Operator analysis of physical states on magnetized T^2/Z_N orbifolds

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Abstract

We discuss an effective way for analyzing the system on the magnetized twisted orbifolds in operator formalism, especially in the complicated cases T^2/Z_3, T^2/Z_4 and T^2/Z_6. We can obtain the exact and analytical results which can be applicable for any larger values of the quantized magnetic flux M, and show that the (non-diagonalized) kinetic terms are generated via our formalism and the number of the surviving physical states are calculable in a rigorous manner by simply following usual procedures in linear algebra in any case. Our approach is very powerful when we try to examine properties of the physical states on (complicated) magnetized orbifolds T^2/Z_3, T^2/Z_4, T^2/Z_6 (and would be in other cases on higher-dimensional torus) and could be an essential tool for actual realistic model construction based on these geometries. (Note: This article is registered under preprint number: arXiv:1409.5421.)

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1. Introduction

Even through the mass of the Higgs boson had been measured precisely at the CERN Large Hadron Collider, some other related topics are still veiled, *e.g.*, the origin of the generations, the mass hierarchy and the mixings of the Standard Model (SM) fermionic particles. Extra dimensions can call us fascinating directions, especially when we try to solve the above problems.

Plenty of models have been proposed to date, and in this paper, we focus on torus compactification with magnetic flux [1–5].[1] This possibility holds lots of captivating aspects. Chiral theory can be realized as a four-dimensional low energy effective theory on the background. Zero-mode equations are analytically solvable, apart from further nontrivial background, *e.g.*, Calabi–Yau manifolds, and, interestingly, their profiles are split and quasi-localized. The former point naturally leads to the nature with three generations, and the latter aspect would promise a natural explanation for the drastic hierarchies in the masses and the mixing patterns in the fermionic sector of the SM.[2] Along this direction, several studies have been done to pursue (further realistic) models and to search for the phenomenological aspects on, namely, Yukawa couplings [5],[3] realization of quark/lepton masses and their mixing angles [20,21], higher order couplings [22], flavor symmetries [23–29],[4] massive modes [33], and others [34–45].

Another important manipulation in higher-dimensional model building is orbifolding. By adding discrete symmetries on original backgrounds via this mechanism, we can realize supersymmetry breaking [46,47], removing exotic particles and breaking down gauge symmetries [48–50]. On two-dimensional torus T^2 (without magnetic flux), not only the simplest Z_2 case, also more complicated twisted orbifolds T^2/Z_N for $N = 3, 4, 6$ can be constructed [47] and their geometrical aspects are discussed [51–53] within the context of string theory. In a higher-dimensional field theory, detailed studies of SU(N) and SO(N) gauge theory have been carried out [54–58]. Furthermore on T^6, which has much amount of degrees of freedom compared with T^2, other complex patterns are possible like T^6/Z_7, T^6/Z_8, T^6/Z_{12} and so on.

Here, we can also consider to combine the two ideas. On orbifolded background geometries, all the states are classified under the eigenvalue (parity) of orbifold, and some zero-mode particles are projected out. Also, mode functions are deformed from the original ones, which could be helpful when we try to realize more realistic SM flavor structure. The simplest case of twisted orbifold with magnetic flux T^2/Z_2 was studied in [59,60],[5] and other cases on higher-dimensional torus (with magnetic flux) T^6/Z_2 and $T^6/(Z_2 \times Z_2')$ were also already analyzed [59, 60]. More nontrivial twisted orbifolds on T^2, namely, T^2/Z_3, T^2/Z_4, T^2/Z_6 with magnetic flux was recently scrutinized in Ref. [63]. On these magnetized orbifolds, nontrivial (discrete) Scherk–Schwarz phases [64,65] and (discrete) Wilson line phases [52,66,67] play an important

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[1] See for string magnetized D-brane models [6,7] and references therein.

[2] Another possible way to tackle these problems is to introduce point interactions (zero-thickness branes) in the bulk space of a five-dimensional theory on S^1 and consider various boundary conditions of fields on them [8–10].

[3] Within the framework of superstring theory, magnetized D-brane models are T-dual of intersecting D-brane models [6,7]. Yukawa couplings are also computed in intersecting D-brane models [11–14]. See also Refs. [15–19]

[4] A similar flavor symmetry can be obtained in heterotic string theory on an orbifold [30] (see also [31,32]).

[5] See for heterotic models on magnetized orbifolds [61] and also for shifted T^2/Z_N orbifold models with magnetic flux [62].

role.[6] Especially in T^2/Z_3 and T^2/Z_6, nonzero values of Scherk–Schwarz phases and/or Wilson line phases are mandatory for defining these magnetized orbifolds consistently [63].

On the T^2/Z_3, T^2/Z_4, T^2/Z_6 magnetized orbifolds, constructing mode functions itself is still formally possible following the usual prescription. However, analyzing the number of surviving physical states and writing down four-dimensional (4D) effective Lagrangians with suitable field normalizations are highly nontrivial because the forms of mode functions on magnetized T^2/Z_N are very entangled, where we should consider (weighted) linear combinations of complicated original functions on magnetized T^2 which include theta functions for all the mass-degenerated states. According to our previous analysis based on numerical computations in [63], the number of degenerated states we should consider tends to be quite large for realizing three generations, especially in the case of T^2/Z_6, it culminates in 24. Only focusing on efficiency, exact analytical evaluation of the above issues is desirable.

In this paper, we declare that operator formalism gives us a remedy for analytic computations of important physical quantities. The way discussed in operator formalism is very powerful when we try to examine properties of the physical states on (complicated) magnetized orbifolds T^2/Z_3, T^2/Z_4, T^2/Z_6 and could be an essential tool for actual (realistic) model construction based on these geometries. We would like to note that we can derive all the results on the number of physical modes in exact and analytic ways based on operator formalism, while it was evaluated in Ref. [63] relying on (huge) numerical calculations.

This paper is constructed as follows. In Section 2, we review the previous wavefunction analysis on magnetized T^2/Z_N orbifolds. In Section 3, with keeping in mind the difficulties of the wavefunction analysis, we show how to describe the system on T^2 with magnetic flux in operator formalism. Based on the knowledge, in Section 4, we analyze the system on magnetized orbifolds T^2/Z_2 and complicated T^2/Z_3, T^2/Z_4, T^2/Z_6 with operator formalism step by step and lead the results in the exact way. After that, in Section 5, we confirm the correspondence between analysis with actual forms of wavefunctions and that in operator formalism. Section 6 is devoted to the conclusions and discussions. In Appendix A, we have a discussion on physical degrees of freedom in the quantum mechanical system in operator formalism through large gauge transformations. In Appendix B, derivations of the formulas which we use in Section 4 is supplied.

2. Brief review of the wavefunction analysis

In this section, we review the wavefunction analysis of the magnetized T^2/Z_N orbifold models [63].

In the previous wavefunction analysis, Z_N orbifolds ($N = 2, 3, 4, 6$) were introduced for the torus T^2 with a homogeneous magnetic flux and the structure of the generations, which is a key ingredients for the generation problem, was evaluated. However, the analysis totally relied on the numerical calculations and has limitations to derive the analytic results. Moreover, there are several unclear points, e.g., the meaning of the degeneracy index j of the wave functions $f_{T^2,\psi_{+,0}}^{(j+\alpha_1,\alpha_\tau)}(z; a_w)$, $g_{T^2,\psi_{+,0}}^{(j+\alpha_1,\alpha_\tau)}(z; a_w)$ (see Eq. (2.11)). The index j is expected to be an eigenvalue of an operator though no one finds such an operator so far.

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[6] Also in intersecting D-brane models, Scherk–Schwarz phases were discussed in [68] and discrete Wilson line phases were studied in [69] (see also [70,71]).

As we will see in this section, the difficulties to derive the exact results appear from the Z_N-rotated zero-mode functions. On the magnetized T^2/Z_N orbifold, the physical states consist of the proper linear combinations of the Z_N-rotated zero modes. The Z_N-rotated zero modes also can be expanded by the original zero modes on T^2 since both of them satisfy the same zero-mode equation and the boundary conditions. Then, we need to compute the expansion coefficients of the Z_N-rotated states to find the number of physical states on T^2/Z_N. However, the derivation of the expansion coefficients with the general magnetic flux is very difficult in an analytic way because the zero-mode functions contain theta functions.

We will see in Section 4 that, in the operator analysis, we can overcome these difficulties and can obtain exact analytic results for the number of physical states and the expansion coefficients. The correspondence between the wavefunction analysis and the operator analysis can be found in Section 5.

2.1. Fermion zero-mode wavefunctions on T^2 and T^2/Z_N with magnetic flux

We consider the 6D action on $M^4 \times T^2$ with magnetic flux [5,72]

$$\int_{M^4} d^4x \int_{T^2} dz d\bar{z} \{ i \bar{\Psi}_+ \Gamma^M D_M \Psi_+ \}, \tag{2.1}$$

where the capital roman indices M, N run over μ $(= 0, 1, 2, 3)$, z and $\bar{z}$. The complex coordinate z $(\bar{z})$, which is very useful for evaluating the actual forms of wavefunctions, is defined as $z = y_1 + i y_2$ $(\bar{z} = y_1 - i y_2)$, where y_1 and y_2 are two Cartesian coordinates. Here, we take the bases of the torus as $\mathbf{u}_1 = (1, 0)^{\mathrm{T}}$, $\mathbf{u}_2 = (\mathrm{Re}\,\tau, \mathrm{Im}\,\tau)^{\mathrm{T}}$ with the modulus parameter τ $(\tau \in \mathbb{C}, \mathrm{Im}\,\tau > 0)$ for convenience.

Ψ_+ is a 6D Weyl fermion with 6D positive chirality and the covariant derivative D_M $(:= \partial_M - i q A_M)$ represents the gauge interaction with a $U(1)$ gauge field A_M with the background configurations $A_z^{(b)}$ and $A_{\bar{z}}^{(b)}$, where q is a $U(1)$ charge. On T^2, the complex coordinate z is identified as $z \sim z + 1 \sim z + \tau$, the counterpart of which in operator formalism is found in Eq. (3.5). When we consider the case with a 6D Weyl fermion with 6D negative chirality Ψ_-, the resultant 4D chirality is simply flipped. We use the notations for representations of 6D Clifford algebra and complex coordinates adopted in Ref. [63].

The vector potential $A^{(b)}$ describing the magnetic flux $b = \int_{T^2} F$ through the field strength $F = \frac{ib}{2\,\mathrm{Im}\,\tau} dz \wedge d\bar{z}$ can be written as

$$\begin{aligned} A^{(b)}(z) &= \frac{b}{2\,\mathrm{Im}\,\tau} \mathrm{Im}\big[(\bar{z} + \bar{a}_w) dz\big] \\ &= \frac{b}{4i\,\mathrm{Im}\,\tau}(\bar{z} + \bar{a}_w) dz - \frac{b}{4i\,\mathrm{Im}\,\tau}(z + a_w) d\bar{z} \\ &=: A_z^{(b)}(z)\, dz + A_{\bar{z}}^{(b)}(z)\, d\bar{z}, \end{aligned} \tag{2.2}$$

where a_w is a complex Wilson line phase. From Eq. (2.2), we obtain

$$\begin{aligned} A^{(b)}(z+1) &= A^{(b)}(z) + \frac{b}{2\,\mathrm{Im}\,\tau}\,\mathrm{Im}\,dz =: A^{(b)}(z) + d\chi_1(z + a_w), \\ A^{(b)}(z+\tau) &= A^{(b)}(z) + \frac{b}{2\,\mathrm{Im}\,\tau}\,\mathrm{Im}(\bar{\tau} dz) =: A^{(b)}(z) + d\chi_\tau(z + a_w), \end{aligned} \tag{2.3}$$

where $\chi_1(z + a_w)$ and $\chi_\tau(z + a_w)$ are given by[7]

$$\chi_1(z + a_w) = \frac{b}{2 \operatorname{Im} \tau} \operatorname{Im}(z + a_w),$$

$$\chi_\tau(z + a_w) = \frac{b}{2 \operatorname{Im} \tau} \operatorname{Im}[\bar{\tau}(z + a_w)]. \tag{2.4}$$

Here, the Lagrangian density in Eq. (2.1) should be single-valued under the identification $z \sim z + 1 \sim z + \tau$, and thereby the field $\Psi_+(x, z)$ should satisfy the pseudo-periodic boundary conditions

$$\Psi_+(x, z + 1) = U_1(z)\Psi_+(x, z), \qquad \Psi_+(x, z + \tau) = U_\tau(z)\Psi_+(x, z), \tag{2.5}$$

with

$$U_1(z) := e^{iq\chi_1(z + a_w) + 2\pi i \alpha_1}, \qquad U_\tau(z) := e^{iq\chi_\tau(z + a_w) + 2\pi i \alpha_\tau}, \tag{2.6}$$

where α_1 and α_τ are allowed to be any real numbers, and correspond to Scherk–Schwarz phases. The consistency of the contractible loops, e.g., $z \rightarrow z + 1 \rightarrow z + 1 + \tau \rightarrow z + \tau \rightarrow z$, requires the magnetic flux quantization condition,

$$\frac{qb}{2\pi} =: M \in \mathbb{Z}. \tag{2.7}$$

Then, $U_1(z)$ and $U_\tau(z)$ satisfy

$$U_1(z + \tau)U_\tau(z) = U_\tau(z + 1)U_1(z). \tag{2.8}$$

It should be emphasized that all of the Wilson line phases and the Scherk–Schwarz phases can be arbitrary on T^2, but are not physically independent because the Wilson line phases can be absorbed into the Scherk–Schwarz phases by a redefinition of fields and vice versa. This fact implies that we can take, for instance, the basis of vanishing Wilson line phases, without any loss of generality. It is then interesting to point out that allowed Scherk–Schwarz phases are severely restricted for T^2/Z_N orbifold models, as we will see in Section 4, while there is no restriction on the Scherk–Schwarz phases for T^2 models.

It is known that the zero-mode states of Ψ_+ become chiral and multiple due to the effect of the magnetic flux

$$\Psi_{+,0}(x, z) = \begin{cases} \sum_{j=0}^{|M|-1} \begin{pmatrix} \psi_{R,0}^{(j)}(x) \\ 0 \end{pmatrix} f_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w) & \text{for } M > 0, \\ \sum_{j=0}^{|M|-1} \begin{pmatrix} 0 \\ \psi_{L,0}^{(j)}(x) \end{pmatrix} g_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w) & \text{for } M < 0, \end{cases} \tag{2.9}$$

where $j = 0, 1, \ldots, |M| - 1$ is just an index for $|M|$-degenerated states and $\psi_{R,0}^{(j)}(x)$ or $\psi_{L,0}^{(j)}(x)$ are 4D chiral zero modes. The mode functions $f_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w)$ and $g_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w)$ are the solutions of the zero-mode equations

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[7] Note that we can freely add constants in the definition of χ_1 and χ_τ without changing the relation (2.3). Here, we chose such constants, as in Eq. (2.4), for later convenience.

$$\left(\partial_{\bar{z}} + \frac{\pi M}{2\operatorname{Im}\tau}(z + a_w)\right)f_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w) = 0,$$

$$\left(\partial_z - \frac{\pi M}{2\operatorname{Im}\tau}(\bar{z} + \bar{a}_w)\right)g_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w) = 0, \tag{2.10}$$

and take the forms

$$f_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w) = \mathcal{N}e^{i\pi M(z+a_w)\frac{\operatorname{Im}(z+a_w)}{\operatorname{Im}\tau}} \cdot \vartheta \begin{bmatrix} \frac{j+\alpha_1}{M} \\ -\alpha_\tau \end{bmatrix}\left(M(z + a_w), M\tau\right),$$

$$g_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w) = \mathcal{N}e^{i\pi M(\bar{z}+\bar{a}_w)\frac{\operatorname{Im}(\bar{z}+\bar{a}_w)}{\operatorname{Im}\bar{\tau}}} \cdot \vartheta \begin{bmatrix} \frac{j+\alpha_1}{M} \\ -\alpha_\tau \end{bmatrix}\left(M(\bar{z} + \bar{a}_w), M\bar{\tau}\right), \tag{2.11}$$

with the normalization factor $\mathcal{N}$. For $a_w = 0$ and $(\alpha_1, \alpha_\tau) = (0,0)$, $\psi_{\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w)$ are reduced to the results obtained in Ref. [5]. Here, $\mathcal{N}$ may be fixed by the orthonormality condition

$$\int_{T^2} dz d\bar{z} \left(f_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w)\right)^* f_{T^2,\Psi_+,0}^{(k+\alpha_1,\alpha_\tau)}(z; a_w) = \delta_{jk},$$

$$\int_{T^2} dz d\bar{z} \left(g_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z; a_w)\right)^* g_{T^2,\Psi_+,0}^{(k+\alpha_1,\alpha_\tau)}(z; a_w) = \delta_{jk}. \tag{2.12}$$

It should be stressed that although j ($= 0, 1, 2, \ldots, |M| - 1$) in Eq. (2.11) is the index that distinguishes the $|M|$ degenerate zero-modes and is expected an eigenvalue of some hermitian operator, the form of such operator is unclear in the wavefunction approach. We will later clarify the operator, which is crucial to evaluate the exact analytic results, in operator formalism.

The ϑ function is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(c\nu, c\tau) = \sum_{l=-\infty}^{\infty} e^{i\pi(a+l)^2 c\tau} e^{2\pi i(a+l)(c\nu+b)}, \tag{2.13}$$

with the properties

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(c(\nu + n), c\tau) = e^{2\pi iacn}\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(c\nu, c\tau),$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(c(\nu + n\tau), c\tau) = e^{-i\pi cn^2\tau - 2\pi in(c\nu+b)}\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(c\nu, c\tau),$$

$$\vartheta \begin{bmatrix} a+m \\ b+n \end{bmatrix}(c\nu, c\tau) = e^{2\pi ian}\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(c\nu, c\tau), \tag{2.14}$$

where a and b are real numbers, c, m and n are integers, and ν and τ are complex numbers with $\operatorname{Im}\tau > 0$.

Now, we explicitly write down a part of the 4D effective Lagrangian describing fermion kinetic terms. Through Eqs. (2.9), (2.11) and (2.12), when $M > 0$, Eq. (2.1) leads to the following zero-mode part

$$\sum_{j,k=0}^{|M|-1}\left\{i\bar{\psi}_{R,0}^{(j)}[\delta_{jk}]\gamma^\mu\partial_\mu\psi_{R,0}^{(k)}\right\} = \sum_{j=0}^{|M|-1}\left\{i\bar{\psi}_{R,0}^{(j)}\gamma^\mu\partial_\mu\psi_{R,0}^{(j)}\right\}, \tag{2.15}$$

where we obtain a $|M|$-generation chiral theory. When we consider $M < 0$, the chirality of the fermions turns out to be left-handed.

