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Supersymmetry in the 6D Dirac action

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We investigate a 6D Dirac fermion on a rectangle. It is found that the 4D spectrum is governed by $N = 2$ supersymmetric quantum mechanics. Then we demonstrate that the supersymmetry is very useful for classifying all the allowed boundary conditions and to expand the 6D Dirac field in Kaluza–Klein modes. A striking feature of the model is that even though the 6D Dirac fermion has non-vanishing bulk mass, the 4D mass spectrum can contain degenerate massless chiral fermions, which may provide a hint to solve the problem of the generation of quarks and leptons. It is pointed out that zero-energy solutions are not affected by the presence of the boundaries, while the boundary conditions work well for determining the positive-energy solutions. We also provide a brief discussion on possible boundary conditions in the general case, especially those on polygons.

Subject Index B15, B33

1. Introduction

The standard model has been completely established by the discovery of the Higgs boson [1,2], and describes well the low-energy physics below the weak scale. Despite the great success of the standard model, it will be natural to regard the standard model as a low-energy effective theory of some more fundamental theories defined at higher energy scales. This is because the standard model leaves various problems to be solved.

Promising candidates beyond the standard model are the models on higher-dimensional space-times with compact extra dimensions. These models could solve the generation problem [3–9] and the fermion mass hierarchy one [10–16], and naturally explain the quark and lepton flavor structure [17–20] of the standard model. Many proposals have been made to explain the quark and lepton mass hierarchies and their flavor structures naturally from an extra-dimensional point of view.

Although extra-dimensional models will be expected to solve the generation problem, phenomenologically realistic models that solve the problem are very limited. A possible mechanism for producing degenerate massless chiral fermions is to put extra dimensions in a homogeneous magnetic field [18,20–29]. Another mechanism is to put point interactions on an extra dimension [30–33]. It would be desirable to find a new mechanism that solves the above problems of the standard model and that can lead to phenomenologically realistic models with a simple setup.

In the context of a five-dimensional (5D) gauge theory, it has been shown that a 4D massless chiral fermion appears from a 5D Dirac fermion with a suitable boundary condition (see, e.g., Ref. [34]). Furthermore, the 5D Dirac mass term plays an important role in the localization of zero-mode functions. Thereby, it can become a source of the observed fermion mass hierarchy. Unfortunately,

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however, in the case of 5D, only one 4D chiral fermion appears from a 5D Dirac field. On the other hand, it would be expected that several 4D massless chiral fermions may emerge in the case of a higher-dimensional Dirac fermion more than 5D that contains more degrees of freedom than those in 5D. Our goal is to solve the generation problem as well as other problems in the standard model from a higher-dimensional Dirac action point of view.

In Ref. [35], the 4D mass spectrum of a 6D Dirac fermion was investigated. An interesting observation is that two 4D massless chiral fermions can appear, even though the 6D Dirac action contains a non-zero bulk mass M. The results strongly suggest that higher-dimensional Dirac fermions can provide more than two 4D massless chiral fermions and could solve the generation problem. Unfortunately, it is not straightforward to extend the analysis given in Ref. [35] to the higher-dimensional Dirac action, because the origin of the degeneracy of the 4D mass spectrum (four for the massive modes, and two for the massless modes) has been obscure, and it is especially unclear how to expand Dirac fields into Kaluza–Klein modes for general higher dimensions.

In this paper, we revisit the 6D Dirac fermion and reveal hidden structures in the 4D mass spectrum from a symmetry point of view, in great detail. We show that the 4D mass spectrum is governed by an $N = 2$ quantum-mechanical supersymmetry, and the degeneracy of the 4D mass spectrum can be explained by the supersymmetry (with an additional symmetry of the action). This supersymmetric structure makes it clear why the 4D massless zero modes become chiral. This is because 4D massive modes always form supermultiplets and then become Dirac fermions, but each massless zero mode does not form a supermultiplet and hence has no chiral partner to form a Dirac fermion. We further find that the supersymmetry is very powerful for analyzing the Kaluza–Klein mode expansions and determining the class of allowed boundary conditions on extra dimensions. We expect that our analysis can apply for general higher-dimensional Dirac fermions and hence hope to answer the question of whether or not Dirac fermions with more than two extra dimensions can solve the generation and fermion mass hierarchy problems.

It is interesting to note that the supersymmetric structure is a common feature in extra dimensions. This is because similar supersymmetric structures have been found in higher-dimensional gauge and gravity theories [36–43] (see also Refs. [44,45]). Thus, it would be of great interest to understand the role of the supersymmetry in extra dimensions thoroughly.

This paper is organized as follows. We first give the setup of our model in Sect. 2, and then show, in Sect. 3, that $N = 2$ supersymmetric quantum mechanics is hidden in the 6D Dirac equation. In Sect. 4, we classify the allowed boundary conditions with the help of the supersymmetry. In Sects. 5 and 6, we explicitly construct positive energy eigenfunctions and point out a problem in determining zero-energy solutions. The degeneracy of positive energy states is explained from symmetry transformations in Sect. 7. In Sect. 8, we provide a brief discussion on possible boundary conditions in the general case, especially those on polygons. Section 9 is devoted to conclusions and discussions.

2. Six-dimensional Dirac fermion on a rectangle

Let us start with the 6D Dirac action

$$S = \int d^4x \int_0^{L_1} dy_1 \int_0^{L_2} dy_2 \overline{\Psi}(x,y)\Big[i\Gamma^A\partial_A - M\Big]\Psi(x,y), \tag{2.1}$$

where $\Psi(x,y)$ is an eight-component Dirac spinor in six dimensions and M is the bulk mass of the Dirac fermion. The 6D space-time is taken to be the direct product of the 4D Minkowski space-time and the 2D rectangle. The coordinates of the 4D Minkowski space-time and the 2D rectangle are denoted by x^μ ($\mu = 0, 1, 2, 3$) and y_j ($j = 1, 2$), respectively. The domain of the rectangle is set as $0 \le y_1 \le L_1$ and $0 \le y_2 \le L_2$.

The Dirac action (2.1) leads to the Dirac equation

$$\left[i\Gamma^\mu \partial_\mu + i\Gamma^{y_1}\partial_{y_1} + i\Gamma^{y_2}\partial_{y_2} - M \right] \Psi(x, y) = 0. \tag{2.2}$$

The 6D gamma matrices Γ^A ($A = 0, 1, 2, 3, y_1, y_2$) are required to satisfy

$$\begin{aligned} &\{\Gamma^A, \Gamma^B\} = -2\eta^{AB}\, \mathrm{I}_8, \qquad (A, B = 0, 1, 2, 3, y_1, y_2), \\ &(\Gamma^A)^\dagger = \begin{cases} +\Gamma^A & A = 0, \\ -\Gamma^A & A \neq 0, \end{cases} \end{aligned} \tag{2.3}$$

with the 6D metric $\mathrm{diag}\,\eta^{AB} = (-1, 1, 1, 1, 1, 1)$. Here, I_n denotes the $n \times n$ identity matrix. The Dirac conjugate $\overline{\Psi}$ is defined by $\overline{\Psi} = \Psi^\dagger \Gamma^0$, as usual.

In order to extract a quantum-mechanical supersymmetric structure from the Dirac equation (2.2), it may be necessary to drive the equation without including the gamma matrices Γ^{y_1} and Γ^{y_2}. For this purpose, it turns out to be convenient to introduce the matrices Γ^5 and Γ^y such as

$$\Gamma^5 \equiv i\Gamma^0\Gamma^1\Gamma^2\Gamma^3, \tag{2.4}$$

$$\Gamma^y \equiv i\Gamma^{y_1}\Gamma^{y_2}, \tag{2.5}$$

where Γ^y is an analogue of γ^5 in the extra dimensions.

Since Γ^y commutes with Γ^5, we can introduce simultaneous eigenstates of Γ^5 and Γ^y defined by

$$\Gamma^5\Psi_{R\pm} = +\Psi_{R\pm}, \qquad \Gamma^5\Psi_{L\pm} = -\Psi_{L\pm}, \tag{2.6}$$

$$\Gamma^y\Psi_{R\pm} = \pm\Psi_{R\pm}, \qquad \Gamma^y\Psi_{L\pm} = \pm\Psi_{L\pm}. \tag{2.7}$$

By use of the projection matrices, $\Psi_{R\pm}$ and $\Psi_{L\pm}$ can be constructed from Ψ as

$$\Psi_{R\pm} \equiv \mathcal{P}_R\mathcal{P}_\pm\Psi, \qquad \Psi_{L\pm} \equiv \mathcal{P}_L\mathcal{P}_\pm\Psi, \tag{2.8}$$

where

$$\mathcal{P}_R = \frac{1}{2}(\mathrm{I}_8 + \Gamma^5), \qquad \mathcal{P}_L = \frac{1}{2}(\mathrm{I}_8 - \Gamma^5), \tag{2.9}$$

$$\mathcal{P}_\pm = \frac{1}{2}(\mathrm{I}_8 \pm \Gamma^y). \tag{2.10}$$

In terms of the eigenstates of Γ^5 and Γ^y, the Dirac equation (2.2) can be decomposed as

$$\begin{aligned} i\Gamma^\mu\partial_\mu\Psi_{R+} &= -i\Gamma^{y_1}(\partial_{y_1} - i\partial_{y_2})\Psi_{L-} + M\Psi_{L+}, \\ i\Gamma^\mu\partial_\mu\Psi_{R-} &= -i\Gamma^{y_1}(\partial_{y_1} + i\partial_{y_2})\Psi_{L+} + M\Psi_{L-}, \\ i\Gamma^\mu\partial_\mu\Psi_{L+} &= -i\Gamma^{y_1}(\partial_{y_1} - i\partial_{y_2})\Psi_{R-} + M\Psi_{R+}, \\ i\Gamma^\mu\partial_\mu\Psi_{L-} &= -i\Gamma^{y_1}(\partial_{y_1} + i\partial_{y_2})\Psi_{R+} + M\Psi_{R-}. \end{aligned} \tag{2.11}$$

Furthermore, in order to remove $i\Gamma^{y_1}$ from the above equations, we may redefine the fields $\Psi_{R\pm}$ and $\Psi_{L\pm}$ as

$$\begin{aligned} \Psi_{R1} &\equiv \Psi_{R+}, \qquad \Psi_{R2} \equiv i\Gamma^{y_1}\Psi_{R-}, \\ \Psi_{L1} &\equiv \Psi_{L+}, \qquad \Psi_{L2} \equiv i\Gamma^{y_1}\Psi_{L-}. \end{aligned} \tag{2.12}$$

Then, we have succeeded in eliminating the gamma matrix Γ^{y_1} from Eq. (2.11) and in rewriting Eq. (2.11) into the form

$$i\Gamma^{\mu}\partial_{\mu}\begin{pmatrix} \Psi_{R1}(x,y) \\ \Psi_{R2}(x,y) \\ \Psi_{L1}(x,y) \\ \Psi_{L2}(x,y) \end{pmatrix} = (Q \otimes \mathrm{I}_2)\begin{pmatrix} \Psi_{R1}(x,y) \\ \Psi_{R2}(x,y) \\ \Psi_{L1}(x,y) \\ \Psi_{L2}(x,y) \end{pmatrix}, \tag{2.13}$$

where I_2 acts on two-dimensional spinors Ψ_{R1}, Ψ_{R2}, Ψ_{L1}, Ψ_{L2}, and the 4×4 matrix Q is defined by

$$Q \equiv \begin{pmatrix} 0 & 0 & M & -(\partial_{y_1} - i\partial_{y_2}) \\ 0 & 0 & \partial_{y_1} + i\partial_{y_2} & -M \\ M & -(\partial_{y_1} - i\partial_{y_2}) & 0 & 0 \\ \partial_{y_1} + i\partial_{y_2} & -M & 0 & 0 \end{pmatrix}. \tag{2.14}$$

It should be emphasized that Q does not act on spinor indices but on the “flavor” space displayed in Eq. (2.13), and satisfies the relation

$$Q^2 = \left[-\partial_{y_1}^2 - \partial_{y_2}^2 + M^2\right]\mathrm{I}_4. \tag{2.15}$$

Thus, the differential operator Q^2 turns out to correspond to a Laplacian on the extra dimensions.

