \tag{43}$$

The first equation is a relativistic Bernoulli condition and will not be dissected in this paper. The second condition, signifying the absence or expulsion of (EV_m), is a generalization of the London equation,[58] an electrodynamic expression of the complete expulsion of canonical vorticity.[59]

Since the London equation is the quintessential electrodynamic expression of a superconducting state, I will borrow the terminology and name the magneto-fluid state represented by

$$\frac{\boldsymbol{\Omega}}{\lambda^2} = -\frac{T\nabla\hat{f}}{\hat{n}} \times (\nabla \times \boldsymbol{\Omega}), \tag{44}$$

the combination of Eqs. (37)–(43), as the Super-Superconducting state (SSS). A formal one-d solution for SSS is worked out in Appendix A and can be fully evaluated for specified profiles of the thermodynamic quantities. It can be seen from Eqs. (A2) and (A3) that magnetic and velocity fields are perpendicular to one another and decay or grow depending on whether the thermodynamic combination $\hat{f}$ decreases or increases as we go away from $x = 0$. But the most interesting feature of the solution is evident from (A1) (repeated here in a slightly different form, for convenience)

$$\frac{1}{\hat{\lambda}^2} = \frac{d\ln\hat{f}}{dx}\frac{d\ln|\Omega|}{dx} \simeq \frac{1}{L_g L_{mag}}, \quad L_{mag} \simeq \frac{\hat{\lambda}^2}{L_g}. \tag{45}$$

Unlike the standard superconducting state for which the scale length for the magnetic field variation equals the skin depth, L_{mag} for SSS is a hybrid, determined by a combination of the skin depth (varying in space) and scale length of the thermodynamic gradients. For very relativistically hot plasmas, the skin depth may be strongly enhanced from its non-relativistic value. Still, the skin depths tend to be often lot shorter than the gradient scale lengths implying that L_G, the characteristic scale of the SSS fields could be considerably smaller than the skin depth.

Notice that the SSS solution is driven and maintained by inhomogeneous thermodynamic, and by construction is free of EV_m and, therefore, of ElectroVortic helicity.

B. A super-Beltrami (SB) equilibrium

The next solution, more complex than SSS, may be obtained by still maintaining

$$\mathcal{E} = 0 = \nabla(\gamma\hat{f}),$$

but solving (39) as

$$\mathcal{B} = \alpha\hat{n}\boldsymbol{U}, \tag{46}$$

where α is a constant "inverse length" and the profile factor $\hat{n}$ is included to ensure that $\nabla\cdot\boldsymbol{\mathcal{B}} = 0$ ($\nabla\cdot(\hat{n}\boldsymbol{U}) = 0$ is demanded by the Maxwell equations). The alignment of an appropriate "magnetic field-vorticity" along the flow (as in (46)) constitutes a Beltrami condition; the resulting magnetic field configurations (the Beltrami states) have been extensively studied in the literature.[60]

When combined with the Amperes law (Eq. (37)), the Beltrami condition (Eq. (46)) yields

$$\boldsymbol{\Omega} + \frac{\lambda^2 T\nabla\hat{f}}{\hat{n}} \times (\nabla\times\boldsymbol{\Omega}) = \alpha\lambda^2\nabla\times\boldsymbol{\Omega}, \tag{47}$$

an equation that, though linear in $\boldsymbol{\Omega}$, looks considerably more elaborate (and contentful) than the generic Beltrami equation, $\nabla\times\boldsymbol{Q} = \alpha\boldsymbol{Q}$.

The exact one-d solution of Eq. (47),

$$\Omega_y = \Omega_0\cos\left(\int_0^x k_R dx'\right) * \exp\left(\int_0^x k_I dx'\right), \tag{48}$$

$$\Omega_z = \Omega_0\sin\left(\int_0^x k_R dx'\right) * \exp\left(\int_0^x k_I dx'\right), \tag{49}$$

where

$$k_R = \frac{L_B}{L_B^2 + \left(\hat{\lambda}^2/L_g\right)^2}, \quad k_I = \frac{\hat{\lambda}^2/L_g}{L_B^2 + \left(\hat{\lambda}^2/L_g\right)^2}, \tag{50}$$

is derived in Appendix B, and combines the characteristics of the Super solutions of Section III A [exponential decay (growth) for negative (positive) $L_g = d\ln\hat{f}/dx$] with the oscillatory behavior peculiar to Beltrami states. These Super-Beltrami (SB) states vary at two distinct scale lengths—the intrinsic scale length L_{mag} and the new scale length L_B that must be set by the value of invariants.