After imposing Z_N-orbifolding on T^2 with $\omega = e^{2\pi i/N}$ in which the modulus parameter τ was identified with ω for T^2/Z_3, T^2/Z_4 and T^2/Z_6, it had been found that the allowed Scherk–Schwarz phases (α_1, α_τ) are restricted to the specific values in the basis of $a_w = 0$ [63]. We should mention that we can always move to the basis in which the Wilson line phase a_w vanishes without any loss of generality through a large gauge transformation. Thus, in the following, we concentrate on the situation

$$a_w = 0. \tag{2.16}$$

On the orbifolds, the Z_N-orbifolded mode functions,

$$f^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(z;0)_\eta = \mathcal{N}^{(j)}_{R,\eta} \sum_{x=0}^{N-1} \bar{\eta}^x f^{(j+\alpha_1,\alpha_\tau)}_{T^2,\Psi_+,0}(\omega^x z;0),$$

$$g^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(z;0)_{\omega\eta} = \mathcal{N}^{(j)}_{L,\omega\eta} \sum_{x=0}^{N-1} (\bar{\omega}\bar{\eta})^x g^{(j+\alpha_1,\alpha_\tau)}_{T^2,\Psi_+,0}(\omega^x z;0), \tag{2.17}$$

with the normalizing factors $\mathcal{N}^{(j)}_{R,\eta} = \mathcal{N}^{(j)}_{L,\omega\eta} = 1/N$,[8] fulfill the following conditions:

$$f^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(\omega z;0)_\eta = \eta f^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(z;0)_\eta,$$

$$g^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(\omega z;0)_{\omega\eta} = \omega\eta \, g^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(z;0)_{\omega\eta}, \tag{2.18}$$

where η is one of the possible eigenvalues $\eta \in \{1, \omega, \omega^2, \dots, \omega^{N-1}\}$. Here, the subscript η of the function $f^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(\omega z;0)_\eta$ denotes the eigenvalue of $f^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(\omega z;0)_\eta$ under the Z_N rotation: $z \to \omega z$.

We note that the Z_N-rotated mode function $f^{(j+\alpha_1,\alpha_\tau)}_{T^2,\Psi_+,0}(\omega^x z;0)_\eta$ $(g^{(j+\alpha_1,\alpha_\tau)}_{T^2,\Psi_+,0}(\omega^x z;0)_{\omega\eta})$ in Eq. (2.17) can be expanded by the original mode functions as

$$f^{(j+\alpha_1,\alpha_\tau)}_{T^2,\Psi_+,0}(\omega^x z;0) = \sum_{k=0}^{|M|-1} C^{(\omega^x)}_{jk} f^{(k+\alpha_1,\alpha_\tau)}_{T^2,\Psi_+,0}(z;0), \tag{2.19}$$

with the expansion coefficients

$$C^{(\omega^x)}_{jk} = \int dz d\bar{z} \, f^{(j+\alpha_1,\alpha_\tau)}_{T^2,\Psi_+,0}(\omega^x z;0)\big(f^{(k+\alpha_1,\alpha_\tau)}_{T^2,\Psi_+,0}(z;0)\big)^*, \tag{2.20}$$

since they satisfy the same zero-mode equation (2.10) and the boundary conditions (2.5). As in the T^2 case of Eq. (2.9), the zero modes on T^2/Z_N are represented as follows:

$$\Psi_{+,0}(x,z) = \begin{cases} \sum_{j=0}^{|M|-1} \begin{pmatrix} \psi^{(j)}_{R,0}(x) \\ 0 \end{pmatrix} f^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(z;0)_\eta & \text{for } M > 0, \\ \sum_{j=0}^{|M|-1} \begin{pmatrix} 0 \\ \psi^{(j)}_{L,0}(x) \end{pmatrix} g^{(j+\alpha_1,\alpha_\tau)}_{T^2/Z_N,\Psi_+,0}(z;0)_{\omega\eta} & \text{for } M < 0. \end{cases} \tag{2.21}$$

In the T^2/Z_N cases with η, the fermion kinetic terms could be evaluated like the previous T^2

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[8] This factor depends on a choice of the range of the integration $\int dz d\bar{z}$. $1/N$ corresponds to the case $\int_{T^2} dz d\bar{z}$ (after the orbifolding).

case as

$$\sum_{j,k=0}^{|M|-1} \{i\bar{\psi}_{R,0}^{(j)}[\mathcal{K}_{jk}^{(Z_N;\eta)}]\gamma^\mu \partial_\mu \psi_{R,0}^{(k)}\}, \tag{2.22}$$

with the kinetic matrix $\mathcal{K}_{jk}^{(Z_N;\eta)}$

$$\mathcal{K}_{jk}^{(Z_N;\eta)} = \int_{T^2} dz d\bar{z} \big(f_{T^2/Z_N,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_\eta\big)^* f_{T^2/Z_N,\Psi_+,0}^{(k+\alpha_1,\alpha_\tau)}(z)_\eta$$

$$= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \eta^x \bar{\eta}^y \sum_{m=0}^{|M|-1} \big(C_{jm}^{(\omega^x)}\big)^* C_{km}^{(\omega^y)}. \tag{2.23}$$

The matrix $\mathcal{K}_{jk}^{(Z_N;\eta)}$ is generally non-diagonal $|M|$-by-$|M|$ matrix because the T^2/Z_N mode functions are constructed as linear combinations of Z_N-rotated zero-mode functions on T^2 (see Eq. (2.17)). An important point is that the rank of the matrix $\mathcal{K}_{jk}^{(Z_N;\eta)}$ indicates the number of physical states on T^2/Z_N orbifold:

$$\text{The number of physical states} = \text{Rank}\big[\mathcal{K}_{jk}^{(Z_N;\eta)}\big]. \tag{2.24}$$

To estimate the matrix $\mathcal{K}_{jk}^{(Z_N;\eta)}$ with general $|M|$, we need to evaluate the expansion coefficients (2.20) thereby the integral (2.23) can be executed. However, it is enormously difficult to do it for large $|M|$ in the wavefunction analysis because of the existence of the theta functions (2.11), though numerical calculations of Eq. (2.20) could be performed for small $|M|$.

As we will see in Section 4, in the operator analysis, we can overcome these difficulties and can obtain the exact analytic results for the number of physical states and the expansion coefficients. The hermitian operator, which has relation to the degeneracy index j in the wavefunction (2.11), also becomes clear in Section 3.

3. Operator formalism for T^2 with magnetic flux

Before we consider Z_N-orbifolds, we formulate a quantum mechanical system on T^2 with a $U(1)$ homogeneous magnetic flux like in Ref. [63]. Its energy spectrum should correspond to the mass spectrum of the six-dimensional (6D) system $M^4 \times T^2$, where M^4 means four-dimensional (4D) Minkowski spacetime. Due to compactness of the system, we face to additional constraints on the system, part of which describes degeneracy of the allowed physical states.

At first, we describe the system with the wavefunction $\psi(\mathbf{y})$, where we adopt the vector notation on T^2 as $\mathbf{y} := (y_1, y_2)^T$ for two Cartesian coordinates y_1 and y_2. We consider the Hamiltonian H and the corresponding Schrödinger equation with energy E as

$$H = \big(-i\nabla - q\mathbf{A}(\mathbf{y})\big)^2, \qquad H\psi(\mathbf{y}) = E\psi(\mathbf{y}), \tag{3.1}$$

where we use the vector notations of $\nabla := (\partial_{y_1}, \partial_{y_2})^T$ and $\mathbf{A} := (A_{y_1}, A_{y_2})^T$. The vector potential $\mathbf{A}$ providing a homogeneous magnetic flux penetrating T^2 can be expressed as

$$\mathbf{A}(\mathbf{y}) = -\frac{1}{2}\Omega(\mathbf{y} + \mathbf{a}), \tag{3.2}$$

or with showing all the indices explicitly

$$A_i(\mathbf{y}) = -\frac{1}{2} \sum_{j=1}^{2} \Omega_{ij}(y_j + a_j) \quad (i = 1, 2). \tag{3.3}$$

Here, we mention that only the antisymmetric part of Ω possesses a physical degree of freedom, while the symmetric part of Ω depending on a choice of gauge is unphysical and can be gauged away. In a later stage, we suitably fix the gauge to make the system simplified.

Two-dimensional torus T^2 is defined from the two-dimensional plane $\mathbb{R}^2$ by modding out the lattice shift Λ with two basis vectors $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$T^2 = \mathbb{R}^2/\Lambda, \quad \Lambda = \left\{ \sum_{a=1}^{2} n_a \mathbf{u}_a \,\bigg|\, n_a \in \mathbb{Z} \right\}, \tag{3.4}$$

where n_1 and n_2 show the numbers of shifting along $\mathbf{u}_1$ and $\mathbf{u}_2$ directions, respectively. This means that the vector coordinate $\mathbf{y}$ should obey the identification

$$\mathbf{y} \sim \mathbf{y} + \sum_{a=1}^{2} n_a \mathbf{u}_a. \tag{3.5}$$

The above condition leads to two requirements for ensuring consistency. The first one puts a constraint on the form of the wavefunction after shifting $\psi(\mathbf{y} + \mathbf{u}_a)$. In order to make the Schrödinger equation well-defined on T^2, the following pseudo-periodic boundary condition should be satisfied,

$$\psi(\mathbf{y} + \mathbf{u}_a) = e^{-i\frac{q}{2}(\mathbf{y}+\mathbf{a})^{\mathrm{T}}\Omega\mathbf{u}_a + 2\pi i \alpha_a} \psi(\mathbf{y}) \quad \text{for } a = 1, 2, \tag{3.6}$$

where α_a shows the y-independent Scherk–Schwarz phase for the $\mathbf{u}_a$ direction.[9] In addition, after considering contractible loops on T^2, *e.g.*, $\mathbf{y} \to \mathbf{y} + \mathbf{u}_1 \to \mathbf{y} + \mathbf{u}_1 + \mathbf{u}_2 \to \mathbf{y} + \mathbf{u}_2 \to \mathbf{y}$, the magnetic flux should be quantized to ensure the single-valuedness of the wavefunction $\psi(\mathbf{y})$ as follows:

$$q\mathbf{u}_a^{\mathrm{T}} B \mathbf{u}_b = 2\pi Q_{ab} \quad (a, b = 1, 2), \tag{3.7}$$

where $B := \frac{1}{2}(\Omega - \Omega^{\mathrm{T}})$ is the gauge-independent antisymmetric part of Ω and $Q_{ab} = -Q_{ba} \in \mathbb{Z}$.

Now, we go to the operator formalism, where we introduce a momentum operator $\widehat{\mathbf{p}} := -i\nabla$ being conjugate to $\mathbf{y}$. The operators satisfy the canonical commutation relations,

$$[\widehat{y_i}, \widehat{p_j}] = i\delta_{i,j} \quad (\text{the others}) = 0 \quad (i, j = 1, 2). \tag{3.8}$$

The wavefunction $\psi(\mathbf{y})$ is represented in the operator formulation as $\langle \mathbf{y} | \psi \rangle$ and the system is rewritten by use of $|\psi\rangle$ as

$$\widehat{H} = \left(\widehat{\mathbf{p}} + \frac{q}{2}\Omega(\widehat{\mathbf{y}} + \mathbf{a}) \right)^2 =: (\widehat{\mathbf{p}}')^2, \quad \widehat{H}|\psi\rangle = E|\psi\rangle. \tag{3.9}$$

---

[9] In this paper, the Scherk–Schwarz phases only represent twisted boundary conditions, not supersymmetry breaking.

The constraints in Eq. (3.6) are also interpreted as

$$e^{i\widehat{T}_a - i\frac{q}{4}\mathbf{u}_a^{\mathrm{T}}\Omega\mathbf{u}_a} \,|\psi\rangle = e^{2\pi i\alpha_a}\,|\psi\rangle\,, \qquad \widehat{T}_a = \mathbf{u}_a^{\mathrm{T}}\left(\widehat{\mathbf{p}} + \frac{q}{2}\Omega^{\mathrm{T}}(\widehat{\mathbf{y}} + \mathbf{a})\right) \quad (a = 1, 2). \tag{3.10}$$

Here, we can find the relation

$$\mathbf{u}_a^{\mathrm{T}}\Omega\mathbf{u}_a = \mathbf{u}_a^{\mathrm{T}}\left(\frac{1}{2}(\Omega + \Omega^{\mathrm{T}})\right)\mathbf{u}_a, \tag{3.11}$$

which says that only unphysical components appear in this part. Hence in the following part, we drop the unphysical symmetric part of Ω,

$$\frac{1}{2}(\Omega + \Omega^{\mathrm{T}}) = 0, \tag{3.12}$$

which leads to the condition $\mathbf{u}_a^{\mathrm{T}}\Omega\mathbf{u}_a = 0$. We can define new operators,

$$\widehat{Y} = -\frac{\sqrt{2}}{\omega}\widehat{p}_2', \qquad \widehat{P} = \sqrt{2}\widehat{p}_1', \tag{3.13}$$

$$\widehat{\widetilde{Y}} = \frac{1}{2\pi M}\widehat{T}_1 - \frac{\alpha_1}{M}, \qquad \widehat{\widetilde{P}} = -\widehat{T}_2 + 2\pi\alpha_2. \tag{3.14}$$

where $\omega = 2qB_{12} = 2q\Omega_{12}$ and $M := Q_{12} \subset \mathbb{Z}$. The transformation $\{\widehat{y}_i, \widehat{p}_i;\ i = 1, 2\} \to \{\widehat{Y}, \widehat{P}, \widehat{\widetilde{Y}}, \widehat{\widetilde{P}}\}$ is canonical and then the operators are suitably quantized with the canonical commutation relations,

$$[\widehat{Y}, \widehat{P}] = i, \qquad [\widehat{\widetilde{Y}}, \widehat{\widetilde{P}}] = i, \qquad (\text{the others}) = 0. \tag{3.15}$$

As apparent from Eqs. (3.9) and (3.14), under the new variables, the Hamiltonian can be rephrased only with the two new variables $\widehat{P}$ and $\widehat{Y}$ as

$$\widehat{H} = \frac{1}{2}\widehat{P}^2 + \frac{\omega^2}{2}\widehat{Y}^2, \tag{3.16}$$

the form of which is the one-dimensional harmonic oscillator. On the other hand, the remaining two ones, $\widehat{\widetilde{P}}$ and $\widehat{\widetilde{Y}}$, work as constraint conditions on the state,

$$e^{i\widehat{\widetilde{P}}}\,|\psi\rangle = |\psi\rangle\,, \qquad e^{2\pi iM\widehat{\widetilde{Y}}}\,|\psi\rangle = |\psi\rangle\,, \tag{3.17}$$

where the two operators are considered to control degenerated states since they do not appear in the Hamiltonian (3.16) and they commute with the Hamiltonian (3.16). In the following part, we check this statement.

After we take an eigenstate of $\widehat{\widetilde{Y}}$, which obeys the relation $\widehat{\widetilde{Y}}\,|\widetilde{Y}\rangle = \widetilde{Y}\,|\widetilde{Y}\rangle$, the second condition in Eq. (3.17) is simplified,

$$e^{2\pi iM\widetilde{Y}}\,|\widetilde{Y}\rangle = |\widetilde{Y}\rangle. \tag{3.18}$$

Operating $e^{2\pi iM\widehat{\widetilde{Y}}}$ on the state $e^{ia\widehat{\widetilde{P}}}\,|\widetilde{Y}\rangle$ $(a \in \mathbb{R})$ and using the relations in Eq. (3.15), we can obtain the relation

$$e^{ia\widehat{\widetilde{P}}}\,|\widetilde{Y}\rangle = |\widetilde{Y} - a\rangle. \tag{3.19}$$

From Eqs. (3.17), (3.18) and (3.19), we can reach the periodic condition,

$$|\widetilde{Y}\rangle = |\widetilde{Y} - 1\rangle, \tag{3.20}$$

and also the coordinate quantization condition,

$$\widetilde{Y} = \frac{j}{M} \quad (j = 0, 1, 2, \ldots, |M| - 1). \tag{3.21}$$

These results imply that every energy state is $|M|$-fold degenerated and the index j discriminates them as

$$\widehat{H} \left| n, \frac{j}{M} \right\rangle = E_n \left| n, \frac{j}{M} \right\rangle, \quad E_n = |\omega| \left(n + \frac{1}{2} \right), \tag{3.22}$$

$$\left\langle m, \frac{i}{M} \middle| n, \frac{j}{M} \right\rangle = \delta_{m,n} \delta_{i,j}, \tag{3.23}$$

where $m, n = 0, 1, 2, \ldots$ and $i, j = 0, 1, 2, \ldots, |M| - 1$.

Here, we note three things. The first one is that there is no constraint on the Scherk–Schwarz phases α_1 and α_2 on the two-dimensional torus with magnetic flux. We will see in Section 4 that only the restricted values are allowed for the Scherk–Schwarz phase in the case of magnetized T^2/Z_N orbifolds.

The second one is that the index j for the degeneracy states is nothing but an eigenvalue of the operator $\widehat{\widetilde{Y}}$. We can easily check that the index j in wavefunction analysis is also the eigenvalue of $\widehat{\widetilde{Y}}$:

$$e^{2\pi i \widehat{\widetilde{Y}}} f_{T^2, \Psi_+, 0}^{(j+\alpha_1, \alpha_\tau)}(z; a_w) = e^{2\pi i \frac{j}{M}} f_{T^2, \Psi_+, 0}^{(j+\alpha_1, \alpha_\tau)}(z; a_w), \tag{3.24}$$

where the explicit form of the operator $\widehat{\widetilde{Y}}$ in the complex coordinate is given by the following,

$$e^{2\pi i \widehat{\widetilde{Y}}} = e^{2\pi i \cdot \frac{1}{2\pi M} \left(-i(\partial_z + \partial_{\bar{z}}) - \frac{\pi M}{\operatorname{Im} \tau} \operatorname{Im}(z + a_w) - 2\pi \alpha_1 \right)}. \tag{3.25}$$

On the other hand, the operator $\widehat{\widetilde{P}}$ acts as the translational operator with respect to the index j,

$$e^{-i \frac{1}{M} \widehat{\widetilde{P}}} f_{T^2, \Psi_+, 0}^{(j+\alpha_1, \alpha_\tau)}(z; a_w) = f_{T^2, \Psi_+, 0}^{((j+1)+\alpha_1, \alpha_\tau)}(z; a_w), \tag{3.26}$$

where the explicit form of the operator $\widehat{\widetilde{P}}$ in the complex coordinate is

$$e^{-i \frac{1}{M} \widehat{\widetilde{P}}} = e^{-i \frac{1}{M} \left(i(\tau \partial_z + \bar{\tau} \partial_{\bar{z}}) - \frac{\pi M}{\operatorname{Im} \tau} \operatorname{Im}[\tau(\bar{z} + \bar{a}_w)] + 2\pi \alpha_2 \right)}. \tag{3.27}$$

Thus, we have succeeded to clarify the form of the hermitian operator which has relation to the index j, as announced before.