In the following sections, we will show that Q can be regarded as a supercharge of $N = 2$ supersymmetric quantum mechanics, and that the 4D mass spectrum of the 6D Dirac fermion system is governed by the supersymmetry.

3. Hidden $N = 2$ supersymmetry

Since we would like to regard Q as a supercharge in supersymmetric quantum mechanics, we may introduce a Hamiltonian H by

$$H = Q^2. \tag{3.1}$$

In order for the system to be supersymmetric, we further need to introduce the “fermion” number operator F which should satisfy the relation [46]

$$\begin{aligned} (-1)^F Q &= -Q(-1)^F, \\ \left[(-1)^F\right]^2 &= \mathrm{I}_4. \end{aligned} \tag{3.2}$$

Then, the operator H, Q, and $(-1)^F$ are assumed to act on four-component wavefunctions

$$\Phi(y) = \begin{pmatrix} f_1(y) \\ f_2(y) \\ g_1(y) \\ g_2(y) \end{pmatrix} \tag{3.3}$$

that depend only on y_1 and y_2.

The operator $(-1)^F$ obeying the relations (3.2) is found to be of the form

$$(-1)^F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.4}$$

In the context of supersymmetry, we might call the $(-1)^F = +1$ (-1) eigenstates "bosonic" ("fermionic") states, though they do not literally mean bosonic or fermionic states in our model. It is worth noting that the eigenstates of $(-1)^F = +1$ (-1) rather correspond to those of $\Gamma^5 = +1$ (-1) from Eq. (2.13), so that $(-1)^F$ may be regarded as a counterpart of the 4D chiral operator.

The Hamiltonian system equipped with Q and $(-1)^F$ is called an $N = 2$ supersymmetric quantum mechanics[1] or a Witten model [46–48] *if* Q and $(-1)^F$ are Hermitian, i.e.

$$Q^\dagger = Q, \tag{3.5}$$

$$\left[(-1)^F\right]^\dagger = (-1)^F. \tag{3.6}$$

It should be emphasized that the above Hermiticity property of Q is *not* trivial because the extra dimensions have boundaries. In fact, we will see in the next section that the Hermiticity requirement (3.5) and the compatibility condition with $(-1)^F$ severely restrict the allowed boundary conditions for the wavefunctions (3.3) at the boundaries of the rectangle.

4. Classification of allowed boundary conditions

4.1. *Requirement of Hermiticity for Q*

Since we have taken the extra dimensions to be a rectangle, the requirement for Hermiticity for the supercharge Q is not trivial. In fact, we will see below that the Hermiticity requirement severely restricts the class of allowed boundary conditions for $\Phi(y)$ at $y_1 = 0, L_1$ and $y_2 = 0, L_2$.

To be more precise, we require that the supercharge Q is Hermitian under the inner product

$$\begin{aligned} \langle \Phi', \Phi \rangle &\equiv \int_0^{L_1} dy_1 \int_0^{L_2} dy_2 \Big(\Phi'(y)\Big)^\dagger \Phi(y) \\ &= \int_0^{L_1} dy_1 \int_0^{L_2} dy_2 \Big\{ \Big(f_1'(y)\Big)^* f_1(y) + \Big(f_2'(y)\Big)^* f_2(y) + \Big(g_1'(y)\Big)^* g_1(y) + \Big(g_2'(y)\Big)^* g_2(y) \Big\}, \end{aligned} \tag{4.1}$$

where

$$\Phi(y) = \begin{pmatrix} f_1(y) \\ f_2(y) \\ g_1(y) \\ g_2(y) \end{pmatrix}, \qquad \Phi'(y) = \begin{pmatrix} f_1'(y) \\ f_2'(y) \\ g_1'(y) \\ g_2'(y) \end{pmatrix}. \tag{4.2}$$

Then, in order for Q to be Hermitian, Q has to satisfy

$$\langle Q\Phi', \Phi \rangle = \langle \Phi', Q\Phi \rangle \tag{4.3}$$

for arbitrary four-component wavefunctions $\Phi(y)$ and $\Phi'(y)$ with appropriate boundary conditions.

[1] If we want to have two supercharges, we may introduce them by $Q_1 \equiv Q$ and $Q_2 \equiv -iQ(-1)^F$. Then, we can show that they form the $N = 2$ supersymmetry algebra, i.e. $\{Q_i, Q_j\} = 2H\delta_{ij}$ $(i, j = 1, 2)$.

To make our analysis tractable, we assume that the probability current in the directions of the extra dimensions terminates at each point of the boundaries of the rectangle. Then, Eq. (4.3) turns out to reduce to the conditions

$$\left(f_1'(y)\right)^* g_2(y) - \left(f_2'(y)\right)^* g_1(y) + \left(g_1'(y)\right)^* f_2(y) - \left(g_2'(y)\right)^* f_1(y) = 0 \quad \text{at } y_1 = 0, L_1, \tag{4.4}$$

$$\left(f_1'(y)\right)^* g_2(y) + \left(f_2'(y)\right)^* g_1(y) + \left(g_1'(y)\right)^* f_2(y) + \left(g_2'(y)\right)^* f_1(y) = 0 \quad \text{at } y_2 = 0, L_2. \tag{4.5}$$

4.2. *Allowed boundary conditions in the y_1-direction*

Let us first investigate condition (4.4). To solve condition (4.4), we first restrict our considerations to the case of $\Phi'(y) = \Phi(y)$, i.e. $f_j'(y) = f_j(y)$ and $g_j'(y) = g_j(y)$ $(j = 1, 2)$. This restriction would give a necessary condition for (4.4). We will, however, verify that the derived boundary conditions are sufficient as well as necessary.

For $f_j'(y) = f_j(y)$ and $g_j'(y) = g_j(y)$ $(j = 1, 2)$, condition (4.4) can be written in the form

$$\rho_1(y)^\dagger \lambda_1(y) + \lambda_1(y)^\dagger \rho_1(y) = 0 \quad \text{at } y_1 = 0, L_1, \tag{4.6}$$

where $\rho_1(y)$ and $\lambda_1(y)$ are two-component vectors defined by

$$\rho_1(y) = \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix}, \qquad \lambda_1(y) = i \begin{pmatrix} -g_2(y) \\ g_1(y) \end{pmatrix}. \tag{4.7}$$

A crucial observation is that the condition (4.6) can be rewritten as

$$|\rho_1(y) + L_0 \lambda_1(y)|^2 = |\rho_1(y) - L_0 \lambda_1(y)|^2 \quad \text{at } y_1 = 0, L_1, \tag{4.8}$$

where L_0 is a non-zero real constant whose value is irrelevant unless L_0 is non-vanishing. General solutions to Eq. (4.8) are easily found in the form

$$\rho_1(y) + L_0 \lambda_1(y) = U_1 \left(\rho_1(y) - L_0 \lambda_1(y)\right) \quad \text{at } y_1 = 0, L_1,$$

or equivalently

$$\left(\mathrm{I}_2 - U_1\right) \rho_1(y) = -L_0 \left(\mathrm{I}_2 + U_1\right) \lambda_1(y) \quad \text{at } y_1 = 0, L_1, \tag{4.9}$$

where U_1 is an arbitrary 2×2 unitary matrix.

We have required the Hermiticity of the supercharge Q in order for the system to be supersymmetric. The Hermiticity of Q is, however, not enough to preserve the supersymmetry. We should further require that the boundary conditions are compatible with the fermion number operator $(-1)^F$.

Since $(-1)^F$ commutes with the Hamiltonian H, $(-1)^F$ can be regarded as a conserved charge. Hence, the eigenvalues of $(-1)^F$ should be conserved, otherwise the supersymmetric structure would be destroyed. Since $\rho_1(y) = (f_1(y), f_2(y))^{\mathrm{T}}$ and $\lambda_1(y) = (-g_2(y), g_1(y))^{\mathrm{T}}$ correspond to $(-1)^F = +1$ and -1, respectively, $\rho_1(y)$ should not be related to $\lambda_1(y)$ at the boundaries in order for eigenvalues of $(-1)^F$ to be conserved.[2] Therefore, the condition (4.9) has to reduce to

$$\left(\mathrm{I}_2 - U_1\right) \rho_1(y) = 0, \quad \text{at } y_1 = 0, L_1, \tag{4.10}$$

[2] It is worth noting that this requirement will correspond to that of the 4D Lorentz invariance in the original 6D action, as discussed in Ref. [35].

$$\Big(\mathrm{I}_2 + U_1\Big)\lambda_1(y) = 0, \quad \text{at } y_1 = 0, L_1. \tag{4.11}$$

In other words, only a class of U_1 that (4.9) reduces to Eqs. (4.10) and (4.11) is permitted.

It is not difficult to show that the condition (4.9) can reduce to Eqs. (4.10) and (4.11) only if the eigenvalues of U_1 are equal to $+1$ or -1. This implies that the diagonalized form of U_1 can be categorized into three types:[3]

$$\text{Type I:} \qquad U_1^{\text{diag}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.12}$$

$$\text{Type II:} \qquad U_1^{\text{diag}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.13}$$

$$\text{Type III:} \qquad U_1^{\text{diag}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.14}$$

In the following, we will derive a general form of U_1 associated with each of Eqs. (4.12), (4.13), and (4.14).

Type I boundary condition: The unitary matrix U_1 can be diagonalized by a unitary matrix V such that

$$VU_1V^{-1} = U_1^{\text{diag}}. \tag{4.15}$$

Since U_1^{diag} is the identity matrix for the Type I case of Eqs. (4.12), (4.15) implies that U_1 is also identity matrix, i.e.

$$U_1 = \mathrm{I}_2. \tag{4.16}$$

Then, condition (4.10) is trivially satisfied, and (4.11) reduces to

$$g_1(y) = g_2(y) = 0 \quad \text{at } y_1 = 0, L_1. \tag{4.17}$$

It will be convenient to rewrite the boundary condition (4.17) in terms of the original four-component wavefunction $\Phi(y)$ as

$$\mathcal{P}_{(-1)^F=-1}\Phi(y) = 0 \quad \text{at } y_1 = 0, L_1 \tag{4.18}$$

for the Type I boundary condition. Here, $\mathcal{P}_{(-1)^F=-1}$ denotes the projection matrix defined by

$$\mathcal{P}_{(-1)^F=\pm 1} \equiv \frac{1}{2}\Big(\mathrm{I}_4 \pm (-1)^F\Big). \tag{4.19}$$

[3] One might add the case of $U_1^{\text{diag}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to the list, but it turns out that this case leads to the same results as for the Type III boundary conditions.

Type II boundary condition: Since $U_1^{\rm diag}$ given in Eq. (4.13) is proportional to the identity matrix, the unitary matrix U_1 is given by

$$U_1 = V^{-1} U_1^{\rm diag} V = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.20}$$

for Type II. Then, the condition (4.11) is trivially satisfied, while (4.10) reduces to

$$f_1(y) = f_2(y) = 0 \quad \text{at } y_1 = 0, L_1. \tag{4.21}$$

In terms of $\Phi(y)$, the above boundary condition can be expressed as

$$\mathcal{P}_{(-1)^F=+1}\Phi(y) = 0 \quad \text{at } y_1 = 0, L_1 \tag{4.22}$$

for the Type II boundary condition.

