

On Quantized Liénard Oscillator and Momentum Dependent Mass

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Abstract

We examine the analytical structure of the nonlinear Liénard oscillator and show that it is a bi-Hamiltonian system depending upon the choice of the coupling parameters. While one has been recently studied in the context of a quantized momentum-dependent mass system, the other Hamiltonian also reflects a similar feature in the mass function and also depicts an isotonic character. We solve for such a Hamiltonian and give the complete solution in terms of a confluent hypergeometric function.

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1 Introduction

Exploring the Schrödinger equation in the momentum space is often advantageous because many quantities of physical interest are more readily evaluated in this representation rather than in the coordinate formulation. This is especially true for some typical scattering problems and form factors of certain kinds [1, 2]. It is worthwhile to recall that for the simple one-dimensional hydrogen atomic system it took well over thirty years to fully appreciate its underlying principles and that too after an analysis was carried out in the momentum space representation [3, 4]. Very recently an interesting aspect has been brought to light that concerns the relevance of a momentum-dependent mass for a quantized nonlinear oscillator of Liénard type [5, 6, 7, 8, 9]

$$\ddot{x} + kx\dot{x} + \omega^2x + \frac{k^2}{9}x^3 = 0, \quad (1)$$

where k and ω are real parameters. It admits a periodic solution for $x(t)$ namely

$$x = \frac{A \sin(\omega t + \delta)}{1 - \frac{kA}{3\omega} \cos(\omega t + \delta)}, \quad (2)$$

where A and δ are arbitrary constants subject to $-1 < kA/3\omega < 1$. An intriguing feature about the Hamiltonian of (1) is that it depicts an interchange of the roles of the position and momentum variables in its kinetic and potential energy terms respectively. However, it is important to realize that its quantization is hard to tackle in the coordinate representation of the Schrodinger equation but can be successfully carried out in the momentum space. Note that the Hamiltonian tied-up with (1) is not unique: rather it points to a bi-Hamiltonian system. This is not surprising since, it also reveals, in the quantum case, a Goldman and Krivchenkov-type isotonic system [10, 11] given by the half-line combination of a harmonic oscillator and centrifugal barrier-like term. The purpose of this communication is to first identify such a Hamiltonian and then give a complete solution of the problem in terms of the confluent hypergeometric function. Such a Hamiltonian as we will demonstrate below may be interpreted to represent a momentum-dependent effective mass quantum system as guided by the choice of an underlying variable parameter η . While the complementary quantum problem of the position-dependent mass has received considerable attention due to its relevance in describing the dynamics of electrons in problems of compositionally graded crystals [12], quantum dots [13] and liquid crystals [14], interest in momentum-dependent mass problems is a somewhat recent curiosity arising from the observation that the parity-time symmetric Liénard type nonlinear oscillator can afford complete solvability in a momentum space description [5]. Motivated by such a revelation we undertake a complete treatment of the Lagrangian description of (1) employing the Jacobi Last Multiplier (JLM) approach

and then moving on to a Hamiltonian formulation. As it will turn out, the general form of the Lagrangian is guided by two choices of η one of which has already been studied. The second one yields a new candidate for the Hamiltonian of (1) an inquiry of which is one of the objectives of the present work.

2 A Lagrangian description of the Liénard equation

Towards this end it is instructive to review the results already available in the literature [5, 6, 7, 8] for a generalized class of Liénard equation given by

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \quad (3)$$

which reduces to (1) for the following specific forms of f and g :

$$f(x) = kx, \quad g(x) = \omega^2 x + \frac{k^2}{9}x^3. \quad (4)$$

Note that both f and g are odd function of x . To study the dynamical aspects of (3) it is convenient to adopt the method of the Jacobi Last Multiplier (JLM) whose relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation, $\ddot{x} = F(t, x, \dot{x})$, is given by [9, 7]