The last one is that E_n correspond to eigenvalues of the Laplace operator with magnetic flux in Eq. (3.1), the value of which expresses the mass square of scalar field $m_n^2 = |\omega|(n + 1/2)$ when we consider the higher-dimensional field theory on $M^4 \times T^2$ with magnetic flux. In the cases of spinor and vector, there is a constant shift from the scalar case originating from their Lorentz structure and the explicit forms are $m_n^2 = |\omega| n$ and $m_n^2 = |\omega|(n - 1/2)$, respectively [5,33].

4. Operator formalism for T^2/Z_N twisted orbifolds with magnetic flux

Based on the knowledge in the previous section, now we discuss the operator formalism describing magnetized T^2/Z_N twisted orbifold. We should emphasize that our results, e.g., the number of physical states, are consistent with the previous numerical results in Ref. [63] and,

moreover, our results are analytically exact. As we mentioned in Section 2, it is quite nontrivial because of the following reasons.

In the wavefunction analysis, the physical states $f_{T^2/Z_N}^{(j)}(z)_\eta$ with a Z_N eigenvalue η on the magnetized T^2/Z_N twisted orbifold consist of linear combinations of Z_N-twisted zero-mode functions $f^{(j)}(\omega^k z)$ $(k = 0, 1, 2, \ldots, N-1)$:

$$f_{T^2/Z_N}^{(j)}(z)_\eta = \mathcal{N}_{R,\eta}^{(j)} \sum_{x=0}^{N-1} \bar{\eta}^x f_{T^2}^{(j)}(\omega^x z), \tag{4.1}$$

where we omit the unimportant subscript. Since the Z_N-twisted mode function $f_{T^2}^{(j)}(\omega^k z)$ satisfies the same equation as $f_{T^2}^{(j)}(z)$, $f_{T^2}^{(j)}(\omega^x z)$ has to be expanded in some linear combination of $f_{T^2}^{(j)}(z)$ as

$$f_{T^2}^{(j)}(\omega^x z) = \sum_{m=0}^{|M|-1} C_{jm}^{(\omega^x)} f_{T^2}^{(m)}(z). \tag{4.2}$$

If we can evaluate the coefficient $C_{jm}^{(\omega^x)}$ analytically, then we can obtain the results, *e.g.*, the number of independent physical states, analytically from Eq. (2.24). However, it is not easy since the mode functions $f_{T^2}^{(j)}(z)$ contain the theta functions (see Eq. (2.11)).

Amazingly, in the operator analysis, we can evaluate the coefficient $C_{jm}^{(\omega^x)}$ analytically as we will see in this section. Furthermore, the results of the operator formalism are available to not only the zero modes but also all KK modes. The correspondence between the numerical results in Ref. [63] and the analytic results of ours will be found in the next section.

The T^2/Z_N twisted orbifold enforces the discrete symmetry under $2\pi/N$-angle rotation, where it is well known that only the $N = 2, 3, 4, 6$ are possible on T^2. In the following part, the unitary operator $\widehat{U}_\theta$ manipulates the rotation with an angle θ around the origin ($y_1 = y_2 = 0$). In the vector coordinate $\mathbf{y}$, the rotation is described by the two-by-two representation matrix R_θ,

$$\mathbf{y} \to R_\theta \mathbf{y}, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{4.3}$$

where the same expression is valid for the operators $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{p}}$,

$$\widehat{\mathbf{y}} \to \widehat{U}_\theta \widehat{\mathbf{y}} \widehat{U}_\theta^\dagger = R_\theta \widehat{\mathbf{y}}, \qquad \widehat{\mathbf{p}} \to \widehat{U}_\theta \widehat{\mathbf{p}} \widehat{U}_\theta^\dagger = R_\theta \widehat{\mathbf{p}}. \tag{4.4}$$

In the following discussion, we take the Wilson line phase $\mathbf{a}$ as zero:

$$\mathbf{a} = \mathbf{0}. \tag{4.5}$$

It has already been discussed in Ref. [63] that we can remove the Wilson line phase by using a large gauge transformation without any loss of generality. We can find the discussion on the issue in operator formalism in Appendix A. By use of the results in Eq. (4.4), the transformation of the Hamiltonian in Eq. (3.9) under the rotation is evaluated as

$$\widehat{H} \to \widehat{U}_\theta \widehat{H} \widehat{U}_\theta^\dagger = \left(\widehat{\mathbf{p}} + \frac{q}{2} R_\theta^{\mathrm{T}} \Omega R_\theta \widehat{\mathbf{y}} \right)^2, \tag{4.6}$$

and hence the following condition is required for invariance:

$$[R_\theta, \Omega] = 0. \tag{4.7}$$

As shown in Eq. (3.11), only the unphysical symmetric part of Ω makes a nonzero contribution to the commutator $[R_\theta, \Omega]$. Therefore, the condition in Eq. (4.7) is realized after the gauge fixing (3.12) irrespective of the value of θ. Then, we can conclude that the Hamiltonian itself is invariant under the rotation in spite of the value of the angle θ.

Although the Hamiltonian is invariant under the rotation U_θ, situations in the constraints on states are found to be nontrivial. Remembering Eq. (3.10), the two operators $\widehat{T}_a$ ($a = 1, 2$) transform under the rotation generated by $\widehat{U}_\theta$ as

$$\widehat{T}_a \to \widehat{U}_\theta \widehat{T}_a \widehat{U}_\theta^\dagger = \left(R_\theta^{\mathrm{T}} \mathbf{u}_a\right)^{\mathrm{T}} \left(\widehat{\mathbf{p}} + \frac{q}{2} \Omega^{\mathrm{T}} \widehat{\mathbf{y}}\right) \quad (a = 1, 2). \tag{4.8}$$

Here, the vector $R_\theta^{\mathrm{T}} \mathbf{u}_a$ is given by a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, and subsequently, $\widehat{T}_1$ and $\widehat{T}_2$ are also mixed each other in general. It is convenient to choose the two basis vectors $\mathbf{u}_a$ in such a way that under the rotations generated by $(R_{\theta=2\pi/N})^{\mathrm{T}}$, the two basis vectors are transformed as

$$
\begin{aligned}
&Z_2 \text{ case:} && (R_{\theta=2\pi/N})^{\mathrm{T}} \mathbf{u}_1 = -\mathbf{u}_1, && (R_{\theta=2\pi/N})^{\mathrm{T}} \mathbf{u}_2 = -\mathbf{u}_2, \\
&Z_3 \text{ case:} && (R_{\theta=2\pi/N})^{\mathrm{T}} \mathbf{u}_1 = -\mathbf{u}_1 - \mathbf{u}_2, && (R_{\theta=2\pi/N})^{\mathrm{T}} \mathbf{u}_2 = +\mathbf{u}_1, \\
&Z_4 \text{ case:} && (R_{\theta=2\pi/N})^{\mathrm{T}} \mathbf{u}_1 = -\mathbf{u}_2, && (R_{\theta=2\pi/N})^{\mathrm{T}} \mathbf{u}_2 = +\mathbf{u}_1, \\
&Z_6 \text{ case:} && (R_{\theta=2\pi/N})^{\mathrm{T}} \mathbf{u}_1 = +\mathbf{u}_1 - \mathbf{u}_2, && (R_{\theta=2\pi/N})^{\mathrm{T}} \mathbf{u}_2 = +\mathbf{u}_1.
\end{aligned} \tag{4.9}
$$

Transformations of $\widehat{T}_a$ under the discrete rotations with angles $\theta = 2\pi/N$ in Z_N orbifoldings ($N = 2, 3, 4, 6$) are easily evaluated by use of the results in Eq. (4.9),

$$
\begin{aligned}
&Z_2 \text{ case:} && \widehat{T}_1 \to -\widehat{T}_1, \quad \widehat{T}_2 \to -\widehat{T}_2, && \tag{4.10} \\
&Z_3 \text{ case:} && \widehat{T}_1 \to -\widehat{T}_1 - \widehat{T}_2, \quad \widehat{T}_2 \to \widehat{T}_1, && \tag{4.11} \\
&Z_4 \text{ case:} && \widehat{T}_1 \to -\widehat{T}_2, \quad \widehat{T}_2 \to \widehat{T}_1, && \tag{4.12} \\
&Z_6 \text{ case:} && \widehat{T}_1 \to \widehat{T}_1 - \widehat{T}_2, \quad \widehat{T}_2 \to \widehat{T}_1. && \tag{4.13}
\end{aligned}
$$

To investigate the number of physical states on the above magnetized T^2/Z_N orbifold, we will construct physical states via the following three steps:

1. Derivation of the allowed Scherk–Schwarz phases (α_1, α_2)
   On T^2/Z_N, any Z_N-transformed state $\widehat{U}_{Z_N} |\psi\rangle$ should satisfy the same constraint condition as $|\psi\rangle$, which is a state on T^2.

$$e^{i\widehat{T}_a} \left(\widehat{U}_{Z_N} |\psi\rangle\right) = e^{2\pi i \alpha_a} \left(\widehat{U}_{Z_N} |\psi\rangle\right) \quad (a = 1, 2), \tag{4.14}$$

   where we already fixed a gauge (3.12) and we adopt a notation $\widehat{U}_{Z_N} := \widehat{U}_{2\pi/N}$. The above consistency condition will produce a restriction to the values of the Scherk–Schwarz phases (α_1, α_2) as a consequence of the Z_N orbifold. We will find that the number of allowed Scherk–Schwarz phases are the same as the number of fixed points on T^2/Z_N orbifold. See Appendix A, for detail.
2. Derivation of the analytic form of a Z_N-transformed state $(\widehat{U}_{Z_N})^x |\psi\rangle$ ($x = 0, 1, 2, \ldots, N - 1$)

Since the physical states $|\psi\rangle_{T^2/Z_N}$ on T^2/Z_N consist of linear combinations of the states $|\psi\rangle$ on T^2 with the projection operator $\widehat{\mathcal{P}}_{T^2/Z_N,\eta}$,

$$|\psi\rangle_{T^2/Z_N,\eta} = \widehat{\mathcal{P}}_{T^2/Z_N,\eta}|\psi\rangle, \tag{4.15}$$

$$\widehat{\mathcal{P}}_{T^2/Z_N,\eta} := \frac{1}{N}\left\{ \sum_{x=0}^{N-1} \bar{\eta}^x (\widehat{U}_{Z_N})^x \right\}, \tag{4.16}$$

we need to obtain the analytic form of the Z_N-transformed state $(\widehat{U}_{Z_N})^x|n,\frac{j}{M}\rangle$. The states $|n,\frac{j}{M}\rangle$ and $\widehat{U}_{Z_N}|n,\frac{j}{M}\rangle$ satisfy the same equation with each other so that $(\widehat{U}_{Z_N})^x|n,\frac{j}{M}\rangle$ should be expanded as follows:

$$(\widehat{U}_{Z_N})^x\left|n,\frac{j}{M}\right\rangle = \sum_{m=0}^{|M|-1} D_{jm}^{(\omega^x)}\left|n,\frac{m}{M}\right\rangle. \tag{4.17}$$

To obtain the analytic form of $(\widehat{U}_{Z_N})^x|n,\frac{j}{M}\rangle$, we need to calculate the coefficient $D_{jm}^{(\omega^x)}$. However, as we mentioned in the beginning of this chapter, it is quite difficult to evaluate $D_{jm}^{(\omega^x)}$ in the wavefunction analysis. It turns out that it can be evaluated analytically and exactly by using operator formalism in this section.

3. Construction of the physical states

With combining the above results, we can construct physical states on T^2/Z_N,

$$\begin{aligned}\left|n,\frac{j}{M}\right\rangle_{T^2/Z_N,\eta} &= \widehat{\mathcal{P}}_{T^2/Z_N,\eta}\left|n,\frac{j}{M}\right\rangle, \\ &= \frac{1}{N}\sum_{x=0}^{N-1} \bar{\eta}^x \sum_{k=0}^{|M|-1} D_{jk}^{(\omega^x)}\left|n,\frac{k}{M}\right\rangle.\end{aligned} \tag{4.18}$$

Thus, in the remaining part of this section, we first derive possible patterns of the two Scherk–Schwarz phases (α_1,α_2) for every Z_N-orbifold ($N = 2, 3, 4, 6$) from Eq. (4.14). We then calculate $D_{jm}^{(\omega^x)}$ in (4.17) and also the coefficients in (4.18) in the case of $N = 2, 3, 4, 6$, separately. Finally, we construct the physical states on the magnetized T^2/Z_N orbifolds and derive the number of physical states.

We emphasize that all the relations discussed in Section 3 should hold since the Z_N discrete symmetry is additional on T^2. As we showed in Eq. (3.20), the system has the periodicity,

$$|n,\widetilde{Y}\rangle = |n,\widetilde{Y}-1\rangle, \tag{4.19}$$

and the coordinate $\widetilde{Y}$ is quantized, where the possible values in Eq. (3.21) differentiate the degenerated states. As shown in Eq. (3.19), the operator $\widehat{\widetilde{P}}$ works as the generator for the translation along $\widetilde{Y}$-direction. The following expression is helpful,

$$\left|n,\frac{j}{M}\right\rangle = e^{-i\frac{j}{M}\widehat{\widetilde{P}}}|n,0\rangle. \tag{4.20}$$

Throughout the following part, the mathematical formula for arbitrary operators $\widehat{A}$ and $\widehat{B}$,

$$\widehat{A}e^{\widehat{B}}\widehat{A}^{-1} = e^{\widehat{A}\widehat{B}\widehat{A}^{-1}} \tag{4.21}$$

is very useful and we use the relations between the operators $\widehat{T}_1$, $\widehat{T}_2$ and $\widehat{\widetilde{Y}}$, $\widehat{\widetilde{P}}$ in Eq. (3.14) many times. Also, we comply with the three "conventions":

- We only consider the case $M > 0$ but we can analyze the case of $M < 0$ in the same way.
- Fundamental regions of α_1, α_2 and $\widetilde{Y}$ (in the eigenstate of $\widehat{\widetilde{Y}}$) are selected as $[0, 1)$ (without loss of generality).
- We omit to describe eigenvalues of the energy since the following discussion is viable in every energy eigenstate,

$$\left| n, \frac{j}{M} \right\rangle \rightarrow \left| \frac{j}{M} \right\rangle. \tag{4.22}$$

4.1. T^2/Z_2

First, we calculate the allowed Scherk–Schwarz phases on T^2/Z_2 by using Eq. (4.14). This case is not complicated since the two operators $\widehat{T}_1$ and $\widehat{T}_2$ are not mixed under the operation of $\widehat{U}_{Z_2}$, where only their signs are flipped. After requesting the coexistence of the conditions in Eqs. (3.17) and (4.14), we obtain the requirements,

$$\alpha_1 = -\alpha_1 \quad \text{and} \quad \alpha_2 = -\alpha_2 \quad (\text{mod } 1), \tag{4.23}$$

where we used the relation in Eqs. (4.10) and (4.21). As shown in [63], the two phases can take individual values and the possibilities are

$$(\alpha_1, \alpha_2) = (0, 0), \left(\frac{1}{2}, 0 \right), \left(0, \frac{1}{2} \right), \left(\frac{1}{2}, \frac{1}{2} \right). \tag{4.24}$$

Next, we consider Z_2-transformed states. Here, we remember the statement in Section 3 for the state on T^2 that degenerated states of an energy can be specified by the eigenvalues of the operator $\widehat{\widetilde{Y}}$. Thereby, we first try to find a state which includes the operator $\widehat{U}_{Z_2}$ and is also an eigenstate of $\widehat{\widetilde{Y}}$. We find such a state as $\widehat{U}_{Z_2} |j/M\rangle$ with the eigenvalue $(-2\alpha_1 - j)/M$ from the following calculation,

$$e^{2\pi i \widehat{\widetilde{Y}}} \left(\widehat{U}_{Z_2} \left| \frac{j}{M} \right\rangle \right) = \widehat{U}_{Z_2} (\widehat{U}_{Z_2}^{\dagger} e^{2\pi i (\frac{\widehat{T}_1}{2\pi M} - \frac{\alpha_1}{M})} \widehat{U}_{Z_2}) \left| \frac{j}{M} \right\rangle$$

$$= \widehat{U}_{Z_2} e^{2\pi i (-\frac{\widehat{T}_1}{2\pi M} - \frac{\alpha_1}{M})} \left| \frac{j}{M} \right\rangle$$

$$= \widehat{U}_{Z_2} e^{-2\pi i \widehat{\widetilde{Y}} - 2\pi i \frac{2\alpha_1}{M}} \left| \frac{j}{M} \right\rangle$$

$$= e^{2\pi i (\frac{-2\alpha_1 - j}{M})} \left(\widehat{U}_{Z_2} \left| \frac{j}{M} \right\rangle \right), \tag{4.25}$$

and then we can write down the state $\widehat{U}_{Z_2} |j/M\rangle$ as follows:

$$\widehat{U}_{Z_2} \left| \frac{j}{M} \right\rangle = e^{2\pi i \eta_j} \left| \frac{-2\alpha_1 - j}{M} \right\rangle, \tag{4.26}$$

where η_j expresses a phase ambiguity to be determined.[10] Since there exists the relation on the translation along $\widetilde{Y}$-direction in Eq. (4.20), we have another expression for the above equation:

$$
\begin{aligned}
\widehat{U}_{Z_2}\left|\frac{j}{M}\right\rangle &= \widehat{U}_{Z_2} e^{-i\frac{j}{M}\widehat{\widetilde{P}}}|0\rangle \\
&= \left(\widehat{U}_{Z_2} e^{-i\frac{j}{M}\widehat{\widetilde{P}}}\widehat{U}_{Z_2}^{\dagger}\right)\widehat{U}_{Z_2}|0\rangle = e^{2\pi i\left(-\frac{j}{M}2\alpha_2+\eta_0\right)}\left|\frac{-2\alpha_1-j}{M}\right\rangle,
\end{aligned}
\tag{4.27}
$$

where we used the relations in Eqs. (3.14), (4.10), (4.20) and (4.26). Comparing the two equations (4.26) and (4.27), we have the relation

$$
\eta_j = \eta_0 - \frac{j}{M}2\alpha_2 \quad (\text{mod } 1).
\tag{4.28}
$$

The Z_2-consistency requires the condition $\widehat{U}_{Z_2}^2 = \widehat{1}$ for every state, where $\widehat{1}$ is the unit operator. This fact leads to the condition,

$$
\widehat{U}_{Z_2}^2\left|\frac{j}{M}\right\rangle = e^{2\pi i(\eta_j+\eta_{-2\alpha_1-j})}\left|\frac{j}{M}\right\rangle = \left|\frac{j}{M}\right\rangle,
\tag{4.29}
$$

which is equivalent to

$$
\eta_j + \eta_{-2\alpha_1-j} = 0 \quad (\text{mod } 1).
\tag{4.30}
$$

Combining the results in Eqs. (4.24), (4.28) and (4.30), we can determine the form of η_j as

$$
\eta_j = -\frac{2\alpha_2}{M}(j+\alpha_1),
\tag{4.31}
$$

where we omit the trivial overall phase. The explicit analytic form of the Z_2-transformed state $\widehat{U}_{Z_2}|\frac{j}{M}\rangle$, which corresponds to Eq. (4.17), is the following:

$$
\widehat{U}_{Z_2}\left|\frac{j}{M}\right\rangle = e^{-2\pi i \cdot \frac{2\alpha_2}{M}(j+\alpha_1)}\left|\frac{-2\alpha_1-j}{M}\right\rangle,
\tag{4.32}
$$

which is equivalent to the following expression:

$$
\begin{aligned}
\widehat{U}_{Z_2}\left|\frac{j}{M}\right\rangle &= \sum_{k=0}^{M-1} D_{jk}^{(\omega)}\left|\frac{k}{M}\right\rangle, \\
D_{jk}^{(\omega)} &= e^{-2\pi i \cdot \frac{2\alpha_2}{M}(j+\alpha_1)}\delta_{-2\alpha_1-j,k}.
\end{aligned}
\tag{4.33}
$$

Finally, we can construct the physical states on T^2/Z_2 by using Eq. (4.18),

$$
\begin{aligned}
\left|\frac{j}{M}\right\rangle_{T^2/Z_2,\eta} &= \frac{1}{2}\sum_{x=0}^{1}\bar{\eta}^x\sum_{k=0}^{M-1}D_{jk}^{(\omega^x)}\left|\frac{k}{M}\right\rangle \\
&=: \sum_{k=0}^{M-1}M_{jk}^{(Z_2;\eta)}\left|\frac{k}{M}\right\rangle \quad (\eta = +1,-1).
\end{aligned}
\tag{4.34}
$$

---

[10] Here, the norm should not be changed under the rotation.