Type III boundary condition: For Type III, the unitary matrix U_1 can generally be written as

$$U_1 = V^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V = V^{-1}\sigma_3 V. \tag{4.23}$$

Since V can be any element of $U(2)$, V could be parameterized as

$$V = e^{ia\,I_2 + ib\,\sigma_3} e^{i\frac{\theta_1}{2}(-\sin\phi_1\,\sigma_1 + \cos\phi_1\,\sigma_2)}. \tag{4.24}$$

However, $e^{ia\,I_2+ib\,\sigma_3}$ trivially acts on σ_3 in Eq. (4.23), so that the relevant part of V in the unitary transformation (4.23) will be given by

$$V = e^{i\frac{\theta_1}{2}(-\sin\phi_1\,\sigma_1 + \cos\phi_1\,\sigma_2)}. \tag{4.25}$$

Then, we find that

$$U_1 = V^{-1}\sigma_3 V = \vec{n}_1 \cdot \vec{\sigma} = \begin{pmatrix} \cos\theta_1 & e^{-i\phi_1}\sin\theta_1 \\ e^{i\phi_1}\sin\theta_1 & -\cos\theta_1 \end{pmatrix}, \tag{4.26}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $\vec{n}_1$ is a unit vector pointing at the position of a unit two-sphere S^2 defined by

$$\vec{n}_1 = (\cos\phi_1 \sin\theta_1, \sin\phi_1 \sin\theta_1, \cos\theta_1). \tag{4.27}$$

The above result shows that the parameter space of the Type III boundary condition is given by $S^2 = U(2)/(U(1)\times U(1))$. Therefore, the Type III boundary condition is expected to possess rich physical implications, because the parameter space is topologically non-trivial [49].

It follows from (4.26) that (4.10) and (4.11) become

$$\begin{aligned} (\mathrm{I}_2 - \vec{n}_1\cdot\vec{\sigma}) \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} &= 0, \\ (\mathrm{I}_2 + \vec{n}_1\cdot\vec{\sigma}) \begin{pmatrix} -g_2(y) \\ g_1(y) \end{pmatrix} &= 0, \quad \text{at } y_1 = 0, L_1. \end{aligned} \tag{4.28}$$

It will be more convenient to express the above boundary condition in terms of the original four-component wavefunction $\Phi(y)$. To this end, we may use the relation

$$\begin{pmatrix} -g_2(y) \\ g_1(y) \end{pmatrix} = -i\sigma_2 \begin{pmatrix} g_1(y) \\ g_2(y) \end{pmatrix}$$

and combine the two conditions of Eq. (4.28) into a single one as

$$\mathcal{P}_{\vec{n}_1\cdot\vec{\Sigma}_1=-1}\Phi(y) = 0 \quad \text{at } y_1 = 0, L_1, \tag{4.29}$$

where $\mathcal{P}_{\vec{n}_1\cdot\vec{\Sigma}_1=-1}$ is defined by

$$\mathcal{P}_{\vec{n}_1\cdot\vec{\Sigma}_1=\pm 1} \equiv \frac{1}{2}\Big(\mathrm{I}_4 \pm \vec{n}_1\cdot\vec{\Sigma}_1\Big), \tag{4.30}$$

$$\vec{\Sigma}_1 \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\sigma_2\vec{\sigma}\sigma_2 \end{pmatrix}. \tag{4.31}$$

Since $(\vec{n}_1\cdot\vec{\Sigma}_1)^2 = \mathrm{I}_4$ with $\vec{n}_1\cdot\vec{n}_1 = 1$, $\mathcal{P}_{\vec{n}_1\cdot\vec{\Sigma}_1=-1}$ can be regarded as the projection matrix on a subspace of $\vec{n}_1\cdot\vec{\Sigma}_1 = -1$.

It is interesting to note that every boundary condition of Type I, II, and III can be expressed by use of the projection matrices $\mathcal{P}_{(-1)^F=-1}$, $\mathcal{P}_{(-1)^F=+1}$, and $\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=-1}$, respectively, and that those representations become important in Sect. 4.4 to verify the sufficiency of the conditions obtained above.

4.3. *Allowed boundary conditions in the y_2-direction*

Let us next investigate the condition (4.5), whose solutions will give possible boundary conditions in the y_2-direction. As before, by taking $\Phi'(y) = \Phi(y)$, (4.5) is found to be written as AQ2

$$|\rho_2(y) + L_0\lambda_2(y)|^2 = |\rho_2(y) - L_0\lambda_2(y)|^2 \quad \text{at } y_2 = 0, L_2, \tag{4.32}$$

where

$$\rho_2(y) = \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix}, \qquad \lambda_2(y) = \begin{pmatrix} g_2(y) \\ g_1(y) \end{pmatrix}. \tag{4.33}$$

Here, L_0 is a non-zero real constant whose value is irrelevant unless L_0 is non-vanishing. General solutions to Eq. (4.32) are given by

$$\Big(\mathrm{I}_2 - U_2\Big)\rho_2(y) = -L_0\Big(\mathrm{I}_2 + U_2\Big)\lambda_2(y) \quad \text{at } y_2 = 0, L_2, \tag{4.34}$$

where U_2 is an arbitrary 2×2 unitary matrix.

Requiring that the boundary conditions have to be compatible with the eigenvalues of $(-1)^F$, we find that (4.34) should reduce to

$$\Big(\mathrm{I}_2 - U_2\Big)\rho_2(y) = 0, \tag{4.35}$$

$$\Big(\mathrm{I}_2 + U_2\Big)\lambda_2(y) = 0 \quad \text{at } y_2 = 0, L_2. \tag{4.36}$$

This implies that the eigenvalues of U_2 have to be $+1$ or -1. As before, we can then show that the form of U_2 is classified into three categories:

$$\text{Type 1:} \qquad U_2 = \mathrm{I}_2, \tag{4.37}$$

$$\text{Type II:} \qquad U_2 = -\mathrm{I}_2, \tag{4.38}$$

$$\text{Type III:} \qquad U_2 = \vec{n}_2 \cdot \vec{\sigma} = \begin{pmatrix} \cos\theta_2 & e^{-i\phi_2}\sin\theta_2 \\ e^{i\phi_2}\sin\theta_2 & -\cos\theta_2 \end{pmatrix},$$
$$\vec{n}_2 = (\cos\phi_2\sin\theta_2, \sin\phi_2\sin\theta_2, \cos\theta_2). \tag{4.39}$$

It follows that allowed boundary conditions are given by:

Type I boundary condition:

$$\mathcal{P}_{(-1)^F=-1}\Phi(y) = 0 \qquad \text{at } \ y_2 = 0, L_2. \tag{4.40}$$

Type II boundary condition:

$$\mathcal{P}_{(-1)^F=+1}\Phi(y) = 0 \qquad \text{at } \ y_2 = 0, L_2. \tag{4.41}$$

Type III boundary condition:

$$\mathcal{P}_{\vec{n}_2\cdot\vec{\Sigma}_2=-1}\Phi(y) = 0 \qquad \text{at } \ y_2 = 0, L_2, \tag{4.42}$$

where $\mathcal{P}_{\vec{n}_2\cdot\vec{\Sigma}_2=-1}$ is a projection matrix defined by

$$\mathcal{P}_{\vec{n}_2\cdot\vec{\Sigma}_2=\pm 1} \equiv \frac{1}{2}(\mathrm{I}_4 \pm \vec{n}_2 \cdot \vec{\Sigma}_2), \tag{4.43}$$

$$\vec{\Sigma}_2 \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\sigma_1\vec{\sigma}\sigma_1 \end{pmatrix}. \tag{4.44}$$

4.4. *Verification of the sufficient condition*

We have succeeded in classifying the allowed boundary conditions into three categories that satisfy (4.4) or (4.5) with the restriction of $\Phi'(y) = \Phi(y)$. In the following, we show that the boundary conditions derived in Sects. 4.2 and 4.3 in fact satisfy (4.4) and (4.5) even for independent $\Phi(y)$ and $\Phi'(y)$. For our purpose, it will be convenient to rewrite Eqs. (4.4) and (4.5) into the form

$$\left(\Phi'(y)\right)^\dagger \widetilde{\Gamma}_1\Phi(y) = 0 \quad \text{at } \ y_1 = 0, L_1, \tag{4.45}$$

$$\left(\Phi'(y)\right)^\dagger \widetilde{\Gamma}_2\Phi(y) = 0 \quad \text{at } \ y_2 = 0, L_2, \tag{4.46}$$

where

$$\widetilde{\Gamma}_1 \equiv \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \qquad \widetilde{\Gamma}_2 \equiv \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \tag{4.47}$$

Type I boundary condition in the y_1-direction: We first investigate the Type I boundary condition in the y_1-direction, i.e.

$$\mathcal{P}_{(-1)^F=-1}\Phi(y) = \mathcal{P}_{(-1)^F=-1}\Phi'(y) = 0 \quad \text{at } y_1 = 0, L_1. \tag{4.48}$$

Important properties for proving the condition (4.45) are

$$\begin{aligned}
&\mathcal{P}_{(-1)^F=+1} + \mathcal{P}_{(-1)^F=-1} = \mathrm{I}_4,\\
&\left(\mathcal{P}_{(-1)^F=\pm1}\right)^2 = \mathcal{P}_{(-1)^F=\pm1}, \qquad \mathcal{P}_{(-1)^F=\pm1}\mathcal{P}_{(-1)^F=\mp1} = 0,\\
&\left(\mathcal{P}_{(-1)^F=\pm1}\right)^\dagger = \mathcal{P}_{(-1)^F=\pm1},\\
&\mathcal{P}_{(-1)^F=\pm1}\widetilde{\Gamma}_1 = \widetilde{\Gamma}_1\mathcal{P}_{(-1)^F=\mp1},
\end{aligned} \tag{4.49}$$

where the last relation follows from $(-1)^F\,\widetilde{\Gamma}_1 = -\widetilde{\Gamma}_1(-1)^F$. With a shorthand notation of $\Phi_\pm(y) \equiv \mathcal{P}_{(-1)^F=\pm1}\Phi(y)$, the condition (4.45) can be verified as follows:

$$\begin{aligned}
\left(\Phi'(y)\right)^\dagger\widetilde{\Gamma}_1\Phi(y) &= \left(\Phi'_+(y) + \Phi'_-(y)\right)^\dagger\widetilde{\Gamma}_1\left(\Phi_+(y) + \Phi_-(y)\right)\\
&= \left(\Phi'_+(y)\right)^\dagger\widetilde{\Gamma}_1\Phi_-(y) + \left(\Phi'_-(y)\right)^\dagger\widetilde{\Gamma}_1\Phi_+(y)\\
&= 0 \quad \text{at } y_1 = 0, L_1,
\end{aligned} \tag{4.50}$$

where we have used the relations (4.48) and (4.49).

Type II boundary condition in the y_1-direction: The above analysis for the Type I boundary condition clearly shows that if $\Phi'(y)$ and $\Phi(y)$ satisfy the Type II boundary condition in the y_1-direction, i.e.

$$\mathcal{P}_{(-1)^F=+1}\Phi'(y) = \mathcal{P}_{(-1)^F=+1}\Phi(y) = 0 \quad \text{at } y_1 = 0, L_1, \tag{4.51}$$

then the condition (4.45) is satisfied for arbitrary wavefunctions $\Phi(y)$ and $\Phi'(y)$ with Eq. (4.51).

Type III boundary condition in the y_1-direction: In order to prove that the Type III boundary condition in the y_1-direction satisfies the condition (4.45), we need the following properties of $\mathcal{P}_{\vec{n}_1\cdot\vec{\Sigma}_1=\pm1}$:

$$\begin{aligned}
&\mathcal{P}_{\vec{n}_1\cdot\vec{\Sigma}_1=+1} + \mathcal{P}_{\vec{n}_1\cdot\vec{\Sigma}_1=-1} = \mathrm{I}_4,\\
&\left(\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=\pm1}\right)^2 = \mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=\pm1}, \qquad \mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=\pm1}\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=\mp1} = 0,\\
&\left(\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=\pm1}\right)^\dagger = \mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=\pm1},\\
&\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=\pm1}\widetilde{\Gamma}_1 = \widetilde{\Gamma}_1\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_1=\mp1},
\end{aligned} \tag{4.52}$$

where the last relation follows from the property $\vec{\Sigma}_1\widetilde{\Gamma}_1 = -\widetilde{\Gamma}_1\vec{\Sigma}_1$. The above relations are enough to show that if $\Phi(y)$ and $\Phi'(y)$ obey the Type III boundary condition in the y_1-direction, they satisfy the condition (4.45).