Once we know $\boldsymbol{\Omega}$, the corresponding velocity field can be calculated using the Maxwell Eq. (37),

$$U_y = -\frac{\lambda^2}{\hat{n}}\frac{d\Omega_z}{dx}, \tag{51}$$

$$U_z = \frac{\lambda^2}{\hat{n}}\frac{d\Omega_y}{dx}. \tag{52}$$

What about the helicity of these SB states? Using Eq. (37), one could rewrite (46) as

$$\mathcal{B} = \alpha\hat{n}\boldsymbol{U} = L_B\nabla\times\boldsymbol{\Omega}, \tag{53}$$

from which one deduces the Electro-Vortical potential to be

$$\mathcal{A} = L_B\boldsymbol{\Omega}. \tag{54}$$

The consequential expression for

$$\mathcal{H} = L_B^2\int\mathcal{A}\cdot\mathcal{B}dx = L_B^2\int\boldsymbol{\Omega}\cdot\nabla\times\boldsymbol{\Omega}\,dx, \tag{55}$$

is amazingly simple and can be readily computed.

Formally, by substituting $\boldsymbol{\Omega}$ into (55), one can relate L_B to the skin depth $\hat{\lambda}$, the gradient scale length L_g, and the helicity $\mathcal{H}$. Since the absolutely conserved $\mathcal{H}$ is a given constant of the system, L_B is fully "determined"; the situation is very similar to the Woltjer-Taylor state for which the scale length is fully set by the ratio of the magnetic helicity to magnetic energy.[10,11]

C. The general solution

After these two special cases of considerable significance, I will now attempt some explorations into the most general set of equations pertinent to the perfect relativistic charged fluid. For carrying out explicit calculations, going back to the original equation (5) turns out to be more convenient, transparent, and revealing. It also means that there are no restrictions on the velocity field; the special velocity field (27) that was needed to formally advance the search for the more encompassing Electro-Vortical field, is no longer needed.

After some exact manipulations including invoking the thermodynamic relation $\nabla f = T\nabla\sigma + n^{-1}\nabla p$, Eq. (5) can be written as (a giant leap backward)

$$\boldsymbol{U}\times\boldsymbol{\Omega} = \frac{\nabla p}{mn} + (\boldsymbol{U}\cdot\nabla)f\boldsymbol{U} \tag{56}$$

that must be solved in conjunction with (37). The system (37)–(56) is highly nonlinear in the field variables $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ in contrast to the "simpler" equilibria SSS and SB. Of the two terms on the righthand side, the first one is the pressure gradient force that is familiar from the "non-relativistic" (NR) formulations (Eq. (56), however, is relativistically correct). The second term could be thought of as the relativistic force because it is dominant in the relativistic regime $|U| \gg 1$ and may be negligible in the NR limit, $|U| \ll 1$. Solving Eq. (56), in conjunction with Eq. (37) is a straightforward numerical exercise using Mathematica, for example. I will, however, spend considerable effort on finding analytic solutions in suitable limits/approximations:

(1) The convective nonlinear term can often be neglected in the NR limit; it may be identically zero for some special choices of velocity fields. What follows, then, is an exact treatment for such situations. In this approximation, Eq. (56) is easily inverted to give (after some simple algebra)

$$\hat{n}\boldsymbol{U} = \frac{\boldsymbol{\Omega}}{\alpha} + \frac{\boldsymbol{\Omega}\times\nabla p}{mn_0\Omega^2}, \tag{57}$$

in which the first term is the (Beltrami-like) component of the velocity field $\boldsymbol{U}$ along the magnetic field ($\boldsymbol{\Omega}\equiv(q/m)\boldsymbol{B}$), and the second term is the standard diamagnetic velocity. Substituting Eq. (57) into Eq. (37), we derive the equilibrium magnetic field equation

$$\nabla\times\boldsymbol{\Omega} = \frac{\boldsymbol{\Omega}}{L_B} + \frac{\boldsymbol{\Omega}\times\nabla p}{mn_0\lambda^2\Omega^2}, \tag{58}$$

where we have introduced the Beltrami scale length L_B like in Section III B. Thus, for the full system, we have arrived at a very interesting nonlinear equation where the current ($\hat{n}\boldsymbol{U}$) that feeds the Maxwell equation has, simultaneously, a linear component along the magnetic field and a nonlinear component perpendicular to it. I believe that this equation is generic and would have a much wider applicability; its 1-d solution is worked out in Appendix C,

$$B_y = \left[\frac{B_{max}^2}{8\pi} - p\right]^{1/2} * \sin\frac{x}{L_B}, \tag{59}$$