The number of independent physical states $|\frac{j}{M}\rangle_{T^2/Z_2, \eta}$ is nothing but the rank of the M-by-M matrix $M_{jk}^{(Z_2; \eta)}$,

$$\text{The number of physical states } \left|n, \frac{j}{M}\right\rangle_{T^2/Z_2, \eta} = \text{Rank}\big[M_{jk}^{(Z_2; \eta)}\big]. \tag{4.35}$$

The analytic results for the T^2/Z_2 have already been given by Ref. [63].

4.2. T^2/Z_3

First, we note that in the cases of T^2/Z_3 and the following T^2/Z_6, a linear combination of the two operators $\widehat{T}_a$ ($a = 1, 2$) appears to a variable of the exponential and the Baker–Campbell–Hausdorff formula (or the Zassenhaus formula) for two operators $\widehat{A}$ and $\widehat{B}$ with $t \in \mathbb{R}$ is helpful:

$$e^{t(\widehat{A}+\widehat{B})} = e^{t\widehat{A}} e^{t\widehat{B}} e^{-\frac{t^2}{2}[\widehat{A}, \widehat{B}]} \quad \left(\text{when } \big[\widehat{A}, [\widehat{A}, \widehat{B}]\big] = 0 \text{ and } \big[\widehat{B}, [\widehat{A}, \widehat{B}]\big] = 0\right). \tag{4.36}$$

Since the commutation relation associated with $\widehat{T}_1$ and $\widehat{T}_2$ is given by

$$[\widehat{T}_1, \widehat{T}_2] = -2\pi i M, \tag{4.37}$$

the precondition in Eq. (4.36) is trivially fulfilled for $\widehat{T}_a$s.

Now, to obtain the allowed Scherk–Schwarz phases, we follow the same strategy as in the case of T^2/Z_2. Then, from Eq. (4.14), α_as should satisfy the conditions

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \alpha_2 = -\alpha_2 - \alpha_1 - \frac{M}{2} \pmod{1}, \tag{4.38}$$

for T^2/Z_3 and the resultant allowed combinations are as follows:

$$\alpha := \alpha_1 = \alpha_2, \tag{4.39}$$

$$\alpha = \begin{cases} 0, \frac{1}{3}, \frac{2}{3} & \text{for } M\text{: even,} \\ \frac{1}{6}, \frac{3}{6}, \frac{5}{6} & \text{for } M\text{: odd.} \end{cases} \tag{4.40}$$

Interestingly, when the value of M is odd, the Z_3-orbifold system cannot be defined without the Scherk–Schwarz phases.

To evaluate the form of the Z_3-transformed state $\widehat{U}_{Z_3} |j/M\rangle$, we act the operator $e^{2\pi i \widehat{\widetilde{Y}}}$ to it. In the case of T^2/Z_3, the state $\widehat{U}_{Z_3} |j/M\rangle$ is not an eigenstate of $\widehat{\widetilde{Y}}$ because the argument of the ket vector gets to be different from the original one as $j/M \to (j+1)/M$ after manipulating $e^{2\pi i \widehat{\widetilde{Y}}}$ to the Z_3-rotated state $\widehat{U}_{Z_3} |j/M\rangle$:

$$\begin{aligned} e^{2\pi i \widehat{\widetilde{Y}}} \widehat{U}_{Z_3} \left|\frac{j}{M}\right\rangle &= \widehat{U}_{Z_3} \left(\widehat{U}_{Z_3}^{\dagger} e^{2\pi i (\frac{\widehat{T}_1}{2\pi M} - \frac{\alpha_1}{M})} \widehat{U}_{Z_3}\right) \left|\frac{j}{M}\right\rangle \\ &= \widehat{U}_{Z_3} e^{2\pi i (\frac{\widehat{T}_2}{2\pi M} - \frac{\alpha_1}{M})} \left|\frac{j}{M}\right\rangle \\ &= \widehat{U}_{Z_3} e^{2\pi i (-\frac{\widehat{\widetilde{P}}}{2\pi M} + \frac{\alpha_2}{M} - \frac{\alpha_1}{M})} \left|\frac{j}{M}\right\rangle \\ &= \widehat{U}_{Z_3} e^{-i \frac{\widehat{\widetilde{P}}}{M}} \left|\frac{j}{M}\right\rangle \\ &= \widehat{U}_{Z_3} \left|\frac{j+1}{M}\right\rangle, \end{aligned} \tag{4.41}$$

where we used Eq. (4.39). This fact gives us a hint for finding a suitable form of an eigenstate of $\widehat{\widetilde{Y}}$. Effects of this shift would be cancelled out after taking summation over the index j. Based on this speculation, we focus on the following state,

$$\sum_{j=0}^{M-1} \widehat{U}_{Z_3} \left| \frac{j}{M} \right\rangle. \tag{4.42}$$

Since the above state satisfies the condition,

$$e^{2\pi i \widehat{\widetilde{Y}}} \sum_{j=0}^{M-1} \widehat{U}_{Z_3} \left| \frac{j}{M} \right\rangle = \sum_{j=0}^{M-1} \widehat{U}_{Z_3} \left| \frac{j}{M} \right\rangle, \tag{4.43}$$

the following representation should be completed with two parameters $\mathcal{A}$ and θ, which show the magnitude and phase parts of the undetermined coefficient,

$$\sum_{j=0}^{M-1} \widehat{U}_{Z_3} \left| \frac{j}{M} \right\rangle = \mathcal{A} e^{i\theta} |0\rangle. \tag{4.44}$$

With casting the shift operator $e^{-i\frac{l}{M}\widehat{\widetilde{P}}}$ on both sides, we obtain

$$e^{-i\frac{\pi}{M}l^2 - i\frac{6\pi\alpha}{M}l} \sum_{j=0}^{M-1} e^{-2\pi i \frac{l \cdot j}{M}} \widehat{U}_{Z_3} \left| \frac{j}{M} \right\rangle = \mathcal{A} e^{i\theta} \left| \frac{l}{M} \right\rangle, \tag{4.45}$$

and subsequently, we can derive the following simple form from the above with summing over l from 0 to $M-1$,

$$\widehat{U}_{Z_3} |0\rangle = \frac{\mathcal{A}}{M} e^{i\theta} \sum_{l=0}^{M-1} e^{i\frac{\pi}{M}l^2 + i\frac{6\pi\alpha}{M}l} \left| \frac{l}{M} \right\rangle. \tag{4.46}$$

Here, we used the formula,

$$\sum_{k=0}^{M-1} e^{2\pi i \frac{s}{M}k} = M\delta_{s,0} \quad (s = 0, 1, \ldots, M-1), \tag{4.47}$$

which is proved with ease via properties of M-th root of unity.

The Z_3-consistency $(\widehat{U}_{Z_3})^3 = \widehat{1}$ pins down values of $\mathcal{A}$ and θ immediately when we utilize the formula,

$$\sum_{s=0}^{M-1} e^{-\pi i \frac{(s+t\pm\beta)^2}{M}} = \sqrt{M} e^{-\frac{1}{4}\pi i} \quad \text{for } t \in \mathbb{Z}, \ \beta = \begin{cases} 0 & \text{for } M: \text{ even,} \\ \frac{1}{2} & \text{for } M: \text{ odd.} \end{cases} \tag{4.48}$$

We mention that the above summation takes the universal form irrespective of the choice of t, β and the sign in front of β (within the shown ranges in Eq. (4.48)). The derivation of this is provided in Appendix B. After some calculations, resultant values are declared, (*e.g.*, via $\widehat{U}_{Z_3}^3 |0\rangle = |0\rangle$) as

$$\mathcal{A} = \sqrt{M}, \qquad \theta = -\frac{\pi}{12} + \frac{3\pi}{M}\alpha^2, \tag{4.49}$$

where we ignored the trivial overall phase in θ.

Now, we can obtain the form of the Z_3-transformed state $\widehat{U}_{Z_3}|\frac{j}{M}\rangle$ for the projection operator from Eq. (4.46) with the shift operation in Eq. (4.20) as

$$\widehat{U}_{Z_3}\left|\frac{j}{M}\right\rangle = (\widehat{U}_{Z_3}^{\dagger})^2\left|\frac{j}{M}\right\rangle = \frac{1}{\sqrt{M}}e^{-i\frac{\pi}{12}+i\frac{3\pi\alpha^2}{M}}\sum_{k=0}^{M-1}e^{i\frac{\pi}{M}k(k+6\alpha)+2\pi i\frac{j\cdot k}{M}}\left|\frac{k}{M}\right\rangle,$$

$$(\widehat{U}_{Z_3})^2\left|\frac{j}{M}\right\rangle = \widehat{U}_{Z_3}^{\dagger}\left|\frac{j}{M}\right\rangle = \frac{1}{\sqrt{M}}e^{i\frac{\pi}{12}-i\frac{3\pi\alpha^2}{M}-i\frac{\pi}{M}j(j+6\alpha)}\sum_{k=0}^{M-1}e^{-2\pi i\frac{j\cdot k}{M}}\left|\frac{k}{M}\right\rangle, \tag{4.50}$$

which are equivalent to the following expressions:

$$(\widehat{U}_{Z_3})^x\left|\frac{j}{M}\right\rangle = \sum_{k=0}^{M-1}D_{jk}^{(\omega^x)}\left|\frac{k}{M}\right\rangle \quad (x=0,1,2),$$

$$D_{jk}^{(\omega)} = \frac{1}{\sqrt{M}}e^{-i\frac{\pi}{12}+i\frac{3\pi\alpha^2}{M}}e^{i\frac{\pi}{M}k(k+6\alpha)+2\pi i\frac{j\cdot k}{M}},$$

$$D_{jk}^{(\omega^2)} = \frac{1}{\sqrt{M}}e^{i\frac{\pi}{12}-i\frac{3\pi\alpha^2}{M}-i\frac{\pi}{M}j(j+6\alpha)}e^{-2\pi i\frac{j\cdot k}{M}}. \tag{4.51}$$

We again comment that the above analytic results of the Z_3-transformed state are nontrivial. In the case of the wavefunction analysis, we need to rely on numerical calculations since the states are given by theta functions. On the other hand, we can evaluate the exact form of the Z_3-transformed states in this case and, moreover, the above results are applicable to all KK-modes since they are irrelevant to the principal quantum number n of the state $|n,\frac{j}{M}\rangle$. We can say the same thing for the following T^2/Z_4 and T^2/Z_6 cases.

Finally, we construct physical states on T^2/Z_3. By using Eq. (4.18), the physical states are represented by

$$\left|\frac{j}{M}\right\rangle_{T^2/Z_3,\eta} = \frac{1}{3}\sum_{x=0}^{2}\bar{\eta}^x\sum_{k=0}^{M-1}D_{jk}^{(\omega^x)}\left|\frac{k}{M}\right\rangle$$

$$=: \sum_{k=0}^{M-1}M_{jk}^{(Z_3;\eta)}\left|\frac{k}{M}\right\rangle \quad (\eta=1,\omega,\omega^2), \tag{4.52}$$

where

$$M_{jk}^{(Z_3;\eta)} = \frac{1}{3}\sum_{x=0}^{2}\bar{\eta}^x D_{jk}^{(\omega^x)}. \tag{4.53}$$

The number of independent physical states $|\frac{j}{M}\rangle_{T^2/Z_3,\eta}$ is nothing but the rank of the matrix $M_{jk}^{(Z_3;\eta)}$,

$$\text{The number of physical states }\left|\frac{j}{M}\right\rangle_{T^2/Z_3,\eta} = \text{Rank}[M_{jk}^{(Z_3;\eta)}]. \tag{4.54}$$

After investigating the rank of the matrix $M_{jk}^{(Z_3;\eta)}$, we obtain the results shown in Tables 1, 2, 3 and 4. We, again, emphasize that the following results are completely consistent with the previous wavefunction analysis [63]. The correspondence between the matrix $M_{jk}^{(Z_3;\eta)}$ and the kinetic matrix $\mathcal{K}_{jk}^{(Z_3;\eta)}$ will be discussed in Section 5.

Table 1
The numbers of linearly independent zero-mode eigenstates with Z_3 eigenvalue η for $M =$ even and $(\alpha_1, \alpha_2) = (0, 0)$ on T^2/Z_3.

η	$|M|$						
	2	4	6	8	10	12	14
1	1	1	3	3	3	5	5
ω	0	2	2	2	4	4	4
$\bar{\omega}$	1	1	1	3	3	3	5

Table 2
The numbers of linearly independent zero-mode eigenstates with Z_3 eigenvalue η for $M =$ even and $(\alpha_1, \alpha_2) = (\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})$ on T^2/Z_3.

η	$|M|$						
	2	4	6	8	10	12	14
1	1	2	2	3	4	4	5
ω	1	1	2	3	3	4	5
$\bar{\omega}$	0	1	2	2	3	4	4

Table 3
The numbers of linearly independent zero-mode eigenstates with Z_3 eigenvalue η for $M =$ odd and $(\alpha_1, \alpha_2) = (\frac{1}{6}, \frac{1}{6}), (\frac{5}{6}, \frac{5}{6})$ on T^2/Z_3.

η	$|M|$						
	1	3	5	7	9	11	13
1	1	1	2	3	3	4	5
ω	0	1	2	2	3	4	4
$\bar{\omega}$	0	1	1	2	3	3	4

Table 4
The numbers of linearly independent zero-mode eigenstates with Z_3 eigenvalue η for $M =$ odd and $(\alpha_1, \alpha_2) = (\frac{3}{6}, \frac{3}{6})$ on T^2/Z_3.

η	$|M|$						
	1	3	5	7	9	11	13
1	0	2	2	2	4	4	4
ω	1	1	1	3	3	3	5
$\bar{\omega}$	0	0	2	2	2	4	4

4.3. T^2/Z_4

In the case of T^2/Z_4, the situation is similar to T^2/Z_2 case but is somewhat complicated. The way for determining allowed values of α_1 and α_2 is the same as in T^2/Z_2 and T^2/Z_3. By using Eqs. (4.12), (4.14) and (4.21), we find that the following two conditions should be fulfilled:

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \alpha_1 = -\alpha_2 \pmod 1. \tag{4.55}$$

Here, we conclude that the consistent values are

$$\alpha := \alpha_1 = \alpha_2 \tag{4.56}$$

$$\alpha = 0 \text{ or } \frac{1}{2}, \tag{4.57}$$

which are a subset of the result in T^2/Z_2 since the Z_4 orbifolding includes the Z_2 operation.

To evaluate the form of the Z_4-transformed state $\widehat{U}_{Z_4}|j/M\rangle$, we act the operator $e^{2\pi i \widehat{\widehat{Y}}}$ to the transformed state. Unlike T^2/Z_2, the state $\widehat{U}_{Z_4}|j/M\rangle$ itself is not an eigenstate of $\widehat{\widehat{Y}}$ as in the case of the T^2/Z_3, i.e.,

$$
\begin{aligned}
e^{2\pi i \widehat{\widehat{Y}}}\left(\widehat{U}_{Z_4}\left|\frac{j}{M}\right\rangle\right) &= \widehat{U}_{Z_4}\left(\widehat{U}_{Z_4}^{\dagger} e^{2\pi i (\frac{\widehat{T}_1}{2\pi M} - \frac{\alpha_1}{M})} \widehat{U}_{Z_4}\right)\left|\frac{j}{M}\right\rangle \\
&= \widehat{U}_{Z_4}\, e^{2\pi i (\frac{\widehat{T}_2}{2\pi M} - \frac{\alpha_1}{M})}\left|\frac{j}{M}\right\rangle \\
&= \widehat{U}_{Z_4}\, e^{2\pi i (-\frac{\widehat{P}}{2\pi M} + \frac{\alpha_2}{M} - \frac{\alpha_1}{M})}\left|\frac{j}{M}\right\rangle \\
&= \widehat{U}_{Z_4}\, e^{-i\frac{\widehat{P}}{M}}\left|\frac{j}{M}\right\rangle \\
&= \widehat{U}_{Z_4}\left|\frac{j+1}{M}\right\rangle.
\end{aligned}
\tag{4.58}
$$

Then we again consider the following state as a candidate for $\widehat{\widehat{Y}}$-eigenstates as T^2/Z_3,

$$\sum_{j=0}^{M-1} \widehat{U}_{Z_4}\left|\frac{j}{M}\right\rangle. \tag{4.59}$$

Using Eq. (4.58), the following condition holds,

$$e^{2\pi i \widehat{\widehat{Y}}} \sum_{j=0}^{M-1} \widehat{U}_{Z_4}\left|\frac{j}{M}\right\rangle = \sum_{j=0}^{M-1} \widehat{U}_{Z_4}\left|\frac{j}{M}\right\rangle. \tag{4.60}$$

Then it should be rewritten as

$$\sum_{j=0}^{M-1} \widehat{U}_{Z_4}\left|\frac{j}{M}\right\rangle = \mathcal{A}e^{i\theta}|0\rangle, \tag{4.61}$$

and casting the operator $e^{-i\frac{l}{M}\widehat{P}}$ on both sides brings us to

$$e^{-i\frac{4\pi\alpha}{M}l} \sum_{j=0}^{M-1} e^{-2\pi i\frac{l\cdot j}{M}}\widehat{U}_{Z_4}\left|\frac{j}{M}\right\rangle = \mathcal{A}e^{i\theta}\left|\frac{l}{M}\right\rangle. \tag{4.62}$$

The parameters $\mathcal{A}$ and θ show the magnitude and phase parts of the undetermined coefficient, respectively as in the case of T^2/Z_3. Here, taking summation over l from 0 to $M-1$ in Eq. (4.62) leads to the simple relation,

$$\widehat{U}_{Z_4}|0\rangle = \frac{\mathcal{A}}{M}e^{i\theta} \sum_{l=0}^{M-1} e^{i\frac{4\pi\alpha}{M}l}\left|\frac{l}{M}\right\rangle, \tag{4.63}$$

where we used the formula (4.47). The form (4.63) is just a part of what we would like to obtain.