The above analysis can also apply to Type I, II, and III boundary conditions in the y_2-direction. In order to verify the condition (4.46) for Type I, II, and III in the y_2-direction, we only need the properties that $\mathcal{P}_{(-1)^F=\pm 1}$ and $\mathcal{P}_{\vec{n}_2\cdot\vec{\Sigma}_2=\pm 1}$ can be regarded as projection matrices and that $\widetilde{\Gamma}_2$ changes the sign of the eigenvalues of $(-1)^F$ and $\vec{n}_2 \cdot \vec{\Sigma}_2$. The proof can be done in a similar way as the case of y_1-direction.

5. Energy spectrum for Type II boundary conditions

In this section, we investigate the energy spectrum of the theory for the Type II boundary condition with the help of supersymmetry.[4] We will show that the Type II boundary condition is enough to determine the positive-energy spectrum completely, but not to determine zero-energy solutions.

5.1. *Supersymmetry relations and boundary conditions*

In this subsection, we summarize the general properties of $N = 2$ supersymmetric quantum mechanics to determine the energy spectrum.

Let $\Phi_{E\pm}(y)$ be simultaneous eigenstates of H and $(-1)^F$, i.e.

$$H\Phi_{E\pm}(y) = E\Phi_{E\pm}(y), \tag{5.1}$$

$$(-1)^F\Phi_{E\pm}(y) = \pm\Phi_{E\pm}(y). \tag{5.2}$$

Since the supercharge Q commutes with H and anticommutes with $(-1)^F$, $Q\Phi_{E\pm}$ turns out to have the same energy E but opposite eigenvalues of $(-1)^F$ if $Q\Phi_{E\pm}$ are non-vanishing. This implies that $Q\Phi_{E\pm}$ should be proportional to $\Phi_{E\mp}$,[5] i.e.

$$Q\Phi_{E+}(y) = \sqrt{E}\Phi_{E-}(y), \tag{5.3}$$

$$Q\Phi_{E-}(y) = \sqrt{E}\Phi_{E+}(y). \tag{5.4}$$

Then, $\{\Phi_{E+}, \Phi_{E-}\}$ turns out to form a supermultiplet (for $E > 0$), and Eqs. (5.3), (5.4) are called the supersymmetry relations or simply SUSY relations. The factor $\sqrt{E}$ on the right-hand-sides ensures that $\langle\Phi_{E+}, \Phi_{E+}\rangle = \langle\Phi_{E-}, \Phi_{E-}\rangle$.

We should emphasize that zero-energy solutions with $E = 0$ do not form supermultiplets, as suggested by the SUSY relations because zero-energy solutions have to satisfy the zero-energy equation[6]

$$Q\Phi_{E=0}(y) = 0. \tag{5.5}$$

In this section, we impose the Type II boundary condition on the wavefunction $\Phi(y)$ in both the y_1- and y_2-directions, i.e.

$$\Phi_+(y) = 0 \quad \text{at} \;\; y_1 = 0, L_1 \;\; \text{and} \;\; y_2 = 0, L_2. \tag{5.6}$$

One might think that Eq. (5.6) is not enough to specify the boundary condition for all the components of $\Phi(y)$ because Eq. (5.6) seems to give no constraint on $\Phi_-(y)$ at the boundaries. This is, however, not the case. The boundary condition for $\Phi_-(y)$ can be obtained through the SUSY relation (5.4). In

[4] The analysis for the Type I boundary condition is almost the same as for Type II.

[5] If the energy spectrum has another kind of degeneracy, we may replace $\Phi_{E\pm}$ by $\Phi^{(i)}_{E\pm}$ with the index i to distinguish degenerate states.

[6] Since the Hamiltonian takes the form $H = Q^2$, the equation $H\Phi_{E=0} = 0$ becomes identical to $Q\Phi_{E=0} = 0$.

order for the boundary condition (5.6) to be consistent with the SUSY relation (5.4), the wavefunction $\Phi_-(y)$ with $(-1)^F = -1$ has to obey the following boundary condition[7]

$$Q\Phi_-(y) = 0 \quad \text{at } y_1 = 0, L_1 \text{ and } y_2 = 0, L_2, \tag{5.7}$$

otherwise the supersymmetry would be lost due to the breakdown of the SUSY relation (5.4). As we will see in the next subsection, the boundary conditions (5.6) and (5.7) work well to determine the positive-energy spectrum.

5.2. *Positive-energy spectrum*

In the following, we clarify the positive-energy spectrum for the Type II boundary condition with the help of supersymmetry.

In terms of the component fields $\Phi(y) = (f_1(y), f_2(y), g_1(y), g_2(y))^{\mathrm{T}}$, the Type II boundary condition (5.6) for $f_1(y)$ and $f_2(y)$ is given by

$$f_1(y) = f_2(y) = 0 \quad \text{at } y_1 = 0, L_1 \text{ and } y_2 = 0, L_2, \tag{5.8}$$

and the boundary condition (5.7) for $g_1(y)$ and $g_2(y)$ is given by

$$\begin{aligned} &Mg_1(y) - (\partial_{y_1} - i\partial_{y_2})g_2(y) = 0, \\ &(\partial_{y_1} + i\partial_{y_2})g_1(y) - Mg_2(y) = 0, \end{aligned} \qquad \text{at } y_1 = 0, L_1 \text{ and } y_2 = 0, L_2. \tag{5.9}$$

Let $\Phi_{E+}(y)$ be an energy eigenstate with $(-1)^F = +1$. In components, the relation $H\Phi_{E+}(y) = E\Phi_{E+}(y)$ is rewritten as

$$\left[-(\partial_{y_1})^2 - (\partial_{y_2})^2 + M^2\right]\begin{pmatrix} f_{1E}(y) \\ f_{2E}(y) \end{pmatrix} = E\begin{pmatrix} f_{1E}(y) \\ f_{2E}(y) \end{pmatrix}. \tag{5.10}$$

Then, the energy eigenfunctions satisfying the Type II boundary condition (5.8) are easily found to be of the form

$$\Phi^{(1)}_{E_{n_1n_2}+}(y) = \begin{pmatrix} f_{n_1n_2}(y) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \Phi^{(2)}_{E_{n_1n_2}+}(y) = \begin{pmatrix} 0 \\ f_{n_1n_2}(y) \\ 0 \\ 0 \end{pmatrix}, \tag{5.11}$$

where

$$f_{n_1n_2}(y) = \frac{2}{\sqrt{L_1L_2}}\sin\Big(\frac{n_1\pi}{L_1}y_1\Big)\sin\Big(\frac{n_2\pi}{L_2}y_2\Big), \tag{5.12}$$

$$E_{n_1n_2} = \Big(\frac{n_1\pi}{L_1}\Big)^2 + \Big(\frac{n_2\pi}{L_2}\Big)^2 + M^2, \tag{5.13}$$

for $n_1, n_2 = 1, 2, 3, \ldots$ The eigenfunctions $f_{n_1n_2}(y)$ satisfy

$$\langle f_{m_1m_2}, f_{n_1n_2}\rangle = \delta_{m_1,n_1}\delta_{m_2,n_2}, \tag{5.14}$$

$$f_{n_1n_2}(y) = 0 \quad \text{at } y_1 = 0, L_1 \text{ and } y_2 = 0, L_2, \tag{5.15}$$

[7] The same situation has been observed in the 5D fermion system on an interval [30,32,34] and also in supersymmetric quantum mechanics with boundaries [49,50].

for $m_1, m_2, n_1, n_2 = 1, 2, 3, \ldots$ It should be noticed that the energy eigenfunctions (5.11) give a complete set of the function $\Phi_+(y)$, since the set of $\{f_{n_1 n_2}(y);\, n_1, n_2 = 1, 2, 3, \ldots\}$ forms a complete set of the function satisfying the boundary condition $f(y) = 0$ at $y_1 = 0, L_1$ and $y_2 = 0, L_2$.

As was explained in Sect. 5.1, the energy eigenfunctions for Φ_{E-} can be obtained through the SUSY relation (5.3), i.e.

$$\Phi^{(1)}_{E_{n_1 n_2}-}(y) = \frac{1}{\sqrt{E_{n_1 n_2}}} Q \Phi^{(1)}_{E_{n_1 n_2}+}(y) = \frac{1}{\sqrt{E_{n_1 n_2}}} \begin{pmatrix} 0 \\ 0 \\ M f_{n_1 n_2}(y) \\ (\partial_{y_1} + i\partial_{y_2}) f_{n_1 n_2}(y) \end{pmatrix},$$

$$\Phi^{(2)}_{E_{n_1 n_2}-}(y) = \frac{1}{\sqrt{E_{n_1 n_2}}} Q \Phi^{(2)}_{E_{n_1 n_2}+}(y) = \frac{1}{\sqrt{E_{n_1 n_2}}} \begin{pmatrix} 0 \\ 0 \\ -(\partial_{y_1} - i\partial_{y_2}) f_{n_1 n_2}(y) \\ -M f_{n_1 n_2}(y) \end{pmatrix}, \tag{5.16}$$

except for zero-energy solutions. We note that the above eigenfunctions satisfy the boundary conditions (5.7) or (5.9), as they should.

5.3. *Zero-energy solutions*

In the previous analysis, we have succeeded in constructing positive-energy solutions, completely. The analysis is, however, insufficient to obtain the whole set of energy eigenfunctions. This is because zero-energy solutions do not form supermultiplets and hence we have to investigate them separately.

As was explained in Sect. 5.1, any zero-energy solution should satisfy the zero-energy equation $Q\Phi_{E=0}(y) = 0$. Since $\Phi_+(y)$ has no zero-energy solution due to the Dirichlet boundary condition (i.e., the Type II boundary condition), zero-energy eigenfunctions will come only from $\Phi_-(y)$ [or $g_1(y)$ and $g_2(y)$] satisfying $Q\Phi_{E=0-}(y) = 0$, or in components,

$$\begin{aligned} M g_{1E=0}(y) - (\partial_{y_1} - i\partial_{y_2})\, g_{2E=0}(y) &= 0, \\ (\partial_{y_1} + i\partial_{y_2})\, g_{1E=0}(y) - M g_{2E=0}(y) &= 0. \end{aligned} \tag{5.17}$$

It is worth pointing out that a strange situation happens here. We have already found that the boundary condition (5.9) for $g_1(y)$ and $g_2(y)$ works properly for positive-energy eigenstates. The boundary condition (5.9), however, gives *no* restriction on zero-energy solutions because any zero-energy solutions *trivially* satisfy the "boundary condition" (5.9), not only at the boundaries but also on the whole space of the rectangle. In fact, the condition (5.9) can be regarded as part of the zero-energy equation (5.17).[8] This implies that the determination of zero-energy solutions might be ambiguous, as we will see below.

A zero-energy solution to (5.17) is found to be of the form

$$\Phi^{(1)}_{E=0-}(y) = \begin{pmatrix} 0 \\ 0 \\ N e^{-i\frac{\theta}{2}} e^{M(\cos\theta\, y_1 + \sin\theta\, y_2)} \\ N e^{i\frac{\theta}{2}} e^{M(\cos\theta\, y_1 + \sin\theta\, y_2)} \end{pmatrix}, \tag{5.18}$$

[8] A similar situation has been observed in the 5D fermion system on an interval [30,32,34].

where θ is an arbitrary real constant[9] and N stands for a normalization constant. We will comment on general zero-energy solutions later.