$$B_z = \left[\frac{B_{max}^2}{8\pi} - p\right]^{1/2} * \cos\frac{x}{L_B}, \tag{60}$$

where I have chosen to display the expressions for the magnetic field components ($\boldsymbol{B} = (m/q)\boldsymbol{\Omega}$). Notice that the essential nature of this nonlinear solution, to be called Super-Beltrami-Nonlinear (SBN), is similar to that of the linear solutions pertinent to the SB equilibria described in Section III B—Beltrami oscillations with a modulating factor that is exponential in the linear case (48) and (49) but algebraic in the nonlinear (59) and (60). The SBN magnetic field is bounded through the pressure profile reaching a maximum amplitude B_{max} at $p = p_{min}$. There are again two scale lengths that define the configuration—the pressure gradient scale length (through currents perpendicular to $\boldsymbol{B}$) and the Beltrami scale-length due to currents along $\boldsymbol{B}$).

As an aside, it should be mentioned that an exact formal solution is possible for the generic system

$$\nabla\times\boldsymbol{\Omega} = \frac{\boldsymbol{\Omega}}{L_B} + \frac{g\boldsymbol{\Omega}\times\hat{\boldsymbol{e}}_x}{\Omega^2}, \tag{61}$$

where $g(x)$ is any arbitrary function of x (and not just a perfect derivative). It is

$$\Omega_y = \left[\Omega_{max}^2 - \int g dx\right]^{1/2} * \sin\frac{x}{L_B}, \tag{62}$$

$$\Omega_z = \left[\Omega_{max}^2 - \int g dx\right]^{1/2} * \cos\frac{x}{L_B}. \tag{63}$$

(2) Finally, I will now construct two distinct 1D formal solutions of the un-approximated Eq. (56). The direction of variation will be the radial direction in an appropriate

cylindrical geometry. The magnetic field will be assumed to have the form (orthogonal to the direction of variation)

$$\boldsymbol{\Omega} = \hat{\boldsymbol{e}}_z G(r) + \hat{\boldsymbol{e}}_\theta \psi(r), \tag{64}$$

where the magnitudes $G(r)$ (the axial component) and $\psi(r)$ (the azimuthal component) are functions of r alone. For this choice (using Eq. (37), the velocity field

$$\boldsymbol{U} = \frac{\lambda^2}{\hat{n}}\left(\frac{1}{r}\frac{d\, r\psi}{dr}\hat{\boldsymbol{e}}_z - \frac{dG}{dr}\hat{\boldsymbol{e}}_\theta\right), \tag{65}$$

is also perpendicular to the radial direction. Substituting Eqs. (64) and (65) into Eq. (56) and remembering that $(d/d\theta)\hat{\boldsymbol{e}}_\theta = -\hat{\boldsymbol{e}}_r$, the radial force balance (the only non-trivial component) yields

$$\frac{d}{dr}\left[\frac{p}{mn_0} + \frac{\lambda^2}{2}\left(\psi^2 + G^2\right)\right] + \frac{\lambda^2\psi^2}{r} = \frac{\lambda^4 f}{r\hat{n}}\left(\frac{dG}{dr}\right)^2, \tag{66}$$

where $n_0 = n/\hat{n}$ is a constant measure of the density. Equation (66) is an exact consequence of the model. Since it is assumed that the pressure and other thermodynamic quantities are give functions of r, (66) constitutes a single equation in two variables (B and ψ):

(a) the first class of solutions I explore have $\psi = 0$, that is, the magnetic (velocity) field is purely axial (azimuthal); its magnitude G satisfies

$$\frac{d\hat{p}}{d\rho} + G\frac{dG}{d\rho} = \frac{f}{\hat{n}}\left(\frac{dG}{d\rho}\right)^2, \tag{67}$$

where $\rho = r^2/2\lambda^2$ is a dimensionless measure of the radial distance and $\hat{p} = p/mn_0\lambda^2$. Various steps in the manipulation and solution of Eq. (67) are detailed in Appendix D. The final exact solution for the exponential profile $p = p_{max}\exp(-\bar{\rho}/\mu)$ is

$$G = 2\left(\frac{p_{max}}{\mu}\right)^{1/2}\exp\left(-\frac{\bar{\rho}}{2\mu}\right)\cosh P, \tag{68}$$

where $\cosh P$ is to be determined from the transcendental algebraic equation

$$\ln\left[\cosh P - \frac{\sinh P}{a}\right] + \frac{P}{a} = \frac{(a^2-1)\bar{\rho}}{2}, \tag{69}$$