The values of $\mathcal{A}$ and θ can be derived when we examine the consistency condition coming from Z_4 symmetry, $\widehat{U}_{Z_4}^4 = \widehat{1}$. After making use of Eq. (4.47), we can reach the result,[11]

$$
\mathcal{A} = \sqrt{M}, \qquad \theta = \frac{2\pi}{M}\alpha^2, \tag{4.64}
$$

where we dropped the overall phase in θ.

Now, the form of the Z_4-transformed state $\widehat{U}_{Z_4}\,|j/M\rangle$ is easily evaluated from Eq. (4.63) by using the shift operation in Eq. (4.20). Here, we summarize the results for constructing the projective operator,

$$
\begin{aligned}
\widehat{U}_{Z_4}\left|\frac{j}{M}\right\rangle &= \left(\widehat{U}_{Z_4}^{\dagger}\right)^3\left|\frac{j}{M}\right\rangle = \frac{1}{\sqrt{M}}e^{2\pi i\frac{\alpha^2}{M}}\sum_{k=0}^{M-1}e^{2\pi i\frac{2\alpha}{M}k+2\pi i\frac{j\cdot k}{M}}\left|\frac{k}{M}\right\rangle,\\
(\widehat{U}_{Z_4})^2\left|\frac{j}{M}\right\rangle &= \left(\widehat{U}_{Z_4}^{\dagger}\right)^2\left|\frac{j}{M}\right\rangle = e^{-2\pi i\frac{2\alpha}{M}(\alpha+j)}\left|\frac{-2\alpha-j}{M}\right\rangle,\\
(\widehat{U}_{Z_4})^3\left|\frac{j}{M}\right\rangle &= \widehat{U}_{Z_4}^{\dagger}\left|\frac{j}{M}\right\rangle = \frac{1}{\sqrt{M}}e^{-2\pi i\frac{\alpha^2}{M}-2\pi i\frac{2\alpha}{M}j}\sum_{k=0}^{M-1}e^{-2\pi i\frac{j\cdot k}{M}}\left|\frac{k}{M}\right\rangle,
\end{aligned} \tag{4.65}
$$

which are equivalent to the following expressions:

$$
\begin{aligned}
(\widehat{U}_{Z_4})^x\left|\frac{j}{M}\right\rangle &= \sum_{k=0}^{M-1}D_{jk}^{(\omega^x)}\left|\frac{k}{M}\right\rangle \quad (x=0,1,2,3),\\
D_{jk}^{(\omega)} &= \frac{1}{\sqrt{M}}e^{2\pi i\frac{\alpha^2}{M}}e^{2\pi i\frac{j\cdot k}{M}+2\pi i\frac{2\alpha}{M}k},\\
D_{jk}^{(\omega^2)} &= e^{-2\pi i\frac{2\alpha}{M}(\alpha+j)}\delta_{-2\alpha-j,k},\\
D_{jk}^{(\omega^3)} &= \frac{1}{\sqrt{M}}e^{-2\pi i\frac{\alpha^2}{M}-2\pi i\frac{2\alpha}{M}j}e^{-2\pi i\frac{j\cdot k}{M}}.
\end{aligned} \tag{4.66}
$$

Finally, we construct physical states on T^2/Z_4 by using Eq. (4.18),

$$
\begin{aligned}
\left|\frac{j}{M}\right\rangle_{T^2/Z_4,\eta} &= \frac{1}{4}\sum_{x=0}^{3}\bar{\eta}^x\sum_{k=0}^{M-1}D_{jk}^{(\omega^x)}\left|\frac{k}{M}\right\rangle\\
&=: \sum_{k=0}^{M-1}M_{jk}^{(Z_4;\eta)}\left|\frac{k}{M}\right\rangle \quad (\eta=1,\omega,\omega^2,\omega^3),
\end{aligned} \tag{4.67}
$$

where

$$
M_{jk}^{(Z_4;\eta)} = \frac{1}{4}\sum_{x=0}^{3}\bar{\eta}^x D_{jk}^{(\omega^x)}. \tag{4.68}
$$

---

[11] The relation $\widehat{U}_{Z_4}^4 = \widehat{1}$ is valid irrespective of operated states. The easiest way to determine $\mathcal{A}$ and θ is to use the state $|0\rangle$ $(\widehat{U}_{Z_4}^4\,|0\rangle = |0\rangle)$.

Table 5
The numbers of linearly independent zero-mode eigenstates with Z_4 eigenvalue η for $(\alpha_1, \alpha_2) = (0, 0)$ on T^2/Z_4.

η	$\|M\|$																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$+1$	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5
$+i$	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4
-1	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
$-i$	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4

Table 6
The numbers of linearly independent zero-mode eigenstates with Z_4 eigenvalue η for $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$ on T^2/Z_4.

η	$\|M\|$															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$+1$	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4
$+i$	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
-1	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4
$-i$	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4

The number of independent physical states $|\frac{j}{M}\rangle_{T^2/Z_4, \eta}$ is nothing but the rank of the matrix $M_{jk}^{(Z_4; \eta)}$,

$$\text{The number of physical states } \left| n, \frac{j}{M} \right\rangle_{T^2/Z_4, \eta} = \text{Rank}\big[M_{jk}^{(Z_4; \eta)}\big]. \tag{4.69}$$

After investigating the rank of the matrix $M_{jk}^{(Z_4; \eta)}$, we obtain the results shown in Tables 5 and 6. We emphasize that the following results are consistent with the previous wavefunction analysis [63].

4.4. T^2/Z_6

For the case of T^2/Z_6, there is no new future to be declared separately and all the calculations are basically the same with those in the previous T^2/Z_3, even though it is somewhat more complicated. Thereby, it suffices to pick up only important points which are different from T^2/Z_3 and to write down components of the projective operator in this case.

From the requirement (4.14), the two Scherk–Schwarz phases α_1 and α_2 should comply with the conditions

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \alpha_2 = \alpha_2 - \alpha_1 - \frac{M}{2} \quad (\text{mod } 1), \tag{4.70}$$

and only the following varieties are realizable,

$$\alpha := \alpha_1 = \alpha_2 \tag{4.71}$$

$$\alpha = \begin{cases} 0 & \text{for } M: \text{ even,} \\ \frac{1}{2} & \text{for } M: \text{ odd.} \end{cases} \tag{4.72}$$

To evaluate the form of the Z_6-transformed state, we can also consider the following state

$$\sum_{j=0}^{M-1} \widehat{U}_{Z_6} \left| \frac{j}{M} \right\rangle. \tag{4.73}$$

Since the above state satisfies the condition

$$e^{2\pi i \widehat{\widetilde{Y}}} \sum_{j=0}^{M-1} \widehat{U}_{Z_6} \left| \frac{j}{M} \right\rangle = \sum_{j=0}^{M-1} \widehat{U}_{Z_6} \left| \frac{j}{M} \right\rangle, \tag{4.74}$$

we can reach to the form

$$\sum_{j=0}^{M-1} \widehat{U}_{Z_6} \left| \frac{j}{M} \right\rangle = \mathcal{A} e^{i\theta} |0\rangle. \tag{4.75}$$

With casting the shift operator $e^{-i \frac{l}{M} \widehat{\widetilde{P}}}$ on both sides, we obtain

$$e^{i \frac{\pi}{M} l^2 - 2\pi i \frac{\alpha}{M} l} \sum_{k=0}^{M-1} e^{-2\pi i \frac{l \cdot k}{M}} \widehat{U}_{Z_6} \left| \frac{k}{M} \right\rangle = \mathcal{A} e^{i\theta} \left| \frac{l}{M} \right\rangle, \tag{4.76}$$

and subsequently, we can derive the following simple form from the above with summing over l from 0 to $M-1$,

$$\widehat{U}_{Z_6} |0\rangle = \frac{\mathcal{A}}{M} e^{i\theta} \sum_{k=0}^{M-1} e^{-i \frac{\pi}{M} k^2 + 2\pi i \frac{\alpha}{M} k} \left| \frac{k}{M} \right\rangle. \tag{4.77}$$

The Z_6-consistency $(\widehat{U}_{Z_6})^6 = \widehat{1}$ will lead to the following values for $\mathcal{A}$ and θ, with the help of Eqs. (4.13), (4.14), (4.20), (4.47), (4.48), (4.77),

$$\mathcal{A} = \sqrt{M}, \qquad \theta = \frac{\pi}{12} + \frac{\pi}{M} \alpha^2. \tag{4.78}$$

In the above, we neglected the trivial overall phase.

Summarizing the above results, we can indicate the forms of the Z_6-transformed states for constructing the projective operator in T^2/Z_6:

$$\widehat{U}_{Z_6} \left| \frac{j}{M} \right\rangle = \left(\widehat{U}_{Z_6}^\dagger \right)^5 \left| \frac{j}{M} \right\rangle = \frac{1}{\sqrt{M}} e^{i \frac{\pi}{12} + i \frac{\pi}{M} \alpha^2} \sum_{k=0}^{M-1} e^{-i \frac{\pi}{M} k^2 + 2\pi i \frac{\alpha}{M} k + 2\pi i \frac{j \cdot k}{M}} \left| \frac{k}{M} \right\rangle,$$

$$(\widehat{U}_{Z_6})^2 \left| \frac{j}{M} \right\rangle = \left(\widehat{U}_{Z_6}^\dagger \right)^4 \left| \frac{j}{M} \right\rangle = \frac{1}{\sqrt{M}} e^{-i \frac{\pi}{12} + i \frac{3\pi\alpha^2}{M} + i \frac{\pi}{M} j^2 + 2\pi i \frac{\alpha}{M} j} \sum_{k=0}^{M-1} e^{i \frac{4\pi\alpha}{M} k + 2\pi i \frac{j \cdot k}{M}} \left| \frac{k}{M} \right\rangle,$$

$$(\widehat{U}_{Z_6})^3 \left| \frac{j}{M} \right\rangle = \left(\widehat{U}_{Z_6}^\dagger \right)^3 \left| \frac{j}{M} \right\rangle = e^{-i \frac{4\pi\alpha^2}{M} - i \frac{4\pi\alpha}{M} j} \left| \frac{-2\alpha - j}{M} \right\rangle,$$

$$(\widehat{U}_{Z_6})^4 \left| \frac{j}{M} \right\rangle = \left(\widehat{U}_{Z_6}^\dagger \right)^2 \left| \frac{j}{M} \right\rangle = \frac{1}{\sqrt{M}} e^{i \frac{\pi}{12} - i \frac{3\pi\alpha^2}{M} - i \frac{4\pi\alpha}{M} j} \sum_{k=0}^{M-1} e^{-i \frac{\pi}{M} k^2 - 2\pi i \frac{\alpha}{M} k - 2\pi i \frac{j \cdot k}{M}} \left| \frac{k}{M} \right\rangle,$$

$$(\widehat{U}_{Z_6})^5 \left| \frac{j}{M} \right\rangle = \widehat{U}_{Z_6}^\dagger \left| \frac{j}{M} \right\rangle = \frac{1}{\sqrt{M}} e^{-i \frac{\pi}{12} - i \frac{\pi}{M} \alpha^2 + i \frac{\pi}{M} j^2 - 2\pi i \frac{\alpha}{M} j} \sum_{k=0}^{M-1} e^{-2\pi i \frac{j \cdot k}{M}} \left| \frac{k}{M} \right\rangle, \tag{4.79}$$

which are equivalent to the following expressions:

$$(\widehat{U}_{Z_6})^x \left| \frac{j}{M} \right\rangle = \sum_{k=0}^{M-1} D_{jk}^{(\omega^x)} \left| \frac{k}{M} \right\rangle \quad (x = 0, 1, 2, 3, 4, 5),$$

$$D_{jk}^{(\omega)} = \frac{1}{\sqrt{M}} e^{i\frac{\pi}{12} + i\frac{\pi}{M}\alpha^2} e^{-i\frac{\pi}{M}k^2 + 2\pi i \frac{\alpha}{M}k + 2\pi i \frac{j\cdot k}{M}},$$

$$D_{jk}^{(\omega^2)} = \frac{1}{\sqrt{M}} e^{-i\frac{\pi}{12} + i\frac{3\pi\alpha^2}{M} + i\frac{\pi}{M}j^2 + 2\pi i \frac{\alpha}{M}j} e^{i\frac{4\pi\alpha}{M}k + 2\pi i \frac{j\cdot k}{M}},$$

$$D_{jk}^{(\omega^3)} = e^{-i\frac{4\pi\alpha^2}{M} - i\frac{4\pi\alpha}{M}j} \delta_{-2\alpha - j, k},$$

$$D_{jk}^{(\omega^4)} = \frac{1}{\sqrt{M}} e^{i\frac{\pi}{12} - i\frac{3\pi\alpha^2}{M} - i\frac{4\pi\alpha}{M}j} e^{-i\frac{\pi}{M}k^2 - 2\pi i \frac{\alpha}{M}k - 2\pi i \frac{j\cdot k}{M}},$$

$$D_{jk}^{(\omega^5)} = \frac{1}{\sqrt{M}} e^{-i\frac{\pi}{12} - i\frac{\pi}{M}\alpha^2 + i\frac{\pi}{M}j^2 - 2\pi i \frac{\alpha}{M}j} e^{-2\pi i \frac{j\cdot k}{M}}. \tag{4.80}$$

Finally, we construct physical states on T^2/Z_6 at last. By using Eq. (4.18), the physical states are represented by the follows:

$$\left| \frac{j}{M} \right\rangle_{T^2/Z_6, \eta} = \frac{1}{6} \sum_{x=0}^{5} \bar{\eta}^x \sum_{k=0}^{M-1} D_{jk}^{(\omega^x)} \left| \frac{k}{M} \right\rangle$$

$$=: \sum_{k=0}^{M-1} M_{jk}^{(Z_6; \eta)} \left| \frac{k}{M} \right\rangle \quad (\eta = 1, \omega, \omega^2, \omega^3, \omega^4, \omega^5), \tag{4.81}$$

where

$$M_{jk}^{(Z_6; \eta)} = \frac{1}{6} \sum_{x=0}^{5} \bar{\eta}^x D_{jk}^{(\omega^x)}. \tag{4.82}$$

The number of independent physical states $\left| \frac{j}{M} \right\rangle_{T^2/Z_6, \eta}$ is nothing but the rank of the matrix $M_{jk}^{(Z_6; \eta)}$,

The number of physical states $\left| n, \frac{j}{M} \right\rangle_{T^2/Z_6, \eta} = \text{Rank}\big[M_{jk}^{(Z_6; \eta)}\big]. \tag{4.83}$

After investigating the rank of the matrix $M_{jk}^{(Z_6; \eta)}$, we obtain the results shown in Tables 7 and 8. We again emphasize that the following results are consistent with the previous wavefunction analysis [63].

5. Correspondence between the operator analysis and the wavefunction analysis

Based on the analytical results that we have obtained in the previous section, we will derive the correspondence between the wavefunction analysis in Section 2 and the operator analysis in Section 4. First, we introduce the description of the magnetized twisted orbifold T^2/Z_N with wavefunctions and rewrite it into the words of the operator analysis. After that, we check the consistency between the operator analysis and the wavefunction analysis.

Table 7
The numbers of linearly independent zero-mode eigenstates with Z_6 eigenvalue η for $M =$ even and $(\alpha_1, \alpha_2) = (0, 0)$ on T^2/Z_6.

η	$|M|$												
	2	4	6	8	10	12	14	16	18	20	22	24	26
1	1	1	2	2	2	3	3	3	4	4	4	5	5
ω	0	1	1	1	2	2	2	3	3	3	4	4	4
ω^2	1	1	1	2	2	2	3	3	3	4	4	4	5
ω^3	0	0	1	1	1	2	2	2	3	3	3	4	4
ω^4	0	1	1	1	2	2	2	3	3	3	4	4	4
ω^5	0	0	0	1	1	1	2	2	2	3	3	3	4

Table 8
The numbers of linearly independent zero-mode eigenstates with Z_6 eigenvalue η for $M =$ odd and $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$ on T^2/Z_6.

η	$|M|$												
	1	3	5	7	9	11	13	15	17	19	21	23	25
1	0	1	1	1	2	2	2	3	3	3	4	4	4
ω	1	1	1	2	2	2	3	3	3	4	4	4	5
ω^2	0	0	1	1	1	2	2	2	3	3	3	4	4
ω^3	0	1	1	1	2	2	2	3	3	3	4	4	4
ω^4	0	0	0	1	1	1	2	2	2	3	3	3	4
ω^5	0	0	1	1	1	2	2	2	3	3	3	4	4

First, we start to discuss the relation when $M > 0$

$$f_{T^2, \psi_+, 0}^{(j+\alpha_1, \alpha_\tau)}(z) = \left\langle z \middle| 0, \frac{j}{M} \right\rangle^{(\alpha_1, \alpha_\tau)}. \tag{5.1}$$

Although we explicitly write down the energy eigenvalue and the Scherk–Schwarz phases in Eq. (5.1) we hereafter omit the information on the T^2 state $|0, j/M\rangle^{(\alpha_1, \alpha_\tau)}$ as $|j/M\rangle$ for simplicity in description.