We would like to know how many independent zero-energy solutions exist in the model. To this end, we may assume a second zero-energy solution to be of the form

$$\Phi^{(2)}_{E=0-}(y) = \begin{pmatrix} 0 \\ 0 \\ N' e^{-i\frac{\theta'}{2}} e^{M(\cos\theta' y_1 + \sin\theta' y_2)} \\ N' e^{i\frac{\theta'}{2}} e^{M(\cos\theta' y_1 + \sin\theta' y_2)} \end{pmatrix}. \tag{5.19}$$

In order for $\Phi^{(1)}_{E=0-}$ and $\Phi^{(2)}_{E=0-}$ to be independent, we require that they are orthogonal, i.e.

$$\langle \Phi^{(1)}_{E=0-}, \Phi^{(2)}_{E=0-} \rangle = 0. \tag{5.20}$$

It follows that the above orthogonality relation is satisfied only if

$$\theta' = \theta + \pi \pmod{2\pi}. \tag{5.21}$$

Then, the second zero-energy solution orthogonal to $\Phi^{(1)}_{E=0-}$ is found to be

$$\Phi^{(2)}_{E=0-}(y) = \begin{pmatrix} 0 \\ 0 \\ N' e^{-i\frac{\theta}{2}} e^{-M(\cos\theta y_1 + \sin\theta y_2)} \\ -N' e^{i\frac{\theta}{2}} e^{-M(\cos\theta y_1 + \sin\theta y_2)} \end{pmatrix}, \tag{5.22}$$

with an appropriate normalization constant N'.

Since there are no more independent zero-energy solutions of the type (5.19), we may conclude that the number of the degeneracy of the zero-energy solutions is two. This result seems to be consistent with the degeneracy of the positive-energy solutions $\Phi^{(i)}_{E_{n_1 n_2}-}$ with $i = 1, 2$.

Before closing this subsection, we would like to comment on a general form of zero-energy solutions. We first note that the wavefunction (5.18) satisfies the zero-energy equation (5.17) even for an arbitrary complex number θ. Then, we can show that a general form of zero-energy solution to (5.17) is given by the superposition of the solution (5.18) with respect to $\theta \in \mathbb{C}$. It follows from this observation that additional conditions (for instance, additional boundary conditions like $\partial_{y_2} g_1(y) = \partial_{y_2} g_2(y) = 0$ at $y_1 = 0,\ L_1$ and $y_2 = 0,\ L_2$) seem to be necessary to determine independent zero-energy solutions definitely.

5.4. *Four-dimensional mass spectrum*

In the previous subsections, we have succeeded in obtaining the energy spectrum of the Hamiltonian system $H = Q^2$, though we have not yet arrived at a definite conclusion for zero-energy solutions. We can use those results to expand the original 6D Dirac field $\Psi(x, y)$ in the 4D Kaluza–Klein modes, and then rewrite the action (2.1) into the four-dimensional effective action that consists of an infinite number of 4D massive fermions and a finite number of 4D massless chiral ones.

[9] It has been shown in Ref. [35] that the origin of the parameter θ in Eq. (5.18) comes from the rotational invariance of the extra dimensions.

As discussed in Sect. 2, the 6D Dirac field $\Psi(x,y)$ can be decomposed into the eigenfunctions of Γ^5 and Γ^y as

$$\Psi(x,y) = \Psi_{R+}(x,y) + \Psi_{R-}(x,y) + \Psi_{L+}(x,y) + \Psi_{L-}(x,y), \tag{5.23}$$

where the subscripts $\pm$ of $\Psi_{R\pm}$ and $\Psi_{L\pm}$ denote the eigenvalues of Γ^y (but not $(-1)^F$).[10] The results given in the previous subsections suggest that $\Psi_{R\pm}(x,y)$ and $\Psi_{L\pm}(x,y)$ may be expanded, in terms of the energy eigenfunctions, as

$$\begin{aligned}
\Psi_{R\pm}(x,y) &= \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \psi_{R\pm}^{(n_1,n_2)}(x) f_{n_1 n_2}(y),\\
\Psi_{L\pm}(x,y) &= \Psi_{L\pm}^{(0)}(x,y)\\
&+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\left\{ i\Gamma^{y_1}\eta_{L\pm}^{(n_1,n_2)}(x)\frac{1}{\sqrt{E_{n_1n_2}}}(\partial_{y_1}\mp i\partial_{y_2})f_{n_1n_2}(y) + \frac{M}{\sqrt{E_{n_1n_2}}}\eta_{L\pm}^{(n_1,n_2)}(x) f_{n_1n_2}(y)\right\},
\end{aligned} \tag{5.24}$$

where

$$\begin{aligned}
\Psi_{L+}^{(0)}(x,y) &= \xi_{L1}^{(0)}(x) N e^{-i\frac{\theta}{2}} e^{M(\cos\theta\, y_1 + \sin\theta\, y_2)} + \xi_{L2}^{(0)}(x) N' e^{-i\frac{\theta}{2}} e^{-M(\cos\theta\, y_1+\sin\theta\, y_2)},\\
\Psi_{L-}^{(0)}(x,y) &= i\Gamma^{y_1}\xi_{L1}^{(0)}(x) N e^{i\frac{\theta}{2}} e^{M(\cos\theta\, y_1+\sin\theta\, y_2)} - i\Gamma^{y_1}\xi_{L2}^{(0)}(x) N' e^{i\frac{\theta}{2}} e^{-M(\cos\theta\, y_1+\sin\theta\, y_2)}.
\end{aligned} \tag{5.25}$$

Here, $\psi_{R\pm}^{(n_1,n_2)}(x)$, $\eta_{L\pm}^{(n_1,n_2)}(x)$, and $\xi_{Li}^{(0)}(x)$ $(i = 1,2)$ denote 4D chiral spinors as depicted by the subscripts R and L. We would like to note that the form of the mode expansion of $\Psi_{L\pm}(x,y)$ is not trivial and that the mode expansions of $\Psi_{R\pm}(x,y)$ and $\Psi_{L\pm}(x,y)$ have to be arranged such that $\psi_{R\pm}^{(n_1,n_2)}(x)$, $\eta_{L\pm}^{(n_1,n_2)}(x)$, and $\xi_{Li}^{(0)}(x)$ give the 4D mass eigenstates.

By inserting the expansions (5.24) and (5.25) into the original action (2.1) and integrating over y_1 and y_2, we find that the action (2.1) becomes[11]

$$\begin{aligned}
S = \int d^4x \Bigg\{ &\sum_{i=1}^{2} \bar{\xi}_{Li}^{(0)}(x)\, i\Gamma^\mu\partial_\mu \xi_{Li}^{(0)}(x)\\
&+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\left[\bar{\psi}_1^{(n_1,n_2)}(x)\left(i\Gamma^\mu\partial_\mu - m_{n_1,n_2}\right)\psi_1^{(n_1,n_2)}(x) + \bar{\psi}_2^{(n_1,n_2)}(x)\left(i\Gamma^\mu\partial_\mu - m_{n_1,n_2}\right)\psi_2^{(n_1,n_2)}(x)\right]\Bigg\},
\end{aligned} \tag{5.26}$$

where $\psi_i^{(n_1,n_2)}(x)$ are 4D Dirac spinors defined by

$$\begin{aligned}
\psi_1^{(n_1,n_2)}(x) &\equiv \psi_{R+}^{(n_1,n_2)}(x) + \eta_{L-}^{(n_1,n_2)}(x),\\
\psi_2^{(n_1,n_2)}(x) &\equiv \psi_{R-}^{(n_1,n_2)}(x) + \eta_{L+}^{(n_1,n_2)}(x),
\end{aligned} \tag{5.27}$$

and

$$m_{n_1n_2} \equiv \sqrt{E_{n_1n_2}} = \sqrt{\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2 + M^2}. \tag{5.28}$$

[10] We hope that readers do not confuse the meanings of the subscripts $\pm$ for $\Psi_{R\pm}$, $\Psi_{L\pm}$ in Eq. (5.23) with $\Phi_{E\pm}(y)$ in Eq. (5.2).

[11] The results are consistent with those given in Ref. [35].

Thus, we conclude that the 4D mass spectrum of the 6D Dirac fermion for the Type II boundary condition consists of infinitely many massive Dirac fermions $\psi_i^{(n_1,n_2)}(x)$ ($n_1, n_2 = 1, 2, 3, \ldots;\ i = 1, 2$) with mass $m_{n_1 n_2}$ and two massless left-handed chiral fermions $\xi_{Li}^{(0)}(x)$ ($i = 1, 2$). It should be emphasized that the appearance of the degenerate massless chiral fermions in the 4D mass spectrum could have important implications for phenomenology to solve the generation problem of the quarks and leptons.

6. Energy spectrum for Type III boundary conditions

In this section, we investigate the energy spectrum for the Type III boundary condition in a slightly different way than in the previous section.

6.1. *Type III boundary conditions and reformulation of SUSY*

The Type III boundary condition has the S^2 parameters at each boundary of $y_1 = 0, L_1$ and $y_2 = 0, L_2$. For simplicity in the following, we restrict our considerations to the simple case of

$$\vec{n}_1 = \vec{n}_2 = (0, 0, -1) \equiv \vec{n} \tag{6.1}$$

for the S^2 parameters. Then, the boundary condition considered in this section is given by

$$\Phi_{\vec{n}\cdot\vec{\Sigma}_1=-1}(y) = \Phi_{\vec{n}\cdot\vec{\Sigma}_2=-1}(y) = \begin{pmatrix} f_1(y) \\ 0 \\ g_1(y) \\ 0 \end{pmatrix} = 0 \quad \text{at } y_1 = 0, L_1 \text{ and } y_2 = 0, L_2. \tag{6.2}$$

Although we could follow the previous analysis for the Type II boundary condition, it will be convenient to reformulate the Hamiltonian with a different supercharge. By decomposing $\Psi(x, y)$ into the eigenstates of Γ^y as $\Psi(x, y) = \Psi_+(x, y) + \Psi_-(x, y)$, we may rewrite the Dirac equation (2.2) into the form

$$\begin{pmatrix} i\Gamma^\mu\partial_\mu - M & 0 \\ 0 & i\Gamma^\mu\partial_\mu + M \end{pmatrix}\begin{pmatrix} \Psi_+(x, y) \\ \widetilde{\Psi}_+(x, y) \end{pmatrix} = \begin{pmatrix} 0 & -(\partial_{y_1} - i\partial_{y_2}) \\ \partial_{y_1} + i\partial_{y_2} & 0 \end{pmatrix}\begin{pmatrix} \Psi_+(x, y) \\ \widetilde{\Psi}_+(x, y) \end{pmatrix}, \tag{6.3}$$

where $\widetilde{\Psi}_+(x, y) \equiv i\Gamma^{y_1}\Psi_-(x, y)$.

We can then define a new Hamiltonian $\widetilde{H}$ by

$$\widetilde{H} \equiv \widetilde{Q}^2 = \Big[-(\partial_{y_1})^2 - (\partial_{y_2})^2\Big]\mathrm{I}_2, \tag{6.4}$$

with a new supercharge

$$\widetilde{Q} \equiv \begin{pmatrix} 0 & -(\partial_{y_1} - i\partial_{y_2}) \\ \partial_{y_1} + i\partial_{y_2} & 0 \end{pmatrix}. \tag{6.5}$$

Here, $\widetilde{H}$ and $\widetilde{Q}$ are represented by 2×2 matrices, instead of 4×4. The differential operators $\widetilde{H}$ and $\widetilde{Q}$ act on the two-component wavefunction

$$\widetilde{\Phi}(y) = \begin{pmatrix} \widetilde{f}(y) \\ \widetilde{g}(y) \end{pmatrix} \tag{6.6}$$

with the boundary condition

$$\widetilde{f}(y) = 0 \quad \text{at } y_1 = 0, L_1 \text{ and } y_2 = 0, L_2, \tag{6.7}$$

which will correspond to (6.2). It should be stressed that the above boundary condition (6.7) guarantees that the supercharge $\widetilde{Q}$ is Hermitian.

The "fermion" number operator $\widetilde{F}$ can be introduced as

$$(-1)^{\widetilde{F}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.8}$$

which satisfies all the desired relations discussed in the previous sections.