where $a = 1 + 1/\mu$, and the boundary condition $P(x=0) = 0$ has been imposed. The $\bar{\rho}$, defined as

$$\bar{\rho} = \int_0^\rho \frac{\hat{n}}{f}d\rho', \quad \frac{d}{d\rho} = \frac{f}{\hat{n}}\frac{d}{d\bar{\rho}} \tag{70}$$

is a modulated ρ. The pressure profile or the gradient scale length $L_g \sim (2\mu\lambda^2)^{1/2}$ sets the scale of variation of the magnetic field. The implicit solution is, perhaps, not so transparent but it allows us to readily calculate the asymptotic values ($\bar{\rho} \to \infty$)

$$P_{asy} = \frac{x}{2\mu} + \frac{1+\mu}{1+2\mu}\ln 2(1+\mu), \tag{71}$$

$$\cosh P_{asy} = \frac{(1+\mu)^{\frac{1+\mu}{1+2\mu}}}{2^{\frac{\mu}{1+2\mu}}}\exp\left(\frac{x}{2\mu}\right), \tag{72}$$

and finally

$$\frac{g_{asy}}{g(0)} = \frac{(1+\mu)^{\frac{1+\mu}{1+2\mu}}}{2^{\frac{\mu}{1+2\mu}}}; \tag{73}$$

the veracity of the above ratio has been fully confirmed by the Mathematica solutions (not displayed) of the ODE (D7). For this class of solutions, the magnetic field (purely axial) and the velocity field (purely azimuthal) are orthogonal.

(b) For the second solution, I will demand that the sum of the magnetic and thermal pressures be a constant,

$$\frac{d}{dr}\left(\frac{p}{mn_0} + \frac{\lambda^2}{2}\left(\psi^2 + G^2\right)\right) = 0,$$
$$\psi^2 + G^2 = \frac{2(p_{max} - p)}{\lambda^2 n_0}, \tag{74}$$

where p_{max} is the pressure maximum. The remaining part of Eq. (66)

$$\frac{\lambda^4 f}{\hat{n}}\left(\frac{dG}{dr}\right)^2 + \lambda^2 G^2 = \frac{2(p_{max} - p)}{mn_0}, \tag{75}$$

is expressible in the dimensionless form

$$\left(\frac{dg}{d\zeta}\right)^2 + g^2 = \left(1 - \frac{p}{p_{max}}\right), \tag{76}$$

in terms of the effective radial variable $\zeta = \int_0^r (\hat{n}/f)^{1/2}dr'$, and $g = G/G_{max}$ with $G_{max} = v_t h/\lambda$ measuring the field strength (the actual magnetic field B = mc/qG).

Equation (76) is deceptively simple looking but exact analytical solutions are not readily accessible for standard assumptions of pressure profiles. What is interesting, however, is that for a wide variety of profiles $[p/p_{max} = \exp(-\zeta/L_g)$, $\exp(-\zeta^2/L_g^2), \zeta^2/(\zeta^2 + L_g^2)$ – – – –, where L_g is the pressure gradient scale length], a simple one parameter Lorentzian approximation,

$$g = \frac{\zeta^2}{\zeta^2 + \mu L_g^2}, \tag{77}$$

seems to yield a remarkably good fit through the tweaking of the parameter μ. This was extensively tested using Mathematica (comparing the solution of the ode (boundary condition b(0) = 0 with (77)). A representative comparison is displayed in Fig. 1.

Most noteworthy generic feature of this class of solutions is that the magnetic fields, starting from zero, smoothly

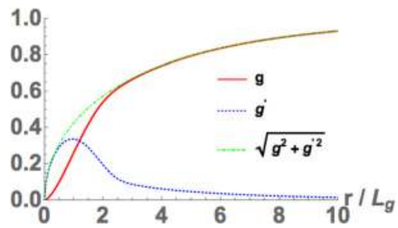

FIG. 1. The exact solution to the ode (76) for the pressure profile $p = p_{max}[\exp - \zeta/5]$ with the boundary condition $g(0) = 0$.

build up to their peak asymptotic value in a distance comparable to L_g. For an exponential pressure profile, the pressure gradient scale length (L_g) is, by definition, a constant. But, in the asymptotic region, since $db/d\zeta \ll b$, Eq. (76) yields the local magnetic scale length $L_m \approx L_g \exp(r/L_g)$, which could be considerably larger than L_g. The spatial variation of these fields is explicitly determined by the fundamental defining length scales of the system—the thermally enhanced skin depth $\hat{\lambda}$ and the gradient scale length L_g. The maximum strength of the sustainable magnetic fields is determined, quite expectedly by $p_{max}^{1/2}$. Without any loss of generality, we have assumed that the pressure peaks at $r = 0$.