The rotated state $|\omega z\rangle$ on T^2/Z_N can be expressed with the operator $\widehat{U}_{Z_N}$

$$|\omega z\rangle = \widehat{U}_{Z_N}^\dagger |z\rangle. \tag{5.2}$$

Now, when we consider the following product

$$C_{jk}^{(\omega^x)} = \int_{T^2} dz d\bar{z} \, f_{T^2, \psi_+, 0}^{(j+\alpha_1, \alpha_\tau)}(\omega^x z) \left(f_{T^2, \psi_+, 0}^{(k+\alpha_1, \alpha_\tau)}(z)\right)^* \quad (x = 0, 1, \ldots, N-1), \tag{5.3}$$

it can be represented in the operator formalism as

$$C_{jk}^{(\omega^x)} = \int_{T^2} dz d\bar{z} \left\langle \frac{k}{M} \middle| z \right\rangle \left\langle z \middle| (\widehat{U}_{Z_N})^x \middle| \frac{j}{M} \right\rangle = \left\langle \frac{k}{M} \middle| (\widehat{U}_{Z_N})^x \middle| \frac{j}{M} \right\rangle, \tag{5.4}$$

where we used the completeness relation $\int_{T^2} dz d\bar{z} \, |z\rangle \langle z| = 1$. As we calculated in the previous section, the state $(\widehat{U}_{Z_N})^x |j/M\rangle$ can be represented as a linear combination of the states on T^2

$$(\widehat{U}_{Z_N})^x \left| \frac{j}{M} \right\rangle = \sum_{m=0}^{M-1} D_{jm}^{(\omega^x)} \left| \frac{m}{M} \right\rangle. \tag{5.5}$$

Then, from the orthonormality relation $\langle \frac{k}{M} | \frac{m}{M} \rangle = \delta_{k,m}$ we can conclude that

$$C_{jk}^{(\omega^x)} = D_{jk}^{(\omega^x)}, \tag{5.6}$$

where the concrete forms of $D_{jk}^{(\omega^x)}$ are found in Eqs. (4.51), (4.66) and (4.80) in the T^2/Z_3, T^2/Z_4 and T^2/Z_6 cases, respectively. Since we already obtained all the analytical forms of $D_{jk}^{(\omega^x)}$ in the previous section, we are now able to express the following relation analytically by use of $D_{jk}^{(\omega^x)}$

$$f_{T^2,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(\omega^x z) = \sum_{k=0}^{M-1} D_{jk}^{(\omega^x)} f_{T^2,\Psi_+,0}^{(k+\alpha_1,\alpha_\tau)}(z). \tag{5.7}$$

We would like to note that the relation in Eq. (5.7) itself is interesting because it brings us lots of nontrivial formulas on the theta functions. By using the numerical calculation for $C_{jk}^{(\omega^x)}$ in Eq. (5.3), we can easily check the validity of the relation Eq. (5.6).

Next, we try to see the correspondence in the orbifold cases directly. The wavefunctions on T^2/Z_N can be shown in operator formalism by use of the projection operator $\widehat{\mathcal{P}}_{T^2/Z_N,\eta}$ for the Z_N eigenstate with eigenvalue η in Eq. (4.16)

$$f_{T^2/Z_N,\Psi_+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_\eta = \left\langle z \left| \widehat{\mathcal{P}}_{T^2/Z_N,\eta} \right| \frac{j}{M} \right\rangle, \tag{5.8}$$

and the kinetic matrix in Eq. (2.23) is rephrased as follows:

$$\begin{aligned}
\mathcal{K}_{jk}^{(Z_N;\eta)} &= \int_{T^2} dz d\bar{z} \left\langle \frac{j}{M} \left| \widehat{\mathcal{P}}_{T^2/Z_N,\eta} \right| z \right\rangle \left\langle z \left| \widehat{\mathcal{P}}_{T^2/Z_N,\eta} \right| \frac{k}{M} \right\rangle \\
&= \left\langle \frac{j}{M} \left| \widehat{\mathcal{P}}_{T^2/Z_N,\eta} \right| \frac{k}{M} \right\rangle \\
&= \left\langle \frac{j}{M} \left| \frac{1}{N} \sum_{x=0}^{N-1} \bar{\eta}^x (\widehat{U}_{Z_N})^x \right| \frac{k}{M} \right\rangle \\
&= \left\langle \frac{j}{M} \left| \frac{1}{N} \sum_{x=0}^{N-1} \bar{\eta}^x \sum_{s=0}^{M-1} D_{ks}^{(\omega^x)} \right| \frac{s}{M} \right\rangle \\
&= M_{kj}^{(Z_N;\eta)}, \tag{5.9}
\end{aligned}$$

where we used the completeness relation on z and the property of the projection operator, $(\widehat{\mathcal{P}}_{T^2/Z_N,\eta})^2 = \widehat{\mathcal{P}}_{T^2/Z_N,\eta}$ with Eq. (5.5). As shown in Eq. (5.9), all the materials of $\mathcal{K}_{jk}^{(Z_N;\eta)}$ are analytically evaluated and then we can now check the consistency between the results of both the analyses by calculating the rank of $\mathcal{K}_{jk}^{(Z_N;\eta)}$ and form of the unitary matrix diagonalizing $\mathcal{K}_{jk}^{(Z_N;\eta)}$. We already re-evaluated the numbers of independent zero-mode eigenstates in all the cases of T^2/Z_2, T^2/Z_3, T^2/Z_4 and T^2/Z_6 in Section 4 with every possible combination of the

Scherk–Schwarz phases. We then confirmed the agreement between the results of both the analyses with exhausting the possibilities of three generations.[12] All the results are totally consistent with the previous ones in Ref. [63] and they are summarized in Tables 1 to 8, where we skipped the T^2/Z_2 because in this case, exact formulas are available in Ref. [63].

Finally, we show some specific examples of the correspondence. The first one is the T^2/Z_6 case with $M = 2$ and $(\alpha_1, \alpha_2) = (0, 0)$. In the Z_6 orbifolding, we should consider linear combinations of six terms and calculations requires lots of efforts. However, when we utilize the operator formalism on T^2/Z_6, situations get to be very clear. We can explicitly evaluate the forms of the inner products $\langle \frac{j}{M} | \widehat{\mathcal{P}}_{T^2/Z_6, \eta} | \frac{k}{M} \rangle$ $(= \mathcal{K}_{jk}^{(Z_6;\eta)} = M_{kj}^{(Z_6;\eta)})$ for all the cases of x by use of the results in Eq. (4.80) and Eq. (4.82). These concrete forms are

$$\mathcal{K}_{\text{op}}^{(Z_6;\omega^0)} = \begin{pmatrix} \frac{1}{6}(2 + \sqrt{2}e^{-\frac{i\pi}{12}} + \sqrt{2}e^{\frac{i\pi}{12}}) & \frac{1}{6}(i\sqrt{2}e^{-\frac{i\pi}{12}} + \sqrt{2}e^{\frac{i\pi}{12}}) \\ \frac{1}{6}(\sqrt{2}e^{-\frac{i\pi}{12}} - i\sqrt{2}e^{\frac{i\pi}{12}}) & \frac{1}{6}(2 - i\sqrt{2}e^{-\frac{i\pi}{12}} + i\sqrt{2}e^{\frac{i\pi}{12}}) \end{pmatrix}$$
$$\approx \begin{pmatrix} 0.789 & 0.289 + 0.289i \\ 0.289 - 0.289i & 0.211 \end{pmatrix}, \tag{5.10}$$

$$\mathcal{K}_{\text{op}}^{(Z_6;\omega^1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.11}$$

$$\mathcal{K}_{\text{op}}^{(Z_6;\omega^2)} = \begin{pmatrix} \frac{1}{6}(2 + \sqrt{2}e^{-\frac{7i\pi}{12}} + \sqrt{2}e^{\frac{7i\pi}{12}}) & \frac{1}{6}(\sqrt{2}e^{-\frac{7i\pi}{12}} + i\sqrt{2}e^{\frac{7i\pi}{12}}) \\ \frac{1}{6}(-i\sqrt{2}e^{-\frac{7i\pi}{12}} + \sqrt{2}e^{\frac{7i\pi}{12}}) & \frac{1}{6}(2 + i\sqrt{2}e^{-\frac{7i\pi}{12}} - i\sqrt{2}e^{\frac{7i\pi}{12}}) \end{pmatrix}$$
$$\approx \begin{pmatrix} 0.211 & -0.289 - 0.289i \\ -0.289 + 0.289i & 0.789 \end{pmatrix}, \tag{5.12}$$

$$\mathcal{K}_{\text{op}}^{(Z_6;\omega^3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.13}$$

$$\mathcal{K}_{\text{op}}^{(Z_6;\omega^4)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.14}$$

$$\mathcal{K}_{\text{op}}^{(Z_6;\omega^5)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.15}$$

where the subscript "op" indicates that these matrices are evaluated in operator formalism explicitly. The above result means that there is no physical degree of freedom in the cases of $\eta = \omega^1, \omega^3, \omega^4, \omega^5$. For $\eta = \omega^0, \omega^2$, the unitary matrix U diagonalizing the "kinetic matrix" $\mathcal{K}^{(Z_6;\eta)}$ can be easily calculated,

$$U|_{x=0} = \begin{pmatrix} \sqrt[4]{\frac{1}{6}(-2 - \sqrt{3})} & -\sqrt[4]{\frac{1}{6}(-2 + \sqrt{3})} \\ \frac{1}{\sqrt{3+\sqrt{3}}} & \sqrt{\frac{1}{6}(3 + \sqrt{3})} \end{pmatrix}, \tag{5.16}$$

$$U|_{x=2} = \begin{pmatrix} -\sqrt[4]{\frac{1}{6}(-2 + \sqrt{3})} & \sqrt[4]{\frac{1}{6}(-2 - \sqrt{3})} \\ \sqrt{\frac{1}{6}(3 + \sqrt{3})} & \frac{1}{\sqrt{3+\sqrt{3}}} \end{pmatrix}. \tag{5.17}$$

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[12] Situations are the same with the right-handed KK modes (in the case of $M > 0$) since all the discussions are valid irrespective of the KK number. We can consider the left-handed KK modes (without corresponding zero mode in the case of $M > 0$) with the help of the Dirac equation for KK fermionic states.

After the manipulation $U^\dagger \mathcal{K}^{(Z_6;\eta)} U$, the kinetic terms are suitably diagonalized as follows:

$$U^\dagger \mathcal{K}_{\mathrm{op}}^{(Z_6;\omega^0)} U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad U^\dagger \mathcal{K}_{\mathrm{op}}^{(Z_6;\omega^2)} U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.18}$$

The two expressions clearly indicate that one mode is physical after the orbifolding in $x = 0, 2$.

Here, we comment on consistency with the numerical calculation with the explicit forms of the theta functions. By calculating Eq. (2.23) numerically with the theta functions in Eq. (2.11), we obtain the following corresponding results,

$$\mathcal{K}_{\mathrm{wf}}^{(Z_6;\omega^0)} = \begin{pmatrix} 0.789 & 0.289 + 0.289i \\ 0.289 - 0.289i & 0.211 \end{pmatrix}, \tag{5.19}$$

$$\mathcal{K}_{\mathrm{wf}}^{(Z_6;\omega^1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.20}$$

$$\mathcal{K}_{\mathrm{wf}}^{(Z_6;\omega^2)} = \begin{pmatrix} 0.211 & -0.289 - 0.289i \\ -0.289 + 0.289i & 0.789 \end{pmatrix}, \tag{5.21}$$

$$\mathcal{K}_{\mathrm{wf}}^{(Z_6;\omega^3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.22}$$

$$\mathcal{K}_{\mathrm{wf}}^{(Z_6;\omega^4)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.23}$$

$$\mathcal{K}_{\mathrm{wf}}^{(Z_6;\omega^5)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.24}$$

where the subscript "wf" is the counterpart of the subscript "op" in the numerical calculation with wavefunctions following the approach in Ref. [63]. We can conclude that they agree with each other within the error of the numerical computation as

$$\mathcal{K}_{\mathrm{op}}^{(Z_6;\eta)} = \mathcal{K}_{\mathrm{wf}}^{(Z_6;\eta)} \quad \text{for } \eta = \omega^0, \omega^1, \ldots, \omega^5, \tag{5.25}$$

and hence, the relation (5.9) is valid.

As the second example, we focus on the T^2/Z_4 case with $M = 2$ and $(\alpha_1, \alpha_2) = (0, 0)$ since the analytical result was already discussed with the exact forms of the mode functions with theta functions as in Eq. (2.11) and some related mathematical relations, and their explicit forms are available in Appendix C of Ref. [63]. The explicit shapes of the kinetic matrix in $\eta = \omega^0, \omega^1, \omega^2, \omega^3$ are

$$\mathcal{K}_{\mathrm{op}}^{(Z_4;\omega^0)} = \begin{pmatrix} \frac{1}{4}(2 + \sqrt{2}) & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{4}(2 - \sqrt{2}) \end{pmatrix}, \tag{5.26}$$

$$\mathcal{K}_{\mathrm{op}}^{(Z_4;\omega^1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.27}$$

$$\mathcal{K}_{\mathrm{op}}^{(Z_4;\omega^2)} = \begin{pmatrix} \frac{1}{4}(2 - \sqrt{2}) & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{4}(2 + \sqrt{2}) \end{pmatrix}, \tag{5.28}$$

$$\mathcal{K}_{\mathrm{op}}^{(Z_4;\omega^3)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.29}$$

Apparently, physical modes survive only in the cases $\eta = \omega^0, \omega^2$. Each unitary matrix for diagonalizing the kinetic matrix takes the following form:

$$U|_{x=0} = \begin{pmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & -\frac{1}{2}\sqrt{2-\sqrt{2}} \\ \frac{1}{\sqrt{2(2+\sqrt{2})}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix},$$ (5.30)

$$U|_{x=2} = \begin{pmatrix} -\frac{1}{2}\sqrt{2-\sqrt{2}} & \frac{\sqrt{2+\sqrt{2}}}{2} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{2(2+\sqrt{2})}} \end{pmatrix},$$ (5.31)

and the diagonalized kinetic terms are calculated as

$$U^\dagger \mathcal{K}_{\mathrm{op}}^{(Z_4;\omega^0)} U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad U^\dagger \mathcal{K}_{\mathrm{op}}^{(Z_4;\omega^2)} U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ (5.32)

After evaluating the ratios of matrix elements of U as

$$\frac{U_{21}|_{x=0}}{U_{11}|_{x=0}} = \sqrt{2} - 1, \qquad \frac{U_{21}|_{x=2}}{U_{11}|_{x=2}} = -\sqrt{2} - 1,$$ (5.33)

these values should correspond to the ratios of coefficients in the construction with mode functions on T^2 discussed in Ref. [63]. We can easily check that this statement is correct.

6. Summary and discussions

We have discussed an effective way for analyzing the system on the magnetized twisted orbifolds in operator formalism, especially in the following complicated cases T^2/Z_3, T^2/Z_4, T^2/Z_6. With the help of mathematical formulas, we have obtained the exact and analytical results which can be applicable for any larger values of the quantized magnetic flux M. The (non-diagonalized) kinetic terms are immediately generated via the formalism and the number of the surviving physical states are straightforwardly calculable in a rigorous manner by simply following usual procedures in linear algebra. We have checked and re-derived all the results in Ref. [63] based on huge numerical computations with ease analytically.

Based on the achievement in this paper, we can consider a few next directions. One is to construct actual (semi-)realistic models based on the magnetized twisted orbifolds of T^2/Z_2, T^2/Z_3, T^2/Z_4 and T^2/Z_6. Even in the simplest T^2/Z_2 case, possibilities with nontrivial Scherk–Schwarz phases are not yet touched. Complex geometries would help us to generate the complicated nature of fermion flavor structure in the SM, and also to eliminate unwanted exotic states in the zero-mode sector. Here, we would like to emphasize that all the technical obstacles in analysis were removed by the formalism which we have discussed. It is also interesting to study non-Abelian flavor symmetries appearing in T^2/Z_N orbifolds with magnetic fluxes [23–28]. (See also [73].)

Another option is to analyze other geometries, *e.g.*, magnetized twisted orbifolds based on higher-dimensional torus in operator formalism [34]. Our strategy which has been used in this paper is expected to be valid in such much more complicated cases. "Dualities" between analyses with wavefunctions and operator formalism are quite interesting since we can excavate mathematical relations like in Eqs. (5.3) and (5.6). Such a study can be a fascinating theme in mathematical physics.

Acknowledgements

Appendix A. Large gauge transformation in operator formalism

Based on the knowledge obtained in Section 3, we reconsider the quantum mechanical system on T^2/Z_N as

$$\widehat{H}\,|\psi\rangle = E\,|\psi\rangle, \qquad e^{i\widehat{T}_a}\,|\psi\rangle = e^{2\pi i\alpha_a}\,|\psi\rangle \quad (a=1,2), \tag{A.1}$$

where the Hamiltonian $\widehat{H}$ and the operator $\widehat{T}_a$ describing the constraints on the state on T^2/Z_N, $|\psi\rangle$, are given as

$$\widehat{H} = \left(\widehat{\mathbf{p}} + \frac{q}{2}\Omega\,(\widehat{\mathbf{y}} + \mathbf{a})\right)^2, \qquad \widehat{T}_a = \mathbf{u}_a^{\mathrm{T}}\left(\widehat{\mathbf{p}} + \frac{q}{2}\Omega^{\mathrm{T}}(\widehat{\mathbf{y}} + \mathbf{a})\right), \tag{A.2}$$

where the gauge-fixing is already done as in Eq. (3.12). α_1 and α_2 represent the Scherk–Schwarz phases in the original coordinate.