6.2. *Energy spectrum*

In order to construct the energy spectrum, it will be convenient to introduce the eigenfunctions of $(-1)^{\widetilde{F}}$, such that

$$(-1)^{\widetilde{F}}\widetilde{\Phi}_\pm(y) = \pm\widetilde{\Phi}_\pm(y), \tag{6.9}$$

where

$$\widetilde{\Phi}_+(y) = \begin{pmatrix} \widetilde{f}(y) \\ 0 \end{pmatrix}, \qquad \widetilde{\Phi}_-(y) = \begin{pmatrix} 0 \\ \widetilde{g}(y) \end{pmatrix}. \tag{6.10}$$

With the boundary conditions (6.7), we can easily find the energy eigenfunctions for $\widetilde{\Phi}_+(y)$. The result is

$$\begin{aligned} &\widetilde{H}\widetilde{\Phi}_{\widetilde{E}_{n_1n_2}+}(y) = \widetilde{E}_{n_1n_2}\widetilde{\Phi}_{\widetilde{E}_{n_1n_2}+}(y), \\ &\widetilde{\Phi}_{\widetilde{E}_{n_1n_2}+}(y) = \begin{pmatrix} f_{n_1n_2}(y) \\ 0 \end{pmatrix} \qquad (n_1, n_2 = 1, 2, 3, \ldots), \end{aligned} \tag{6.11}$$

where $f_{n_1n_2}(y)$ are defined in Eq. (5.12) and

$$\widetilde{E}_{n_1n_2} = \left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2 \qquad (n_1, n_2 = 1, 2, 3, \ldots). \tag{6.12}$$

In order to obtain the positive-energy spectrum for $\widetilde{\Phi}_-(y)$, we use the SUSY relations

$$\sqrt{\widetilde{E}_{n_1n_2}}\,\widetilde{\Phi}_{\widetilde{E}_{n_1n_2}\mp}(y) = \widetilde{Q}\widetilde{\Phi}_{\widetilde{E}_{n_1n_2}\pm}(y). \tag{6.13}$$

It follows that the positive-energy eigenfunctions $\widetilde{\Phi}_{\widetilde{E}_{n_1n_2}-}(y)$ are given by

$$\widetilde{\Phi}_{\widetilde{E}_{n_1n_2}-}(y) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{\widetilde{E}_{n_1n_2}}}(\partial_{y_1} + i\partial_{y_2})f_{n_1n_2}(y) \end{pmatrix}. \tag{6.14}$$

The SUSY relations (6.13) also imply that $\widetilde{\Phi}_-(y)$ should satisfy the boundary condition

$$\widetilde{Q}\widetilde{\Phi}_-(y) = 0 \quad \text{at } y_1 = 0, L_1 \text{ and } y_2 = 0, L_2,$$

or equivalently

$$(\partial_{y_1} - i\partial_{y_2})\widetilde{g}(y) = 0 \quad \text{at } y_1 = 0, L_1 \text{ and } y_2 = 0, L_2. \tag{6.15}$$

This is not the end of the story. The set of $\{\widetilde{\Phi}_{\widetilde{E}_{n_1 n_2}\pm}(y);\ n_1, n_2 = 1, 2, 3, \ldots\}$ gives a complete spectrum for the positive-energy state, but we have not yet obtained zero-energy eigenfunctions for $\widetilde{\Phi}_{E=0-}(y)$.

Since $\widetilde{\Phi}_{+}(y)$ obeys the Dirichlet boundary condition, it cannot possess any zero-energy state. Therefore, any zero-energy solution to $\widetilde{H} = \widetilde{Q}^2$ should appear from an eigenstate of $(-1)^{\widetilde{F}} = -1$ and satisfies $\widetilde{Q}\widetilde{\Phi}_{E=0-}(y) = 0$, i.e.

$$(\partial_{y_1} - i\partial_{y_2})\widetilde{g}_{E=0}(y) = 0. \tag{6.16}$$

A general solution to Eq. (6.16) is given by

$$\widetilde{g}_{E=0}(y) = \rho(\bar{z}), \tag{6.17}$$

where $\rho(\bar{z})$ is an arbitrary anti-holomorphic function of $\bar{z} = y_1 - iy_2$.

Here, we face a strange situation again. The Type III boundary condition (6.7) for $\widetilde{\Phi}_{+}(y)$ and (6.15) for $\widetilde{\Phi}_{-}(y)$ turns out to work well to determine the positive-energy solutions. On the other hand, the boundary condition (6.15) for $\widetilde{\Phi}_{-}(y)$ or $\widetilde{g}(y)$ does not work properly for zero-energy solutions because any zero-energy solution to Eq. (6.16) trivially satisfies the boundary condition (6.15), and in fact the boundary condition does not give any restriction on zero-energy solutions.

It is worth commenting on a general form of zero-energy solutions (6.17). The zero-energy equation $\widetilde{Q}\widetilde{\Phi}_{E=0}(y) = 0$ possesses two-dimensional conformal invariance because $\tilde{Q}$ includes no massive parameter. Therefore, it is reasonable that a general solution to the conformal invariant equation $\widetilde{Q}\widetilde{\Phi}(y) = 0$ is given by any anti-holomorphic function (without specifying non-trivial boundary conditions).

7. Mapping between degenerate states

In Sect. 5, we have found that positive-energy eigenfunctions are four-fold degenerate for the Type II boundary condition. The purpose of this section is to understand the degeneracy of the energy eigenfunctions, especially for the positive-energy states. In the following analysis, we will restrict our considerations to the energy spectrum for the Type II boundary condition.

As already discussed, every pair of positive-energy eigenfunctions Φ_{E+} and Φ_{E-} forms a supermultiplet. This implies that the positive-energy solutions $\Phi^{(i)}_{E_{n_1 n_2}+}$ $(n_1, n_2 = 1, 2, 3, \ldots;\ i = 1, 2)$ are related to $\Phi^{(i)}_{E_{n_1 n_2}-}$ by supersymmetry, i.e.

$$\begin{aligned}
\Phi^{(1)}_{E_{n_1 n_2}+} &\overset{Q}{\longleftrightarrow} \Phi^{(1)}_{E_{n_1 n_2}-},\\
\Phi^{(2)}_{E_{n_1 n_2}+} &\overset{Q}{\longleftrightarrow} \Phi^{(2)}_{E_{n_1 n_2}-}.
\end{aligned} \tag{7.1}$$

To clarify the relations between $\Phi^{(1)}_{E_{n_1 n_2}\pm}$ and $\Phi^{(2)}_{E_{n_1 n_2}\pm}$, let us consider the $\mathcal{C}$ transformation defined by

$$\Phi(y) \xrightarrow{\mathcal{C}} \mathcal{C}\Phi(y) \equiv C\Big(\Phi(y)\Big)^*, \tag{7.2}$$

where C is the 4×4 matrix

$$C \equiv \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{7.3}$$

Interestingly, we can show that the $\mathcal{C}$ transformation satisfies the following relations:

$$\begin{aligned} \mathcal{C}(-1)^F &= (-1)^F \mathcal{C}, \\ \mathcal{C}Q &= Q\mathcal{C}, \\ \mathcal{C}H &= H\mathcal{C}, \\ (\mathcal{C})^2 &= 1. \end{aligned} \tag{7.4}$$

It follows from Eq. (7.4) that if $\Phi_{E\pm}(y)$ are any eigenfunctions of $H = E$ and $(-1)^F = \pm 1$, then the states $\mathcal{C}\Phi_{E\pm}(y)$ also have the same eigenvalues as $\Phi_{E\pm}(y)$, i.e.

$$H\Big(\mathcal{C}\Phi_{E\pm}(y)\Big) = E\Big(\mathcal{C}\Phi_{E\pm}(y)\Big), \tag{7.5}$$

$$(-1)^F\Big(\mathcal{C}\Phi_{E\pm}(y)\Big) = \pm\Big(\mathcal{C}\Phi_{E\pm}(y)\Big). \tag{7.6}$$

If $\mathcal{C}\Phi_{E\pm}$ are not proportional to $\Phi_{E\pm}$ themselves, $\Phi_{E\pm}(y)$ and $\mathcal{C}\Phi_{E\pm}(y)$ can be independent of each other with the same energy eigenvalue E. This observation implies that the set of $\{\Phi_{E\pm}, \mathcal{C}\Phi_{E\pm}\}$ gives four-fold degenerate eigenstates of $H = E$. In fact, the eigenfunctions $\{\Phi^{(1)}_{E_{n_1 n_2}\pm}, \Phi^{(2)}_{E_{n_1 n_2}\pm}\}$ turn out to be related as

$$\begin{tikzcd} \Phi^{(1)}_{E_{n_1 n_2}+} \arrow[r, leftrightarrow, "Q"] \arrow[d, leftrightarrow, "\mathcal{C}"] & \Phi^{(1)}_{E_{n_1 n_2}-} \arrow[d, leftrightarrow, "\mathcal{C}"] \\ \Phi^{(2)}_{E_{n_1 n_2}+} \arrow[r, leftrightarrow, "Q"] & \Phi^{(2)}_{E_{n_1 n_2}-} \end{tikzcd} \tag{7.7}$$

For the zero-energy eigenfunctions $\Phi^{(1)}_{E=0-}$ and $\Phi^{(2)}_{E=0-}$ given in Eqs. (5.18) and (5.22), we find

$$\mathcal{C} \circlearrowright \Phi^{(1)}_{E=0-} \xrightarrow{Q} 0 \xleftarrow{Q} \Phi^{(2)}_{E=0-} \circlearrowleft \mathcal{C}, \tag{7.8}$$

where $\Phi^{(1)}_{E=0-}$ and $\Phi^{(2)}_{E=0-}$ are found to be eigenfunctions of $\mathcal{C} = -1$ and $\mathcal{C} = +1$, respectively.

In the following part, we show that this $\mathcal{C}$ transformation for mode functions originates from a CP transformation in a 6D sense. Let us consider a CP transformation that consists of the 6D charge conjugation C and parity transformation P with $(t, \boldsymbol{x}, y) \to (t, -\boldsymbol{x}, y)$. The 6D charge conjugation is given by

$$C : \Psi(x, y) \to \Psi^{(C)}(x, y) = C\overline{\Psi}^{\mathrm{T}}(x, y), \tag{7.9}$$

where C is an 8×8 unitary matrix. The concrete definition and properties of 6D charge conjugation are given in Appendix B. This transformation flips both the 4D chirality R/L and the inner chirality $\pm$ (see Appendix A) as

$$\Psi^{(C)}_{R/L,\pm} \sim \Psi^{*}_{L/R,\mp}. \tag{7.10}$$

Since components with the same 4D chiralities (but opposite inner chiralities) are related by the $\mathcal{C}$ transformation, the 6D charge conjugation C itself cannot be the origin of the $\mathcal{C}$ transformation. Here, we focus on the fact that the parity transformation P,[12]

$$P: \Psi(t,\boldsymbol{x},y) \to \ \Psi^{(P)}(t,\boldsymbol{x},y) = \Gamma^0\Gamma^y\Psi(t,-\boldsymbol{x},y), \tag{7.11}$$

flips only the 4D chirality R/L as

$$\Psi^{(P)}_{R/L,\pm} \sim \Psi_{L/R,\pm}, \tag{7.12}$$

so that the 6D CP transformation, which is the combination of the 6D charge conjugation C and the parity transformation P, flips only the inner chirality $\pm$ and can correspond to the $\mathcal{C}$ transformation,[13]

$$CP: \Psi(t,\boldsymbol{x},y) \to \ \Psi^{(CP)}(t,\boldsymbol{x},y) = \Gamma^0\Gamma^y C\overline{\Psi}^T(t,-\boldsymbol{x},y). \tag{7.13}$$