For this more general class of solutions, the velocity field (both axial and azimuthal components) has components along as well as perpendicular to the magnetic field.

To the best of my knowledge, these may be the first set of general (even though 1-D) and complete solution derived for an equilibrium state accessible to a fully relativistic charged fluid (Fig. 2).

IV. SUMMING UP-DISCUSSION

This paper deals with several different aspects of the physics associated with the dynamics of a perfect isotropic fluid which is relativistic thermally as well as kinematically.

The first objective is formal and somewhat esoteric but is of great significance; it reduces the rather complex dynamics of a charged relativistic fluid to a mathematical structure that mimics MHD, one of the very simplest (but profoundly useful) models to describe charged fluids. The centre piece of the formalism is the construction of an Electro-Vortical Field $\mathcal{M}^{\mu\nu}$ that is the weighted sum of the electromagnetic and all the fluid and thermal forces present in the model. To affect this macroscopic unification, the non-electromagnetic forces had to be cast in the electromagnetic clothing; the fluid-thermal field $\mathcal{S}^{\mu\nu}$ is a fully antisymmetric tensor of second rank like the Faraday tensor $F^{\mu\nu}$ (Both these fields are four curls of a four potential). The unification program, initiated in Ref. 1, was, in a sense, completed in this paper

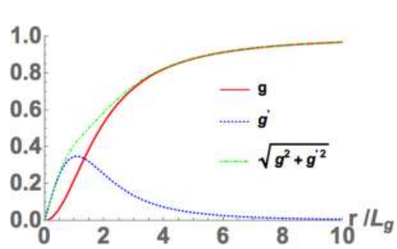

FIG. 2. Analytic approximation (77) with $\mu = 0.14$.

for the model dynamics of (4) and (5). Seeking a similar unifying formulation for more elaborate models that include, for example, the general-relativistic effects following the approach of Ref. 61, or the quantum-mechanical effects along the lines of Refs. 8 and 62, will be both interesting and revealing.

The Electro-Vortical field has associated "electric" ($\mathcal{E}$) and "magnetic"($\mathcal{B}$) fields that reduce to the ordinary electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields in the MHD limit of the theory. One of the more significant attributes of this theory is that the highly complicated $\mathcal{E}$ and $\mathcal{B}$ obey the ideal Ohm's law just as their MHD counterparts do.

Perhaps, the most important result of this effort is the emergence of a new helicity $\mathcal{H}$ that is absolutely conserved for all thermodynamics; there are no (including baroclinic) sources that can generate $\mathcal{H}$ from nothing.

The second objective was to conduct an in-depth examination of the problem of magneto genesis, i.e., the generation of the magnetic field $\boldsymbol{B}$ in the context of theories more advanced than MHD. The physics underlying this examination is closely connected to the notions of vorticity, helicity, etc., that form the sum and substance of the Electro-Vortical (EV) formalism. Thus, it makes sense that magneto genesis problem be studied in the Electro-Vortical context. One must wonder which physical quantity aught to interest us most, the magnetic field $\boldsymbol{B}$ or the EV "magnetic" component $\mathcal{B}$. The answer is clearly contextual, and the standard context is the problem of magnetic field generation in the astrophysical and cosmic settings. From various observations, one does infer $\boldsymbol{B}$ *and not* $\mathcal{B}$; the problem of magneto genesis, in the conventional ordinary sense, is surely the problem to solve.

One of the standard subproblems in the theories of magneto genesis is the creation of a seed field that could start a dynamo action which, in turn, could amplify the seed. This subproblem has its origin in the MHD based dynamo theories where the conservation of magnetic helicity ($h_m = \int \boldsymbol{A} \cdot \boldsymbol{B}\, d^3x$) imposed the constraint that finite $\boldsymbol{B}$ could not emerge from a state with $\boldsymbol{B} = 0$. Thus, a seed creation mechanism like, a baroclinic thermodynamics, had to be invoked. However, it was shown in Sec. I B that in all theories beyond MHD, in particular, the model investigated in this paper, there existed natural kinematic and thermal sources that could create and sustain a finite magnetic field starting from a field free initial state. This paper, then, suggests a qualitative, perhaps, paradigmatic shift in our approach to astro-cosmic magneto genesis; instead of inventing fancy seed-creation mechanisms, we must simply find the self-consistent solutions to the fluid and Maxwell system.