When we examine the gauge transformation,

$$\psi(\mathbf{y}) = e^{-i\frac{q}{2}\mathbf{y}^{\mathrm{T}}(\Omega\mathbf{a})}\psi'(\mathbf{y}), \qquad \langle\mathbf{y}\,|\,\psi\rangle = \psi(\mathbf{y}), \tag{A.3}$$

the Schrödinger equation (3.1) and the pseudo-periodic boundary condition (3.6) are modified as

$$\left(-i\nabla + \frac{q}{2}\Omega\mathbf{y}\right)^2\psi'(\mathbf{y}) = E\psi'(\mathbf{y}), \tag{A.4}$$

$$\begin{aligned}
\psi'(\mathbf{y} + \mathbf{u}_a) &= e^{-i\frac{q}{2}(\mathbf{y}+\mathbf{0})^{\mathrm{T}}\Omega\mathbf{u}_a + 2\pi i\alpha_a - iq\mathbf{a}^{\mathrm{T}}\Omega\mathbf{u}_a}\psi'(\mathbf{y}) \\
&=: e^{-i\frac{q}{2}(\mathbf{y}+\mathbf{0})^{\mathrm{T}}\Omega\mathbf{u}_a + 2\pi i\alpha'_a}\psi'(\mathbf{y}) \quad \text{for } a=1,2,
\end{aligned} \tag{A.5}$$

and the forms of the operators $\widehat{H}$ and $\widehat{T}_a$ get morphed as follows:

$$\widehat{H} \to \widehat{H'} = \left(\widehat{\mathbf{p}} + \frac{q}{2}\Omega\widehat{\mathbf{y}}\right)^2, \qquad \widehat{T}_a \to \widehat{T'_a} = \mathbf{u}_a^{\mathrm{T}}\left(\widehat{\mathbf{p}} + \frac{q}{2}\Omega^{\mathrm{T}}\widehat{\mathbf{y}}\right), \tag{A.6}$$

where the Wilson line phases are gauged away from the Hamiltonian $\widehat{H'}$ and the translational operators $\widehat{T'_a}$. Besides, due to the modification in the operators $\widehat{T'_a}$, the values of the Scherk–Schwarz phases in the gauge-transformed system α'_1 and α'_2 turn out to be

$$2\pi\alpha'_a = 2\pi\alpha_a - q\mathbf{a}^{\mathrm{T}}\Omega\mathbf{u}_a, \quad \text{mod } 2\pi. \tag{A.7}$$

The above expression is equivalent to the following expression:

$$\begin{cases} 2\pi\alpha'_1 = 2\pi\alpha_1 + 2\pi M a_2, \\ 2\pi\alpha'_2 = 2\pi\alpha_2 - 2\pi M a_1, \end{cases} \quad \text{for } T^2/Z_2,$$

$$\begin{cases} 2\pi\alpha_1' = 2\pi\alpha_1 + \dfrac{2\pi}{\sin(2\pi/N)}Ma_2, \\[3mm] 2\pi\alpha_2' = 2\pi\alpha_2 - \dfrac{2\pi}{\sin(2\pi/N)}\big(Ma_1\sin(2\pi/N) - Ma_2\cos(2\pi/N)\big), \\[3mm] \text{for } T^2/Z_N \ (N=3,4,6), \end{cases} \tag{A.8}$$

where we used the choices of $\mathbf{u}_1 = (1,0)^{\mathrm{T}}$, $\mathbf{u}_2 = (0,1)^{\mathrm{T}}$ for $N=2$ and $\mathbf{u}_1 = (1,0)^{\mathrm{T}}$, $\mathbf{u}_2 = (\cos(2\pi/N), \sin(2\pi/N))^{\mathrm{T}}$ for $N = 3, 4, 6$, respectively. We also used

$$\Omega = \frac{2\pi M}{q\sin(2\pi/N)}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.9}$$

which is obtained from Eq. (3.7) with $M := Q_{12}$.

Here, we understand that the Wilson line phases and the Scherk–Schwarz phases are correlated under the large gauge transformation and not independent degrees of freedom. Now, we conclude that we can take

$$\mathbf{a} = \mathbf{0}, \tag{A.10}$$

without any loss of generality.

When we take $\alpha_a = 0$, we can see the correspondence in a direct way. Thus, from Eq. (A.8), the following relations should hold

Z_2 case: $\quad Ma_1 = -\alpha_2', \quad Ma_2 = \alpha_1',$ (A.11)

Z_N case: $\quad Ma_1 = \alpha_1'\cos(2\pi/N) - \alpha_2', \quad Ma_2 = \alpha_1'\sin(2\pi/N) \quad (N=3,4,6),$ (A.12)

or

Z_2 case: $\quad Ma_1 = -\alpha_2', \quad Ma_2 = \alpha_1',$ (A.13)

Z_3 case: $\quad Ma_1 = -\dfrac{3}{2}\alpha', \quad Ma_2 = \dfrac{\sqrt{3}}{2}\alpha',$ (A.14)

Z_4 case: $\quad Ma_1 = -\alpha', \quad Ma_2 = \alpha',$ (A.15)

Z_6 case: $\quad Ma_1 = -\dfrac{1}{2}\alpha', \quad Ma_2 = \dfrac{\sqrt{3}}{2}\alpha'.$ (A.16)

As we discussed in Section 4 in the system with $\mathbf{a} = \mathbf{0}$, the two Scherk–Schwarz phases should take the same value $\alpha' := \alpha_1' = \alpha_2'$ on T^2/Z_3, T^2/Z_4 and T^2/Z_6. After we reflect on the fact that α_1' and α_2' have the period 1, we check that the differences of the allowed values of $M(a_1, a_2)$ correspond to the positions of the fixed points of T^2/Z_N, which strongly indicate that the number of the allowed Scherk–Schwarz phases is connected to the number of the fixed points:

$$\begin{aligned} Z_2 \text{ case:} \quad M(a_1, a_2) &= (0,0) \quad (\alpha_1' = 0, \alpha_2' = 0) \\ &= (1/2, 0) \quad (\alpha_1' = 1/2, \alpha_2' = 0) \\ &= (0, 1/2) \quad (\alpha_1' = 0, \alpha_2' = 1/2) \\ &= (1/2, 1/2) \quad (\alpha_1' = 1/2, \alpha_2' = 1/2), \\ Z_3 \text{ case:} \quad M(a_1, a_2) &= (0,0) \quad (M: \text{even}, \ \alpha' = 0) \\ &= (1/2, \sqrt{3}/6) \quad (M: \text{even}, \ \alpha' = 2/6) \\ &= (0, \sqrt{3}/3) \quad (M: \text{even}, \ \alpha' = 4/6) \end{aligned} \tag{A.17}$$

$$= (3/4, \sqrt{3}/12) \quad \left(M: \text{ odd}, \ \alpha' = 1/6\right)$$
$$= (1/4, \sqrt{3}/4) \quad \left(M: \text{ odd}, \ \alpha' = 3/6\right)$$
$$= (-1/4, 5\sqrt{3}/12) \quad \left(M: \text{ odd } \alpha' = 5/6\right), \tag{A.18}$$

Z_4 case:
$$M(a_1, a_2) = (0, 0) \quad \left(\alpha' = 0\right)$$
$$= (1/2, 1/2) \quad \left(\alpha' = 1/2\right), \tag{A.19}$$

Z_6 case:
$$M(a_1, a_2) = (0, 0) \quad \left(M: \text{ even}, \ \alpha' = 0\right)$$
$$= (3/4, \sqrt{3}/4) \quad \left(M: \text{ odd}, \ \alpha' = 1/2\right). \tag{A.20}$$

Note that the results are totally consistent with those in Ref. [63] with wavefunctions.

Appendix B. Derivation of formulas

B.1. Eq. (4.48)

In this part, we derive the formula (4.48)

$$I_\pm(t, \beta) := \sum_{s=0}^{M-1} e^{-\pi i \frac{(s+t\pm\beta)^2}{M}} = \sqrt{M} e^{-\frac{1}{4}\pi i} \quad \text{for } t \in \mathbb{Z},$$
$$\beta = \begin{cases} 0 & \text{for } M: \text{ even}, \\ \frac{1}{2} & \text{for } M: \text{ odd}, \end{cases} \tag{B.1}$$

where M is a positive integer and the resultant form is independent of t, β and the sign in front of β. First, we show the relation $I_\pm(t, \beta) = I_\pm(0, \beta)$, which implies the independence of t.

$$I_\pm(t, \beta) = \sum_{s'=t}^{M-1+t} e^{-\pi i \frac{(s'\pm\beta)^2}{M}} \quad \left(s' := s + t\right)$$
$$= \left[\sum_{s'=t}^{M-1} + \sum_{s'=M}^{M-1+t}\right] e^{-\pi i \frac{(s'\pm\beta)^2}{M}}$$
$$= \sum_{s'=t}^{M-1} e^{-\pi i \frac{(s'\pm\beta)^2}{M}} + \sum_{s''=0}^{t-1} e^{-i \frac{\pi}{M}[(s'\pm\beta)^2 + M(M\pm2\beta+2s'')]} \quad \left(s'' := s' - M\right). \tag{B.2}$$

After noticing that $M\pm2\beta$ is always an even integer, we can justify the manipulation,

$$e^{-i \frac{\pi}{M}[M(M\pm2\beta+2s'')]} = e^{-i\pi(M\pm2\beta+2s'')} \to 1. \tag{B.3}$$

Then, the following result is obtained

$$I_\pm(t, \beta) = \left[\sum_{s'=t}^{M-1} + \sum_{s'=0}^{t-1}\right] e^{-\pi i \frac{(s'\pm\beta)^2}{M}} = \sum_{s=0}^{M-1} e^{-\pi i \frac{(s\pm\beta)^2}{M}} = I_\pm(0, \beta). \tag{B.4}$$

From now on, we can set $t = 0$ without loss of generality and examine the $\beta = 0$ (M is even) and $\beta = 1/2$ (M is odd) separately.

The former case ($\beta = 0$) is evaluated straightforwardly. Using the periodicity of the exponential functions (when M is even), the following deformation is realized

$$I_\pm(0,0) = \frac{1}{2}\left[\sum_{k=0}^{M-1} e^{-\pi i \frac{k^2}{M}} + \sum_{k=M}^{2M-1} e^{-\pi i \frac{(k-M)^2}{M}}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{M-1} + \sum_{k=M}^{2M-1}\right] e^{-\pi i \frac{k^2}{M}}$$

$$= \frac{1}{2}\sum_{k=0}^{N-1} e^{-2\pi i \frac{k^2}{N}} \quad (N := 2M). \tag{B.5}$$

After using the mathematical relation, which is proved in the next subsection,

$$\sum_{s=0}^{N-1} e^{-2\pi i \frac{s^2}{N}} = \sqrt{\frac{N}{2}}\, e^{-\frac{1}{4}\pi i}\left(1 + e^{\frac{N}{2}\pi i}\right) \quad (N \in \mathbb{N}), \tag{B.6}$$

we can reach the final form,

$$I_\pm(0,0) = \frac{1}{2}\sqrt{M}\, e^{-\frac{1}{4}\pi i}\left(1 + e^{M\pi i}\right) = \sqrt{M}\, e^{-\frac{1}{4}\pi i}, \tag{B.7}$$

where M is even and then $e^{M\pi i} = 1$.

In the latter case $(\beta = 1/2)$ is somewhat complicated. By use of the fact that $2M$ is even, the following deformation is possible,

$$I_+\left(0, \frac{1}{2}\right) = \frac{1}{2}\sum_{s=0}^{N-1} e^{-2\pi i \frac{(2s+1)^2}{4N}} \quad (N := 2M)$$

$$= \frac{1}{2}\sum_{\substack{k=+1,\\k:\text{odd}}}^{2N-1} e^{-2\pi i \frac{k^2}{4N}} \quad (k := 2s+1)$$

$$= \frac{1}{2}\left\{\left[\sum_{k=+1}^{2N-1} - \sum_{\substack{k=+1,\\k:\text{even}}}^{2N-1}\right] e^{-2\pi i \frac{k^2}{4N}}\right\}$$

$$= \frac{1}{2}\left\{\left[\sum_{k=0}^{2N-1} - \sum_{\substack{k=0,\\k:\text{even}}}^{2N-1}\right] e^{-2\pi i \frac{k^2}{4N}}\right\}$$

$$= \frac{1}{2}\left\{\sum_{k=0}^{2N-1} e^{-2\pi i \frac{k^2}{4N}} - \sum_{l=0}^{N-1} e^{-2\pi i \frac{l^2}{N}}\right\} \quad \left(l := \frac{k}{2}\right), \tag{B.8}$$

$$I_-\left(0, \frac{1}{2}\right) = \frac{1}{2}\sum_{s=0}^{N-1} e^{-2\pi i \frac{(2s-1)^2}{4N}} \quad (N := 2M)$$

$$= \frac{1}{2}\sum_{\substack{k=-1,\\k:\text{odd}}}^{2N-3} e^{-2\pi i \frac{k^2}{4N}} \quad (k := 2s-1)$$

$$= \frac{1}{2}\left\{\left[\sum_{k=-1}^{2N-1} - \sum_{\substack{k=-1,\\k:\text{even}}}^{2N-1}\right] e^{-2\pi i \frac{k^2}{4N}} - e^{-2\pi i \frac{(2N-1)^2}{4N}}\right\}$$

$$
= \frac{1}{2} \left\{ \left[\sum_{k=0}^{2N-1} e^{-2\pi i \frac{k^2}{4N}} + e^{-2\pi i \frac{(-1)^2}{4N}} \right] \right.
$$

$$
\left. - \sum_{\substack{k=0, \\ k: \text{ even}}}^{2N-2} e^{-2\pi i \frac{k^2}{4N}} - e^{-2\pi i \frac{1}{4N}[4N(N-1)+1]} \right\}
$$

$$
= \frac{1}{2} \left\{ \sum_{k=0}^{2N-1} e^{-2\pi i \frac{k^2}{4N}} - \sum_{l=0}^{N-1} e^{-2\pi i \frac{l^2}{N}} \right\} \quad \left(l := \frac{k}{2} \right), \tag{B.9}
$$

where the two final forms get to be the same. Here, the second term of the last line of Eq. (B.8) or (B.9) can be calculated with the help of the formula in Eq. (B.6). The first term of the last line in Eq. (B.8) or (B.9) needs additional transformations to be evaluated,

$$
\sum_{k=0}^{2N-1} e^{-2\pi i \frac{k^2}{4N}} = \frac{1}{2} \left[\sum_{k=0}^{2N-1} e^{-2\pi i \frac{k^2}{4N}} + \sum_{k=2N}^{4N-1} e^{-2\pi i \frac{(k-2N)^2}{4N}} \right]
$$

$$
= \frac{1}{2} \left[\sum_{k=0}^{2N-1} e^{-2\pi i \frac{k^2}{4N}} + \sum_{k=2N}^{4N-1} e^{-2\pi i \frac{(k^2+4N(N-k))}{4N}} \right]
$$

$$
= \frac{1}{2} \left[\sum_{k=0}^{2N-1} + \sum_{k=2N}^{4N-1} \right] e^{-2\pi i \frac{k^2}{4N}}
$$

$$
= \frac{1}{2} \sum_{k=0}^{4N-1} e^{-2\pi i \frac{k^2}{4N}}, \tag{B.10}
$$

in which the final form is applicable for the formula in Eq. (B.6). Now, we can show the final result as

$$
I_\pm \left(0, \frac{1}{2} \right) = \frac{1}{4} \sqrt{4M} e^{-\frac{1}{4}\pi i} \left(1 + e^{4M\pi i} \right) - \frac{1}{2} \sqrt{M} e^{-\frac{1}{4}\pi i} \left(1 + e^{M\pi i} \right)
$$

$$
= \sqrt{M} e^{-\frac{1}{4}\pi i}, \tag{B.11}
$$

where we use the oddness of M as $e^{4M\pi i} = 1$, $e^{M\pi i} = -1$.

B.2. Eq. (B.6)

In this part, we prove an essential formula in the previous subsection. Firstly, we consider the following function with a positive integer M,

$$
F(z) = \frac{e^{2\pi i z^2 / M}}{e^{2\pi i z} - 1}, \tag{B.12}
$$

where it contains the shift properties,

$$
F(z + M) = e^{4\pi i z} F(z),
$$
$$
F(z + M) - F(z) = e^{2\pi i z^2 / M + 2\pi i z} + e^{2\pi i z^2 / M}. \tag{B.13}
$$

The integral contour is considered in Fig. B.1, which consists of four paths C_1, C_2, C_3 and C_4 with the condition on θ of $0 < \theta < \pi/4$. Here, we set $a \notin \mathbb{Z}$ to avoid poles of $F(z)$.
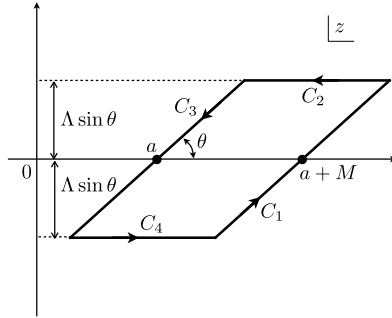
Fig. B.1. A description for the integral contour consisting of four paths C_1, C_2, C_3 and C_4 with the condition on θ of $0 < \theta < \pi/4$.

The complex integrals on C_2 and C_4 can be ignored in the limit $\Lambda \to \infty$ since $F(z)$ gets to be zero in this limit (when $0 < \theta < \pi/4$), and then the remaining integrals are

$$I(\Lambda) := \int_{C_1} dz F(z) + \int_{C_3} dz F(z) = \int_{-\Lambda}^{\Lambda} dr e^{i\theta} \big[F(M + a + re^{i\theta}) - F(a + re^{i\theta}) \big]. \quad \text{(B.14)}$$

After using the shift properties in Eq. (B.13), we obtain

$$I(\Lambda) = \int_{-\Lambda}^{\Lambda} dr e^{i\theta} \big[e^{2\pi i (a + re^{i\theta})^2 / M + 2\pi i (a + re^{i\theta})} + e^{2\pi i (a + re^{i\theta})^2 / M} \big]. \quad \text{(B.15)}$$

When we take a notice of $\sin 2\theta / M > 0$, changing variable from r to $x := (re^{i\theta} + a)e^{\pi i/4}$ is available to perform the integration $I(\Lambda)$. Also, we can use generalized Fresnel integrals,

$$\int_0^\infty dx e^{-(b^2 \cot \phi) x^2} \cos(b^2 x^2) = \frac{\sqrt{\pi}}{2b} \sqrt{\sin \phi} \cos(\phi/2), \quad \text{(B.16)}$$

$$\int_0^\infty dx e^{-(b^2 \cot \phi) x^2} \sin(b^2 x^2) = \frac{\sqrt{\pi}}{2b} \sqrt{\sin \phi} \sin(\phi/2), \quad \text{(B.17)}$$

where $b > 0$ and $0 < \phi < \pi/2$.

The limiting value

$$I(\Lambda \to \infty) = \sqrt{\frac{M\pi}{2\pi}} e^{i\pi/4 - i\pi M/2} + \sqrt{\frac{M\pi}{2\pi}} e^{i\pi/4} \quad \text{(B.18)}$$

is independent of a. Since the poles of F in the contour are located in $z = [a] + 1, \ldots, [a] + M$, we use the residue theorem on complex integral,

$$\frac{1}{2\pi i} \sum_{k=[a]+1}^{[a]+M} (2\pi i) e^{2\pi i k^2 / M} = \sqrt{\frac{M\pi}{2\pi}} e^{i\pi/4 - i\pi M/2} + \sqrt{\frac{M\pi}{2\pi}} e^{i\pi/4}. \quad \text{(B.19)}$$

Here, the right-hand side of Eq. (B.19) is independent of a and still, $e^{2\pi i([a]+M+1)^2/M} = e^{2\pi i([a]+1)^2/M}$. When we set a in the range of $-1 < a < 0$, the following sum formula can be derived,

$$\sum_{k=0}^{M-1} e^{2\pi i k^2/M} = \sqrt{\frac{M}{2}} e^{i\pi/4 - i\pi M/2} + \sqrt{\frac{M}{2}} e^{i\pi/4} = \sqrt{\frac{M}{2}} e^{i\pi/4}\left(1 + e^{-\pi i M/2}\right), \qquad (B.20)$$

which is just (complex-conjugated) what we would like to show.