In fact, multiplying Γ^5 and Γ^y and using the properties of the 6D charge conjugation given in Appendix B, we can easily check that the 6D CP transformation only flips the inner chirality $\pm$:

$$\Psi^{(CP)}_{R/L,\pm} \sim \Psi^*_{R/L,\mp}. \tag{7.14}$$

We should mention that the action (2.1) is invariant under the 6D CP transformation (7.13), and the CP-transformed Dirac fermion $\Psi^{(CP)}(t,\boldsymbol{x},y)$ satisfies the same 6D Dirac equation (2.2) as the original Dirac fermion $\Psi(x,y)$. This implies that the 6D CP transformation does not change the spectrum and could connect the degenerate solutions of the Dirac equation if they exist, as the $\mathcal{C}$ transformation. In the chiral representation of 6D Gamma matrices (see Appendix A for details), the 6D CP transformation (7.13) is represented in the following concrete form by regarding $\xi_{R/L,\pm}(x,y)$ as two-component spinors:

$$\begin{pmatrix} \xi^{(CP)}_{R+} \\ \xi^{(CP)}_{L+} \\ \xi^{(CP)}_{R-} \\ \xi^{(CP)}_{L-} \end{pmatrix}(t,\boldsymbol{x},y) = C \begin{pmatrix} \xi_{R+} \\ \xi_{L+} \\ -\xi_{R-} \\ -\xi_{L-} \end{pmatrix}^* (t,-\boldsymbol{x},y), \tag{7.15}$$

where

$$C = i\sigma_2 \otimes C^{(\mathrm{4D})}. \tag{7.16}$$

$C^{(\mathrm{4D})} = i\gamma^2\gamma^0$ is the (ordinary) 4D charge conjugation. We can see from Eq. (7.15) that the 6D CP transformation contains the 4D CP transformation to connect $\Psi_{R,+}$ ($\Psi_{L,+}$) and $\Psi_{R,-}$ ($\Psi_{L,-}$) without changing the 4D chirality as the 4D CP transformation. In the basis defined in Eq. (2.12), rearranging

[12] The gamma matrix Γ^y in the parity transformation (7.11) plays the role of the π-rotation in the y_1y_2-plane. The $\mathcal{C}$ transformation does not change the sign of the extra dimension coordinates; we multiplied Γ^y instead of the replacement $y \to -y$.

[13] Note that this CP transformation is not equal to the "modified" CP transformation which is useful for discussing CP violation from the 4D point of view [51–53] in $4+2n$ ($n = 1, 2, \ldots$) dimensions.

the order of the components in Eq. (7.15), we can rewrite it in the form

$$\begin{pmatrix} -\xi_{R+}^{(CP)} \\ \xi_{R-}^{(CP)} \\ \xi_{L+}^{(CP)} \\ \xi_{L-}^{(CP)} \end{pmatrix} (t, \boldsymbol{x}, y) = \left[\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} \otimes (-\mathrm{I}_2) \right] \begin{pmatrix} i\sigma_2 \xi_{R+}^* \\ i\sigma_2 (-\xi_{R-})^* \\ -i\sigma_2 (\xi_{L+})^* \\ -i\sigma_2 (\xi_{L-})^* \end{pmatrix} (t, -\boldsymbol{x}, y), \tag{7.17}$$

where $i\sigma_2\xi_{R\pm}^*$ and $-i\sigma_2\xi_{L\pm}^*$ are CP-transformed fields in the 4D sense. The above transformation with respect to the extra dimensions is found to correspond to the $\mathcal{C}$ transformation (7.2). Thus, we can understand that the 4×4 $\mathcal{C}$ matrix originates from the 6D CP transformation, where $(-\mathrm{I}_2)$ shows the trivial rotation of two-component spinors with an unphysical overall minus sign.

8. Six-dimensional Dirac fermion on arbitrary flat surfaces with boundaries

So far, we have restricted our considerations to the rectangle as a two-dimensional extra space. For phenomenological applications, it will be useful to extend our analysis to arbitrary flat surfaces S with boundaries like polygons, a disk, etc. To this end, we introduce the inner product for four-component wavefunctions $\Phi'(y)$ and $\Phi(y)$ on S as

$$\langle \Phi', \Phi \rangle_S = \int_S dy_1 dy_2 \left(\Phi'(y)\right)^\dagger \Phi(y). \tag{8.1}$$

The requirement is that the supercharge Q is given by

$$\langle Q\Phi', \Phi \rangle_S = \langle \Phi', Q\Phi \rangle_S. \tag{8.2}$$

By expressing the supercharge Q defined in Eq. (2.14) in the form

$$Q = i\partial_{y_j} \widetilde{\Gamma}_j + M\widetilde{\Gamma}_M \qquad (j = 1, 2) \tag{8.3}$$

with

$$\widetilde{\Gamma}_1 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \widetilde{\Gamma}_2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \widetilde{\Gamma}_M = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \tag{8.4}$$

we have found that the condition (8.2) leads to

$$\oint_{\partial S} dy_{/\!/} \left(\Phi'(y)\right)^\dagger \left(n_j^\chi \widetilde{\Gamma}_j\right) \Phi(y) = 0, \tag{8.5}$$

where ∂S denotes the boundary of the surface S, $(n_1^\chi, n_2^\chi) = (\cos\chi, \sin\chi)$ is a unit normal vector orthogonal to the boundary ∂S, and $dy_{/\!/}$ is a line element along ∂S, as depicted in Fig. 1.

Since it is hard to solve the non-local equation (8.5) in general, we will here restrict our considerations to the case that the local condition

$$\left(\Phi'(y)\right)^\dagger \left(n_j^\chi \widetilde{\Gamma}_j\right) \Phi(y) = 0 \quad \text{at } (y_1, y_2) \in \partial S \tag{8.6}$$

is satisfied at each point of the boundary ∂S, as was done in Sect. 4.

Although the condition (8.6) should be satisfied for arbitrary four-component wavefunctions $\Phi'(y)$ and $\Phi(y)$, it is actually sufficient to solve Eq. (8.6) for $\Phi'(y) = \Phi(y)$, as was shown in Sect. 4. Inserting

$$\left(n_1^\chi, n_2^\chi\right) = (\cos\chi, \sin\chi) \tag{8.7}$$

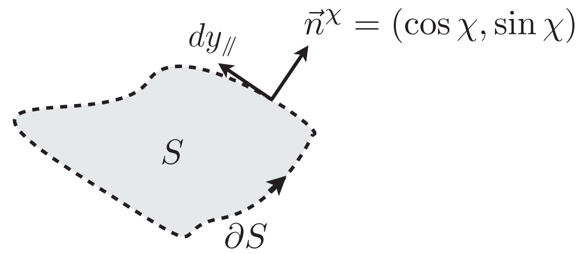

Fig. 1. ∂S denotes the boundary of S. $\vec{n}^{\chi}$ is a unit normal vector orthogonal to ∂S, and $dy_{/\!/}$ is a line element along the boundary ∂S.

into Eq. (8.6) with $\Phi'(y) = \Phi(y) = (f_1(y), f_2(y), g_1(y), g_2(y))^{\mathrm{T}}$ leads to

$$\begin{aligned} 0 &= \left(\Phi'(y)\right)^{\dagger} \left(n_j^{\chi} \widetilde{\Gamma}_j\right) \Phi(y) \\ &= (\rho(y))^{\dagger} \sigma_{\chi} \lambda(y) + \left(\sigma_{\chi} \lambda(y)\right)^{\dagger} \rho(y), \end{aligned} \tag{8.8}$$

where

$$\rho(y) \equiv \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix}, \quad \lambda(y) \equiv \begin{pmatrix} g_1(y) \\ g_2(y) \end{pmatrix}, \tag{8.9}$$

and

$$\sigma_{\chi} \equiv -\cos\chi\, \sigma_2 + \sin\chi\, \sigma_1 = \left(\sigma_{\chi}\right)^{\dagger}. \tag{8.10}$$

A crucial observation is that the condition (8.8) can be rewritten as

$$\left|\rho(y) + L_0 \sigma_{\chi} \lambda(y)\right|^2 = \left|\rho(y) - L_0 \sigma_{\chi} \lambda(y)\right|^2, \tag{8.11}$$

where L_0 is a non-zero real constant whose value is irrelevant unless L_0 is non-vanishing.

General solutions to Eq. (8.11) are easily found in the form

$$\rho(y) + L_0 \sigma_{\chi} \lambda(y) = U \left(\rho(y) - L_0 \sigma_{\chi} \lambda(y)\right), \tag{8.12}$$

or equivalently

$$(\mathrm{I}_2 - U)\rho(y) = -L_0(\mathrm{I}_2 + U)\sigma_{\chi} \lambda(y), \tag{8.13}$$

where U is an arbitrary two-by-two unitary matrix. Following the arguments given in Sect. 4, we conclude that the condition (8.13) has to reduce to

$$(\mathrm{I}_2 - U)\rho(y) = 0, \tag{8.14}$$

$$(\mathrm{I}_2 + U)\sigma_{\chi} \lambda(y) = 0, \tag{8.15}$$

and, further, that the allowed boundary conditions are classified into three types:

Type I boundary condition:

$$U_{\text{Type I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8.16}$$

It follows that the condition (8.14) is trivially satisfied, and the condition (8.15) reduces to

$$g_1(y) = g_2(y) = 0 \quad \text{at } (y_1, y_2) \in \partial S. \tag{8.17}$$

It will be convenient to rewrite the boundary condition (8.17) in terms of the original four-component wavefunction $\Phi(y)$ as

$$\mathcal{P}_{(-1)^F=-1}\Phi(y) = 0 \quad \text{at } (y_1, y_2) \in \partial S, \tag{8.18}$$

with

$$\mathcal{P}_{(-1)^F=\pm 1} = \frac{1}{2}\left(\mathrm{I}_4 \pm (-1)^F\right). \tag{8.19}$$

Type II boundary condition:

$$U_{\text{Type II}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.20}$$

It follows that the condition (8.15) is trivially satisfied, while the condition (8.14) reduces to

$$f_1(y) = f_2(y) = 0 \quad \text{at } (y_1, y_2) \in \partial S. \tag{8.21}$$

In terms of $\Phi(y)$, the above boundary condition can be expressed as

$$\mathcal{P}_{(-1)^F=+1}\Phi(y) = 0 \quad \text{at } (y_1, y_2) \in \partial S. \tag{8.22}$$

Type III boundary condition:

$$U_{\text{Type III}} = \vec{n}\cdot\vec{\sigma} = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}, \tag{8.23}$$

with

$$\vec{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta). \tag{8.24}$$

It follows from Eq. (8.23) that Eqs. (8.14) and (8.15) become

$$\begin{aligned} (\mathrm{I}_2 - \vec{n}\cdot\vec{\sigma})\rho(y) &= (\mathrm{I}_2 - \vec{n}\cdot\vec{\sigma})\begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} = 0, \\ (\mathrm{I}_2 + \vec{n}\cdot\vec{\sigma}')\lambda(y) &= (\mathrm{I}_2 + \vec{n}\cdot\vec{\sigma}')\begin{pmatrix} g_1(y) \\ g_2(y) \end{pmatrix} = 0 \quad \text{at } (y_1, y_2) \in \partial S, \end{aligned} \tag{8.25}$$

with

$$\begin{aligned} \vec{\sigma}' &\equiv \sigma_\chi \vec{\sigma} \sigma_\chi \\ &= (-\cos(2\chi)\sigma_1 - \sin(2\chi)\sigma_2, \cos(2\chi)\sigma_2 - \sin(2\chi)\sigma_1, -\sigma_3). \end{aligned} \tag{8.26}$$

Here, we used the property $(\sigma_\chi)^2 = \mathrm{I}_2$. It will be convenient to express the above boundary condition in terms of the original four-component wavefunction $\Phi(y)$. The result is given by

$$\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}=-1}\Phi(y) = 0 \quad \text{at } (y_1, y_2) \in \partial S, \tag{8.27}$$

where $\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}=\pm1}$ are projection matrices defined by

$$\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}=\pm1} \equiv \frac{1}{2}\left(\mathrm{I}_4 \pm \vec{n}\cdot\vec{\Sigma}\right), \tag{8.28}$$

$$\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma}' \end{pmatrix}. \tag{8.29}$$

We have succeeded in classifying the allowed boundary conditions at each point of the boundary ∂S. We should note that the results given in this section are consistent with those in Sect. 4. Actually, for $\chi = \pm\pi$ ($\chi = \pm\pi/2$), the above results reduce to those given in Sect. 4.2 (Sect. 4.3).