The third objective of this paper was to, precisely, work out a variety of such self-consistent equilibrium configurations. In the process, exact one-d solutions for successively more complicated defining equations were found. It was assumed that the thermodynamics is specified, and then the spatial structures of the magnetic and velocity fields were derived. The derived magneto fluid configurations fall in the following principle classes:

(1) The super "superconducting" or the SSS state that is an appropriate generalization (the constitutive relation is

$\mathcal{B} = 0$) of the London state of superconductors (consequence of the vanishing of the canonical $\hat{\boldsymbol{B}}$ (22)). The SSS fields vary on a hybrid scale L_{mag} determined by the relativistically enhanced skin depth ($\hat{\lambda}$) and the gradient scale length (L_g) of the thermodynamic quantities. For $L_g > \hat{\lambda}$, the SSS fields could vary on scale shorter than the skin depth. This is, perhaps, the introduction of a qualitatively new scale length in the charged fluid theories.

(2) The Super-Beltrami (SB) states obtained by imposing the Beltrami condition of aligning the Electro-Vortical magnetic field along the flow ($\mathcal{B} = \alpha\hat{n}\boldsymbol{U}$). The resulting configurations combine the exponential variation of the SSS with the sinusoidal characteristics of the Beltrami states. The SB solutions, therefore, are endowed with two distinct scale lengths—the intrinsic scale length L_{mag}, and the new scale length L_B that is determined by the total helicity $\mathcal{H}$. Notice that the $\mathcal{H}$ of SSS states is identically zero.

(3) The Super-Beltrami-Nonlinear (SBN) represents a more general solution from the exact equations (37) and (56). Unlike the SSS and SB systems, the exact system is highly nonlinear. It is interesting, therefore, that exact one-d solutions could, still, be extracted. The SBN is obtained by placing the velocity field in a direction perpendicular to the direction of variation ensuring the vanishing of the convective nonlinearity (the system is still nonlinear). The SBN may be viewed as the nonlinear counterpart of the linear SB states–Beltrami oscillations with a modulating factor that is exponential for the linear SB and is algebraic in the nonlinear SBN. Leading to bounded fields with two characteristic scales (L_g and L_B), this solution is likely to be of immense interest in both relativistic and non-relativistic theories of magneto genesis.

(4) The general solutions in cylindrical geometry with the convective nonlinearity playing a basic role. Barring the Beltrami behavior (absent because the flow (in this case the current) is perpendicular to the magnetic field), this solution is more encompassing and more complicated. The amplitude and the spatial variation of the resulting fields are related to the defining lengths scales of the system—the thermally enhanced skin depth $\hat{\lambda}$, the gradient scale length L_g (for instance the pressure gradient scale length), and the system length L_s.

A final remark concerning all these magneto-fluid configurations is that none requires baroclinic thermodynamics; most of them, however, are driven and sustained by inhomogeneous thermodynamics.

There are, of course, several new research directions that this paper can point to. From a formal point of view, it may be about time that one could go past the perfect isotropic relativistic fluid embodied in (4) and begin to deal with fluids that have anisotropic pressure, heat flow,[63,64] and dissipations of various kinds. It is not likely that a new more encompassing Electro-Vortical field will emerge or that there will be a conserved helicity (generalizing $\mathcal{H}$), but one could certainly learn how these pillars of the perfect model will be affected in real fluids.

Perhaps, the most important practical achievement of this paper is the delineation of a whole class of exact solutions of magnetic fields and flows that are accessible within the perfect isotropic fluid paradigm. To match these solutions with appropriate astrophysical system will be both interesting and useful. These solutions should also provide a fundamental guide and check for numerical simulations on studying the broad class of problems in magneto genesis and the generation of outflows. One is likely to find considerable overlap in the field-flow structures uncovered in this paper with, for instance, the investigations on relativistic batteries[65] and on the general forms of reconnection in relativistic plasmas.[66,67]

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APPENDIX A: THE SUPER "SUPERCONDUCTING" STATE

For all these model calculations, the thermodynamic quantities vary in the x-direction, and so do physical variables; $\nabla = \hat{\boldsymbol{e}}_x d/dx$. Equation (43), then, reads ($\hat{\boldsymbol{e}}_x \cdot \boldsymbol{\Omega} = 0$, $\boldsymbol{\Omega}$ is in the y-z plane)

$$\frac{\boldsymbol{\Omega}}{\hat{\lambda}^2} = \frac{d\ln\hat{f}}{dx}\frac{d\boldsymbol{\Omega}}{dx}. \tag{A1}$$

where $\hat{\lambda}^2 = T\hat{f}\lambda^2/\hat{n}$ is the profile modified, x-dependent skin depth. The formal solution is