References

[1] C. Bachas, A way to break supersymmetry, arXiv:hep-th/9503030.
[2] R. Blumenhagen, L. Goerlich, B. Kors, D. Lust, Noncommutative compactifications of type I strings on tori with magnetic background flux, J. High Energy Phys. 0010 (2000) 006, http://dx.doi.org/10.1088/1126-6708/2000/10/006, arXiv:hep-th/0007024.
[3] C. Angelantonj, I. Antoniadis, E. Dudas, A. Sagnotti, Type I strings on magnetized orbifolds and brane transmutation, Phys. Lett. B 489 (2000) 223–232, http://dx.doi.org/10.1016/S0370-2693(00)00907-2, arXiv:hep-th/0007090.
[4] R. Blumenhagen, B. Kors, D. Lust, Type I strings with F flux and B flux, J. High Energy Phys. 0102 (2001) 030, http://dx.doi.org/10.1088/1126-6708/2001/02/030, arXiv:hep-th/0012156.
[5] D. Cremades, L. Ibanez, F. Marchesano, Computing Yukawa couplings from magnetized extra dimensions, J. High Energy Phys. 0405 (2004) 079, http://dx.doi.org/10.1088/1126-6708/2004/05/079, arXiv:hep-th/0404229.
[6] R. Blumenhagen, M. Cvetic, P. Langacker, G. Shiu, Toward realistic intersecting D-brane models, Annu. Rev. Nucl. Part. Sci. 55 (2005) 71–139, http://dx.doi.org/10.1146/annurev.nucl.55.090704.151541, arXiv:hep-th/0502005.
[7] R. Blumenhagen, B. Kors, D. Lust, S. Stieberger, Four-dimensional string compactifications with D-branes, orientifolds and fluxes, Phys. Rep. 445 (2007) 1–193, http://dx.doi.org/10.1016/j.physrep.2007.04.003, arXiv:hep-th/0610327.
[8] Y. Fujimoto, T. Nagasawa, K. Nishiwaki, M. Sakamoto, Quark mass hierarchy and mixing via geometry of extra dimension with point interactions, PTEP, Proces. Teh. Energ. Poljopr. 2013 (2013) 023B07, http://dx.doi.org/10.1093/ptep/pts097, arXiv:1209.5150.
[9] Y. Fujimoto, K. Nishiwaki, M. Sakamoto, CP phase from twisted Higgs VEV in extra dimension, Phys. Rev. D 88 (2013) 115007, http://dx.doi.org/10.1103/PhysRevD.88.115007, arXiv:1301.7253.
[10] Y. Fujimoto, K. Nishiwaki, M. Sakamoto, R. Takahashi, Realization of lepton masses and mixing angles from point interactions in an extra dimension, arXiv:1405.5872.
[11] D. Cremades, L. Ibanez, F. Marchesano, Yukawa couplings in intersecting D-brane models, J. High Energy Phys. 0307 (2003) 038, http://dx.doi.org/10.1088/1126-6708/2003/07/038, arXiv:hep-th/0302105.
[12] M. Cvetic, I. Papadimitriou, Conformal field theory couplings for intersecting D-branes on orientifolds, Phys. Rev. D 68 (2003) 046001, http://dx.doi.org/10.1103/PhysRevD.68.046001, arXiv:hep-th/0303083.
[13] S. Abel, A. Owen, Interactions in intersecting brane models, Nucl. Phys. B 663 (2003) 197–214, http://dx.doi.org/10.1016/S0550-3213(03)00370-5, arXiv:hep-th/0303124.
[14] G. Honecker, J. Vanhoof, Yukawa couplings and masses of non-chiral states for the Standard Model on D6-branes on T6/Z6′, J. High Energy Phys. 1204 (2012) 085, http://dx.doi.org/10.1007/JHEP04(2012)085, arXiv:1201.3604.
[15] S. Abel, A. Owen, N point amplitudes in intersecting brane models, Nucl. Phys. B 682 (2004) 183–216, http://dx.doi.org/10.1016/j.nuclphysb.2003.11.032, arXiv:hep-th/0310257.
[16] S. Abel, M. Goodsell, Intersecting brane worlds at one loop, J. High Energy Phys. 0602 (2006) 049, http://dx.doi.org/10.1088/1126-6708/2006/02/049, arXiv:hep-th/0512072.
[17] S. Abel, M. Goodsell, Realistic Yukawa couplings through instantons in intersecting brane worlds, J. High Energy Phys. 0710 (2007) 034, http://dx.doi.org/10.1088/1126-6708/2007/10/034, arXiv:hep-th/0612110.
[18] I. Pesando, Green functions and twist correlators for N branes at angles, Nucl. Phys. B 866 (2013) 87–123, http://dx.doi.org/10.1016/j.nuclphysb.2012.08.016, arXiv:1206.1431.
[19] I. Pesando, Correlators of arbitrary untwisted operators and excited twist operators for N branes at angles, Nucl. Phys. B 886 (2014) 243–287, http://dx.doi.org/10.1016/j.nuclphysb.2014.06.010, arXiv:1401.6797.
[20] H. Abe, T. Kobayashi, H. Ohki, A. Oikawa, K. Sumita, Phenomenological aspects of 10D SYM theory with magnetized extra dimensions, Nucl. Phys. B 870 (2013) 30–54, http://dx.doi.org/10.1016/j.nuclphysb.2013.01.014, arXiv:1211.4317.

[21] H. Abe, T. Kobayashi, K. Sumita, Y. Tatsuta, Gaussian Froggatt–Nielsen mechanism on magnetized orbifolds, arXiv:1405.5012.

[22] H. Abe, K.-S. Choi, T. Kobayashi, H. Ohki, Higher order couplings in magnetized brane models, J. High Energy Phys. 0906 (2009) 080, http://dx.doi.org/10.1088/1126-6708/2009/06/080, arXiv:0903.3800.

[23] H. Abe, K.-S. Choi, T. Kobayashi, H. Ohki, Non-Abelian discrete flavor symmetries from magnetized/intersecting brane models, Nucl. Phys. B 820 (2009) 317–333, http://dx.doi.org/10.1016/j.nuclphysb.2009.05.024, arXiv:0904.2631.

[24] H. Abe, K.-S. Choi, T. Kobayashi, H. Ohki, Magnetic flux, Wilson line and orbifold, Phys. Rev. D 80 (2009) 126006, http://dx.doi.org/10.1103/PhysRevD.80.126006, arXiv:0907.5274.

[25] H. Abe, K.-S. Choi, T. Kobayashi, H. Ohki, Flavor structure from magnetic fluxes and non-Abelian Wilson lines, Phys. Rev. D 81 (2010) 126003, http://dx.doi.org/10.1103/PhysRevD.81.126003, arXiv:1001.1788.

[26] M. Berasaluce-Gonzalez, P. Camara, F. Marchesano, D. Regalado, A. Uranga, Non-Abelian discrete gauge symmetries in 4d string models, J. High Energy Phys. 1209 (2012) 059, http://dx.doi.org/10.1007/JHEP09(2012)059, arXiv:1206.2383.

[27] G. Honecker, W. Staessens, To tilt or not to tilt: discrete gauge symmetries in global intersecting D-brane models, J. High Energy Phys. 1310 (2013) 146, http://dx.doi.org/10.1007/JHEP10(2013)146, arXiv:1303.4415.

[28] F. Marchesano, D. Regalado, L. Vazquez-Mercado, Discrete flavor symmetries in D-brane models, J. High Energy Phys. 1309 (2013) 028, http://dx.doi.org/10.1007/JHEP09(2013)028, arXiv:1306.1284.

[29] H. Abe, T. Kobayashi, H. Ohki, K. Sumita, Y. Tatsuta, Non-Abelian discrete flavor symmetries of 10D SYM theory with magnetized extra dimensions, J. High Energy Phys. 1406 (2014) 017, http://dx.doi.org/10.1007/JHEP06(2014)017, arXiv:1404.0137.

[30] T. Kobayashi, H.P. Nilles, F. Ploger, S. Raby, M. Ratz, Stringy origin of non-Abelian discrete flavor symmetries, Nucl. Phys. B 768 (2007) 135–156, http://dx.doi.org/10.1016/j.nuclphysb.2007.01.018, arXiv:hep-ph/0611020.

[31] T. Kobayashi, S. Raby, R.-J. Zhang, Searching for realistic 4d string models with a Pati–Salam symmetry: orbifold grand unified theories from heterotic string compactification on a Z(6) orbifold, Nucl. Phys. B 704 (2005) 3–55, http://dx.doi.org/10.1016/j.nuclphysb.2004.10.035, arXiv:hep-ph/0409098.

[32] P. Ko, T. Kobayashi, J.-h. Park, S. Raby, String-derived $D(4)$ flavor symmetry and phenomenological implications, Phys. Rev. D 76 (2007) 035005, http://dx.doi.org/10.1103/PhysRevD.76.035005, arXiv:0704.2807.

[33] Y. Hamada, T. Kobayashi, Massive modes in magnetized brane models, Prog. Theor. Phys. 128 (2012) 903–923, http://dx.doi.org/10.1143/PTP.128.903, arXiv:1207.6867.

[34] M. Sakamoto, S. Tanimura, An extension of Fourier analysis for the n torus in the magnetic field and its application to spectral analysis of the magnetic Laplacian, J. Math. Phys. 44 (2003) 5042–5069, http://dx.doi.org/10.1063/1.1616203, arXiv:hep-th/0306006.

[35] I. Antoniadis, T. Maillard, Moduli stabilization from magnetic fluxes in type I string theory, Nucl. Phys. B 716 (2005) 3–32, http://dx.doi.org/10.1016/j.nuclphysb.2005.03.026, arXiv:hep-th/0412008.

[36] I. Antoniadis, A. Kumar, B. Panda, Fermion wavefunctions in magnetized branes: theta identities and Yukawa couplings, Nucl. Phys. B 823 (2009) 116–173, http://dx.doi.org/10.1016/j.nuclphysb.2009.08.002, arXiv:0904.0910.

[37] K.-S. Choi, T. Kobayashi, R. Maruyama, M. Murata, Y. Nakai, et al., $E(6, 7, 8)$ magnetized extra dimensional models, Eur. Phys. J. C 67 (2010) 273–282, http://dx.doi.org/10.1140/epjc/s10052-010-1275-9, arXiv:0908.0395.

[38] T. Kobayashi, R. Maruyama, M. Murata, H. Ohki, M. Sakai, Three-generation models from E_8 magnetized extra dimensional theory, J. High Energy Phys. 1005 (2010) 050, http://dx.doi.org/10.1007/JHEP05(2010)050, arXiv:1002.2828.

[39] P. Di Vecchia, R. Marotta, I. Pesando, F. Pezzella, Open strings in the system D5/D9, J. Phys. A 44 (2011) 245401, http://dx.doi.org/10.1088/1751-8113/44/24/245401, arXiv:1101.0120.

[40] H. Abe, T. Kobayashi, H. Ohki, K. Sumita, Superfield description of 10D SYM theory with magnetized extra dimensions, Nucl. Phys. B 863 (2012) 1–18, http://dx.doi.org/10.1016/j.nuclphysb.2012.05.012, arXiv:1204.5327.

[41] L. De Angelis, R. Marotta, F. Pezzella, R. Troise, More about branes on a general magnetized torus, J. High Energy Phys. 1210 (2012) 052, http://dx.doi.org/10.1007/JHEP10(2012)052, arXiv:1206.3401.

[42] H. Abe, T. Kobayashi, H. Ohki, K. Sumita, Y. Tatsuta, Flavor landscape of 10D SYM theory with magnetized extra dimensions, arXiv:1307.1831.

[43] E.J. Ferrer, V. de la Incera, Mass eigenvalues of the open charged string in a magnetic background, Phys. Rev. D 52 (1995) 1011–1018, http://dx.doi.org/10.1103/PhysRevD.52.1011.

[44] E.J. Ferrer, V. de la Incera, Global symmetries of open strings in an electromagnetic background, Phys. Rev. D 49 (1994) 2926–2932, http://dx.doi.org/10.1103/PhysRevD.49.2926.

[45] P. Di Vecchia, A. Liccardo, R. Marotta, I. Pesando, F. Pezzella, Wrapped magnetized branes: two alternative descriptions?, J. High Energy Phys. 0711 (2007) 100, http://dx.doi.org/10.1088/1126-6708/2007/11/100, arXiv:0709.4149.

[46] L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Strings on orbifolds, Nucl. Phys. B 261 (1985) 678–686, http://dx.doi.org/10.1016/0550-3213(85)90593-0.

[47] L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Strings on orbifolds. 2, Nucl. Phys. B 274 (1986) 285–314, http://dx.doi.org/10.1016/0550-3213(86)90287-7.

[48] Y. Kawamura, Gauge symmetry breaking from extra space S^1/Z_2, Prog. Theor. Phys. 103 (2000) 613–619, http://dx.doi.org/10.1143/PTP.103.613, arXiv:hep-ph/9902423.

[49] Y. Kawamura, Split multiplets, coupling unification and extra dimension, Prog. Theor. Phys. 105 (2001) 691–696, http://dx.doi.org/10.1143/PTP.105.691, arXiv:hep-ph/0012352.

[50] Y. Kawamura, Triplet doublet splitting, proton stability and extra dimension, Prog. Theor. Phys. 105 (2001) 999–1006, http://dx.doi.org/10.1143/PTP.105.999, arXiv:hep-ph/0012125.

[51] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo, Y. Ono, et al., $Z(N)$ orbifold models, Nucl. Phys. B 341 (1990) 611–640, http://dx.doi.org/10.1016/0550-3213(90)90542-L.

[52] T. Kobayashi, N. Ohtsubo, Geometrical aspects of $Z(N)$ orbifold phenomenology, Int. J. Mod. Phys. A 9 (1994) 87–126, http://dx.doi.org/10.1142/S0217751X94000054.

[53] K.-S. Choi, J.E. Kim, Quarks and leptons from orbifolded superstring, Lect. Notes Phys. 696 (2006) 1–406, http://dx.doi.org/10.1007/b11681670.

[54] Y. Kawamura, T. Kinami, T. Miura, Equivalence classes of boundary conditions in gauge theory on $Z(3)$ orbifold, Prog. Theor. Phys. 120 (2008) 815–831, http://dx.doi.org/10.1143/PTP.120.815, arXiv:0808.2333.

[55] Y. Kawamura, T. Miura, Equivalence classes of boundary conditions in SU(N) gauge theory on 2-dimensional orbifolds, Prog. Theor. Phys. 122 (2010) 847–864, http://dx.doi.org/10.1143/PTP.122.847, arXiv:0905.4123.

[56] Y. Kawamura, T. Kinami, K.-y. Oda, Orbifold family unification, Phys. Rev. D 76 (2007) 035001, http://dx.doi.org/10.1103/PhysRevD.76.035001, arXiv:hep-ph/0703195.

[57] Y. Kawamura, T. Miura, Orbifold family unification in SO($2N$) gauge theory, Phys. Rev. D 81 (2010) 075011, http://dx.doi.org/10.1103/PhysRevD.81.075011, arXiv:0912.0776.

[58] Y. Goto, Y. Kawamura, T. Miura, Orbifold family unification on 6 dimensions, Phys. Rev. D 88 (2013) 055016, http://dx.doi.org/10.1103/PhysRevD.88.055016, arXiv:1307.2631.

[59] H. Abe, T. Kobayashi, H. Ohki, Magnetized orbifold models, J. High Energy Phys. 0809 (2008) 043, http://dx.doi.org/10.1088/1126-6708/2008/09/043, arXiv:0806.4748.

[60] H. Abe, K.-S. Choi, T. Kobayashi, H. Ohki, Three generation magnetized orbifold models, Nucl. Phys. B 814 (2009) 265–292, http://dx.doi.org/10.1016/j.nuclphysb.2009.02.002, arXiv:0812.3534.

[61] S. Groot Nibbelink, P.K. Vaudrevange, Schoen manifold with line bundles as resolved magnetized orbifolds, J. High Energy Phys. 1303 (2013) 142, http://dx.doi.org/10.1007/JHEP03(2013)142, arXiv:1212.4033.

[62] Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, M. Sakamoto, Shifted orbifold models with magnetic flux, Phys. Rev. D 87 (2013) 086001, http://dx.doi.org/10.1103/PhysRevD.87.086001, arXiv:1302.5768.

[63] T.-H. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, et al., Z_N twisted orbifold models with magnetic flux, J. High Energy Phys. 1401 (2014) 065, http://dx.doi.org/10.1007/JHEP01(2014)065, arXiv:1309.4925.

[64] J. Scherk, J.H. Schwarz, Spontaneous breaking of supersymmetry through dimensional reduction, Phys. Lett. B 82 (1979) 60, http://dx.doi.org/10.1016/0370-2693(79)90425-8.

[65] J. Scherk, J.H. Schwarz, How to get masses from extra dimensions, Nucl. Phys. B 153 (1979) 61–88, http://dx.doi.org/10.1016/0550-3213(79)90592-3.

[66] L.E. Ibanez, H.P. Nilles, F. Quevedo, Orbifolds and Wilson lines, Phys. Lett. B 187 (1987) 25–32, http://dx.doi.org/10.1016/0370-2693(87)90066-9.

[67] T. Kobayashi, N. Ohtsubo, Analysis on the Wilson lines of $Z(N)$ orbifold models, Phys. Lett. B 257 (1991) 56–62, http://dx.doi.org/10.1016/0370-2693(91)90858-N.

[68] C. Angelantonj, M. Cardella, N. Irges, Scherk–Schwarz breaking and intersecting branes, Nucl. Phys. B 725 (2005) 115–154, http://dx.doi.org/10.1016/j.nuclphysb.2005.07.025, arXiv:hep-th/0503179.

[69] R. Blumenhagen, M. Cvetic, F. Marchesano, G. Shiu, Chiral D-brane models with frozen open string moduli, J. High Energy Phys. 0503 (2005) 050, http://dx.doi.org/10.1088/1126-6708/2005/03/050, arXiv:hep-th/0502095.

[70] C. Angelantonj, C. Condeescu, E. Dudas, M. Lennek, Stringy instanton effects in models with rigid magnetised D-branes, Nucl. Phys. B 818 (2009) 52–94, http://dx.doi.org/10.1016/j.nuclphysb.2009.03.018, arXiv:0902.1694.

[71] S. Forste, G. Honecker, Rigid D6-branes on $T^6/(Z_2 \times Z_{2M} \times \Omega R)$ with discrete torsion, J. High Energy Phys. 1101 (2011) 091, http://dx.doi.org/10.1007/JHEP01(2011)091, arXiv:1010.6070.

[72] A. Hashimoto, W. Taylor, Fluctuation spectra of tilted and intersecting D-branes from the Born–Infeld action, Nucl. Phys. B 503 (1997) 193–219, http://dx.doi.org/10.1016/S0550-3213(97)00399-4, arXiv:hep-th/9703217.

[73] Y. Hamada, T. Kobayashi, S. Uemura, Flavor structure in D-brane models: Majorana neutrino mases, arXiv:1402.2052.