Let us examine an n-sided polygon as an application of the analysis given above. Let $\vec{n}^{\chi_a} = (\cos\chi_a, \sin\chi_a)$ ($a = 1, 2, \ldots, n$) be a normal unit vector orthogonal to the ath side of the polygon. Then, we can impose one of the following boundary conditions on the ath side of the polygon:

$$\begin{aligned} \text{Type I} : \mathcal{P}_{(-1)^F=-1}\Phi(y) &= 0, \\ \text{Type II} : \mathcal{P}_{(-1)^F=+1}\Phi(y) &= 0, \\ \text{Type III} :\ \mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_a=-1}\Phi(y) &= 0, \end{aligned} \tag{8.30}$$

with

$$\vec{\Sigma}_a = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\sigma_{\chi_a}\vec{\sigma}\sigma_{\chi_a} \end{pmatrix}. \tag{8.31}$$

If we would like to impose a single boundary condition on every side of the polygon, the possible boundary conditions are restricted to

$$\begin{aligned} &(1)\ \ g_1(y) = g_2(y) = 0, \\ &(2)\ \ f_1(y) = f_2(y) = 0, \\ &(3)\ \ f_1(y) = g_1(y) = 0, \\ &(4)\ \ f_2(y) = g_2(y) = 0 \end{aligned} \tag{8.32}$$

on every side of the polygon. The above boundary conditions (1), (2), (3), and (4) correspond to Type I, Type II, Type III with $\theta = \pi$, and Type III with $\theta = 0$, respectively. We note that the allowed Type III boundary conditions are limited to $\theta = \pi$ and 0, where ϕ does not contribute to the boundary conditions at $\theta = \pi$ and 0. This is because the normal unit vector $\vec{n}^{\chi_a}$ ($a = 1, 2, \ldots, n$) on the ath side is independent of $\vec{n}^{\chi_b}$ for $a \neq b$, in general, so that $\mathcal{P}_{\vec{n}\cdot\vec{\Sigma}_a=-1}$ ($a = 1, 2, \ldots, n$) cannot be identical for all sides of the polygon expect for $\theta = \pi$ and 0, irrespective of ϕ.

Let us finally discuss a disk as the extra dimensions. For a disk, we may impose a single boundary condition on every point of the edge of the disk. It then follows from the analysis of the n-sided polygon that the boundary condition on the edge of the disk has to be chosen from one of the four boundary conditions (8.32), otherwise the Hermiticity of the supercharge would be lost.

9. Conclusions and discussions

We have succeeded in revealing the supersymmetric structure hidden in the 6D Dirac action on a rectangle. The supersymmetry turns out to be very useful to classify the class of allowed boundary conditions, and to clarify the 4D mass spectrum of the Kaluza–Klein modes for the 6D Dirac fermion.

In fact, the allowed boundary conditions are derived by demanding the Hermiticity of the supercharge and are classified into three types. We have furthermore extended our analysis to arbitrary flat surfaces as the two-dimensional extra space. We have then found that the supersymmetric structure is still realized there and have succeeded in classifying the allowed boundary conditions, in general.

An important observation in our results is that two massless chiral fermions appear in the 4D mass spectrum for the Type I or Type II boundary conditions. This result seems to be surprising because the 6D Dirac fermion is non-chiral and furthermore has the non-vanishing bulk mass M.[14] Then, one might naively expect that the 4D mass spectrum would consist of only massive states with masses heavier than M. Actually, positive-energy eigenstates correspond to massive 4D Dirac fermions with masses $m_{n_1 n_2} > M$ for $n_1, n_2 = 1, 2, 3, \ldots$

On the other hand, we have found that the 4D massless chiral fermions correspond to zero-energy solutions, which are bound states and possess a topological nature in supersymmetric quantum mechanics. The appearance of the degenerate 4D massless chiral fermions will become crucially important in solving the generation problem and also the fermion mass hierarchy problem of the quarks and leptons, though the 4D massless chiral fermions are two-fold degenerate but not three- in the present 6D model.

In our analysis, we have found the remarkable feature that zero-energy solutions are not affected by the presence of the boundaries, while the boundary conditions work well for determining the positive-energy solutions. Even though we have explicitly constructed a one-parameter family of zero-energy solutions (5.18) and (5.22) for the Type II boundary condition and shown that the number of the degeneracy is two, the analysis seems to be insufficient. This is because the general class of zero-energy solutions is much wider than considered here, and we have not succeeded in determining a complete set of zero-energy solutions definitely.[15] Since zero-energy solutions are directly related to massless 4D chiral fermions, it would be of great importance to clarify the structure of the zero-energy solutions for higher-dimensional Dirac systems with more than or equal to two extra dimensions, phenomenologically as well as mathematically.[16]

One extension of our analysis is to introduce potential terms in the Hamiltonian. This can be done by replacing the bulk mass M by a superpotential $W(y)$ in the supercharge Q in Eq. (2.14). Even with the superpotential $W(y)$, the supercharge is still Hermitian for Type I, II, and III boundary conditions. Interestingly, the superpotential may naturally be introduced through a Yukawa interaction $g\langle\phi(y)\rangle\overline{\Psi}(x,y)\Psi(x,y)$ with a non-trivial background $\langle\phi(y)\rangle$ of a scalar field $\phi(x,y)$.

Another important extension of our analysis is to investigate higher-dimensional Dirac actions. In the case of a 6D Dirac fermion, only two massless chiral fermions appear in the 4D mass spectrum, which is not sufficient to solve the generation problem. However, more than two 4D massless chiral fermions may appear in the case of higher dimensions, equal to or more than eight dimensions,

[14] It should be emphasized that no zero-energy solution or 4D massless chiral fermion appears for the non-vanishing bulk mass M if we take the torus as the two-dimensional extra space, instead of the rectangle.

[15] It is worth noting that no trouble appears in 5D fermion systems with a single extra dimension, though a similar situation happens there [30–32,34]. Any zero-energy solution is not degenerate in one dimension, so it can be determined uniquely.

[16] Determining the size and the shape of the extra dimensions, known as moduli stabilization, would be issues closely related to gravitational effects in higher-dimensional space-time, which is absent in the present flat setup. Though this subject is of importance for a complete discussion on models in the context of extra dimensions, we will leave it as a topic for future studies.

Another extension is to consider curved extra dimensions. Even for this situation, the supersymmetric structure is expected to be realized [39,42]. It would be of interest to study the above subjects.

even though it is naively expected that 2^n massless chiral fermions would appear in the case of $D = 4 + 2n$. This may imply that it is very important to perform a comprehensive analysis of the allowed boundary conditions in higher-dimensional Dirac actions, as done in this paper, because a suitable choice of boundary conditions could reduce the possible 2^n massless chiral fermions to three massless ones. Thus, it would be of great interest to extend our analysis to higher-dimensional Dirac fermions and to search for the possibility of producing a three-generation model. This work will be reported elsewhere.

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Appendix A. Chiral representation of 6D Gamma matrices

In this appendix, we represent our choice of the chiral representation of the 6D Gamma matrices:

$$\Gamma^\mu = \mathrm{I}_2 \otimes \gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & \sigma^\mu & & \\ \overline{\sigma}^\mu & 0 & & \\ \hline & & 0 & \sigma^\mu \\ & & \overline{\sigma}^\mu & 0 \end{array}\right), \tag{A.1}$$

$$\Gamma^{y_1} = i\sigma_1 \otimes \gamma^5 = \begin{pmatrix} 0 & i\gamma^5 \\ i\gamma^5 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} & & i\mathrm{I}_2 & 0 \\ & & 0 & -i\mathrm{I}_2 \\ \hline i\mathrm{I}_2 & 0 & & \\ 0 & -i\mathrm{I}_2 & & \end{array}\right), \tag{A.2}$$

$$\Gamma^{y_2} = i\sigma_2 \otimes \gamma^5 = \begin{pmatrix} 0 & \gamma^5 \\ -\gamma^5 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} & & \mathrm{I}_2 & 0 \\ & & 0 & -\mathrm{I}_2 \\ \hline -\mathrm{I}_2 & 0 & & \\ 0 & \mathrm{I}_2 & & \end{array}\right), \tag{A.3}$$

with $\sigma^\mu = (1_2, -\sigma_1, -\sigma_2, -\sigma_3)$ and $\bar{\sigma}^\mu = (1_2, \sigma_1, \sigma_2, \sigma_3)$. In this basis, the 4D chirality and the inner chirality are expressed with the following diagonal forms:

$$\begin{aligned} \Gamma^5 &\equiv i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \mathrm{I}_2 \otimes (i\gamma^0\gamma^1\gamma^2\gamma^3) \\ &= \mathrm{I}_2 \otimes \gamma^5 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & \gamma^5 \end{pmatrix} = \left(\begin{array}{cc|cc} \mathrm{I}_2 & 0 & & \\ 0 & -\mathrm{I}_2 & & \\ \hline & & \mathrm{I}_2 & 0 \\ & & 0 & -\mathrm{I}_2 \end{array}\right), \end{aligned} \tag{A.4}$$

$$\Gamma^y \equiv i\Gamma^{y_1}\Gamma^{y_2}$$

$$= \sigma_3 \otimes \mathrm{I}_4 = \begin{pmatrix} \mathrm{I}_4 & 0 \\ 0 & -\mathrm{I}_4 \end{pmatrix} = \left(\begin{array}{cc|cc} \mathrm{I}_2 & 0 & & \\ 0 & \mathrm{I}_2 & & \\ \hline & & -\mathrm{I}_2 & 0 \\ & & 0 & -\mathrm{I}_2 \end{array}\right). \tag{A.5}$$

As a result, the eight-component spinors $\Psi_{R/L,\pm}$, which are simultaneous eigenstates of Γ^5 and Γ^y, are expressed in terms of two-component spinors $\xi_{R/L,\pm}$ as

$$\Psi_{R,+} = \begin{pmatrix} \xi_{R,+} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_{L,+} = \begin{pmatrix} 0 \\ \xi_{L,+} \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_{R,-} = \begin{pmatrix} 0 \\ 0 \\ \xi_{R,-} \\ 0 \end{pmatrix}, \quad \Psi_{L,-} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi_{L,-} \end{pmatrix}. \tag{A.6}$$

Appendix B. Six-dimensional charge conjugation

In this appendix, we show the definition of the 6D charge conjugation, C:

$$\begin{aligned} C : \Psi(x,y) \to \ \Psi^{(C)}(x,y) &= C\overline{\Psi}^{\mathrm{T}}(x,y) \\ &= C(\Gamma^0)^{\mathrm{T}}\Psi^*(x,y). \end{aligned} \tag{B.1}$$

In the 6D case, the charge conjugation matrix C satisfies the following relations:

$$C^{-1}\Gamma^M C = -(\Gamma^M)^{\mathrm{T}}, \tag{B.2}$$

$$C^\dagger C = \mathrm{I}_8, \tag{B.3}$$

$$C^{\mathrm{T}} = C. \tag{B.4}$$

In general, we have two choices for 6D charge conjugation:

$$C_\eta^{-1}\Gamma^M C_\eta = \eta(\Gamma^M)^{\mathrm{T}}, \tag{B.5}$$

$$C_\eta^\dagger C_\eta = \mathrm{I}_8, \tag{B.6}$$

$$C_\eta^{\mathrm{T}} = -\eta^3 C_\eta \quad (\eta = \pm 1). \tag{B.7}$$

For concrete discussions, we adopt the form

$$C = i\sigma_2 \otimes C^{(4\mathrm{D})},$$

where $C^{(4\mathrm{D})} = i\gamma^2\gamma^0$ is the 4D charge conjugation matrix.

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