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}^0 \exp\left[\int_0^x \frac{dx'}{\hat{\lambda}^2 \dfrac{d\ln\hat{f}}{dx'}}\right], \tag{A2}$$

where $\boldsymbol{\Omega}^0 = \boldsymbol{\Omega}(x=0)$. The velocity field for this solution is

$$\boldsymbol{U} = \frac{\hat{\boldsymbol{e}}_x \times \boldsymbol{\Omega}^0}{T\hat{f}\dfrac{d\ln\hat{f}}{dx}} \exp\left[\int_0^x \frac{dx'}{\hat{\lambda}^2 \dfrac{d\ln\hat{f}}{dx'}}\right]. \tag{A3}$$

APPENDIX B: SUPER BELTRAMI STATE

We seek a one-d solution for Eq. (46) with $\nabla = \hat{\boldsymbol{e}}_x d/dx$. With the same notation as in Appendix A, (46) is written as

$$\boldsymbol{\Omega} - \hat{\lambda}^2 \frac{d\ln\hat{f}}{dx}\frac{d\boldsymbol{\Omega}}{dx} = L_B\left(\hat{\boldsymbol{e}}_x \times \frac{d\boldsymbol{\Omega}}{dx}\right), \tag{B1}$$

where $L_B = \alpha\lambda^2$ is a length scale measuring the strength of the "Beltrami" term on the right hand side of (46). For L_B going to zero, (B1) reduces to (A1) of Appendix A, and for homogeneous thermodynamics ($d\ln\hat{f}/dx = 0$), it reduces to a standard Beltrami equation. Interestingly, the solution of this equation, very neatly, combines both these features. The ansatz ($\hat{\boldsymbol{\Omega}}$ is a complex amplitude)

$$\boldsymbol{\Omega} = \hat{\boldsymbol{\Omega}} \exp\left(i \int_0^x k(x')dx' \right)$$

converts (B1) into

$$\hat{\boldsymbol{\Omega}} = ik\hat{\lambda}^2 \frac{d \ln \hat{f}}{dx} \hat{\boldsymbol{\Omega}} + ikL_B (\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{\Omega}}). \quad \text{(B2)}$$

Notice that, in addition to $\hat{\boldsymbol{e}}_x \cdot \hat{\boldsymbol{\Omega}} = 0$, we have

$$\hat{\boldsymbol{\Omega}} \cdot \hat{\boldsymbol{\Omega}} = 0, \quad \text{(B3)}$$

implying $\hat{\boldsymbol{\Omega}}_R = \pm \hat{\boldsymbol{\Omega}}_I$, $\hat{\boldsymbol{\Omega}}_R \cdot \hat{\boldsymbol{\Omega}}_I = 0$. Without loss of generality, one may choose

$$\hat{\boldsymbol{\Omega}} = \Omega_0 (\hat{\boldsymbol{e}}_y + i\hat{\boldsymbol{e}}_z), \quad \text{(B4)}$$

leading to the evaluation

$$k(x) = \frac{L_B - i\hat{\lambda}^2/L_g}{L_B^2 + \left(\hat{\lambda}^2/L_g\right)^2} \equiv k_R - ik_I, \quad \text{(B5)}$$

the x dependence comes from $\hat{\lambda}^2$ and $L_g = d \ln \hat{f}/dx$. The exact solution can, now, be totally spelled out

$$\Omega_y = \Omega_0 \left(\cos \int_0^x k_R dx' \right) \left(\exp \int_0^x k_I dx' \right), \quad \text{(B6)}$$

$$\Omega_z = \Omega_0 \left(\sin \int_0^x k_R dx' \right) \left(\exp \int_0^x k_I dx' \right). \quad \text{(B7)}$$

APPENDIX C: GENERAL SOLUTION OF EVF EQUATIONS—CARTESIAN GEOMETRY

With

$$\boldsymbol{\Omega} = \hat{\boldsymbol{e}}_y \Omega_y + \hat{\boldsymbol{e}}_z \Omega_z, \quad \text{(C1)}$$

the ansatz

$$\Omega_y = Q(x) \sin \frac{x}{L_B}, \quad \Omega_z = Q(x) \cos \frac{x}{L_B}, \quad \text{(C2)}$$

collapses Eq. (61) to

$$\frac{d}{dx}\left(\frac{Q^2}{2} + \frac{p}{mn_0 \lambda^2} \right) = 0 \equiv \frac{d}{dx}\left(\frac{B^2}{8\pi} + p \right), \quad \text{(C3)}$$

reproducing the constancy of the magnetic plus thermal pressures and balancing the magnetic and thermal pressures. Let $Q = \Omega_{max}$ at $p = p_{min}$, then the general solution is

$$\Omega_y = \sqrt{\Omega_{max}^2 - \frac{2p}{mn_0\lambda^2}} * \sin \frac{x}{L_B}, \quad \text{(C4)}$$

$$\Omega_z = \sqrt{\Omega_{max}^2 - \frac{2p}{mn_0\lambda^2}} * \cos \frac{x}{L_B}. \quad \text{(C5)}$$

The solutions could also be written directly for the magnetic field

$$B_y = \left[\frac{B_{max}^2}{8\pi} - p \right]^{1/2} * \sin \frac{x}{L_B}, \quad \text{(C6)}$$

$$B_z = \left[\frac{B_{max}^2}{8\pi} - p \right]^{1/2} * \cos \frac{x}{L_B}. \quad \text{(C7)}$$

APPENDIX D: GENERAL SOLUTION OF EVF EQUATIONS—CYLINDRICAL GEOMETRY

In terms of the variable $\bar{\rho}$ defined as

$$\bar{\rho} = \int_0^\rho \frac{\hat{n}}{f} d\rho', \ \frac{d}{d\rho} = \frac{f}{\hat{n}} \frac{d}{d\bar{\rho}}. \quad \text{(D1)}$$

Eq. (67) simplifies to

$$\frac{d\hat{p}}{d\bar{\rho}} + G\frac{dG}{d\bar{\rho}} = \left(\frac{dG}{d\bar{\rho}} \right)^2 \quad \text{(D2)}$$

from which we derive

$$\frac{dG}{d\bar{\rho}} = \frac{G}{2} \pm \left[\frac{G^2}{4} + \frac{d\hat{p}}{d\bar{\rho}} \right]^{1/2}, \quad \text{(D3)}$$

a nonlinear first order ODE. If pressure gradients were neglected, (D3) yields a set of trivial solutions

$$\frac{dG}{d\bar{\rho}} = 0 => G = Const., \quad \text{(D4)}$$

and

$$\frac{dG}{d\bar{\rho}} = G => G = e^{\bar{\rho}}. \quad \text{(D5)}$$

The exponential increasing solution could only pertain to a very finite physical system. I will not purse this anymore.

The problem becomes much more interesting when the pressure gradient is switched back on. Without any loss of generality, one can assume that the pressure peaks at $\bar{\rho} = 0$ and is of the general form

$$p = p_{max} F(\bar{\rho}), \quad \text{(D6)}$$

where $F \leq 1$ is, in general, a monotonically decreasing function of $\bar{\rho}$, $dF/d\bar{\rho} < 0$. Eventually, I will display some Mathematica solutions of this system for a representative F, but a lot is gained by deeper analysis. I will concentrate on the $(-)$ of the two branches of (D3) because the $(+)$ branch, again, will be monotonically increasing. The normalized $g = G/(p_{max})^{1/2}$ obeys the differential equation

$$\frac{dg}{d\bar{\rho}} = \frac{g}{2} - \left| \frac{g^2}{4} + \frac{dF}{d\bar{\rho}} \right|^{1/2} \equiv \frac{g}{2} - \left[\frac{g^2}{4} - s^2 \right]^{1/2}, \quad \text{(D7)}$$

where $s^2 = -dF/d\bar{\rho}$. Real solutions are possible only if $g^2 > 4s^2$; there is, thus, a lower bound on g. One also notices that $dg/d\bar{\rho}$ is always positive but must decrease with $\bar{\rho}$

becoming very small when $g^2 \gg s^2$; it will be shown that there is also an upper bound on g. The substitution $g = s\cosh P$ converts (D7) to

$$2\sinh P\frac{dP}{d\bar{\rho}} = \left(1 - 2\frac{d\ln s}{d\bar{\rho}}\right)\cosh P - \sinh P. \quad \text{(D8)}$$

For a pressure profile, exponential in $\bar{\rho}$ ($F(\bar{\rho}) = \exp(-\bar{\rho}/\mu)$, $2d\ln s/d\bar{\rho} = -1/\mu$). For this profile

$$g = \frac{2}{\mu^{1/2}}\exp\left(-\frac{\bar{\rho}}{2\mu}\right)\cosh P, \quad \text{(D9)}$$

and (D8) integrates exactly to

$$\ln\left[\cosh P - \frac{\sinh P}{a}\right] + \frac{P}{a} = \frac{(a^2-1)\bar{\rho}}{2}, \quad \text{(D10)}$$

where $a = 1 + 1/\mu$, and the boundary condition $P(x=0) = 0$ has been imposed.

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