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \quad (5)$$

$M = M(t, x, \dot{x})$, the JLM, satisfies the following equation

$$\frac{d}{dt}(\log M) + \frac{\partial F}{\partial \dot{x}} = 0. \quad (6)$$

In the present case (6) reads

$$\frac{d}{dt}(\log M) - f(x) = 0, \quad (7)$$

whose formal solution in terms of a nonlocal variable u is

$$M(t, x) = \exp\left(\int f(x)dt\right) := u^\eta, \quad (8)$$

where η is a variable parameter. From (8) it is obvious that, $\eta\dot{u} = uf(x)$ and we assume equation (3) can be recast as a pair of coupled first-order differential equations

$$\dot{u} = \frac{1}{\eta}u f(x), \quad \dot{x} = u + W(x). \quad (9)$$

The functional form of $W(x)$ may be determined by differentiating the latter equation of (9) with respect to t and using the former to eliminate the nonlocal variable u , which yields

$$\ddot{x} - \left(\frac{1}{\eta}f(x) + W'(x)\right)\dot{x} + \frac{1}{\eta}Wf(x) = 0.$$

A comparison with (3) shows that $W = \eta g/f$ and the functions f and g are subject to the following constraint involving η .

$$\frac{d}{dx} \left(\frac{g}{f} \right) = -\frac{1}{\eta} \left(\frac{1}{\eta} + 1 \right) f(x). \quad (10)$$

The ratio g/f can be integrated out to

$$\eta \frac{g}{f} = \left[-\frac{\eta+1}{\eta} \int^x f(s) ds + \nu \right], \quad (11)$$

where ν is a constant of integration. It is interesting to note here that when (11) is coupled with the choices for f and g in (4) it leads to a quadratic equation determining the parameter η , viz. $2\eta^2 + 9\eta + 9 = 0$, and yields two plausible solutions for η :

$$\eta = -3 \quad \text{and} \quad -3/2, \quad (12)$$

with $\nu = -3\omega^2/k$ and $-3\omega^2/2k$ respectively. While $\eta = -3$ was investigated in [5], the second solution $\eta = -3/2$ is new and is the point of focus in this paper.

Turning to (9) we have from (5) and (8)

$$\frac{\partial^2 L}{\partial \dot{x}^2} = \left(\dot{x} - \eta \frac{g}{f} \right)^\eta, \quad (13)$$

which works out to the following explicit form for L :

$$L(x, \dot{x}, t) = \frac{\left(\dot{x} - \eta \frac{g}{f} \right)^{\eta+2}}{(\eta+1)(\eta+2)} + h_1(x, t)\dot{x} + h_2(x, t). \quad (14)$$

In (14) $h_1(x, t)$ and $h_2(x, t)$ are arbitrary functions. However, consistency with (3) demands that h_1 and h_2 satisfy the constraint $h_{1t} - h_{2x} = 0$. Hence there exists an auxiliary function $G(x, t)$ in terms of which h_1 and h_2 are expressed by $h_1(x, t) = G_x$ and $h_2(x, t) = G_t$. As a result the Lagrangian (14) assumes the form

$$L = \frac{\left(\dot{x} - \eta \frac{g}{f} \right)^{\eta+2}}{(\eta+1)(\eta+2)} + \frac{dG}{dt}, \quad (15)$$

where the total derivative term can be discarded without loss of any generality.

3 Hamiltonian formulations of the Liénard equation

The conjugate momentum corresponding to L is then defined through

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{\left(\dot{x} - \eta \frac{g}{f}\right)^{\eta+1}}{(\eta+1)}, \quad (16)$$

implying $\dot{x} = \eta g/f + ((\eta+1)p)^{1/(\eta+1)}$. The associated Hamiltonian, H , using the standard Legendre transformation, turns out to be

$$H = p\dot{x} - L = \eta p \frac{g}{f} + \frac{(\eta+1)^{\frac{\eta+2}{\eta+1}}}{(\eta+2)} p^{\frac{\eta+2}{\eta+1}}. \quad (17)$$

We can also express H in terms of a scaled variable, $\tilde{p} = (\eta+1)p$, whence (17) reads

$$H(x, \tilde{p}, \eta) = \frac{1}{\eta+2} \tilde{p}^{\frac{\eta+2}{\eta+1}} + \frac{\eta}{\eta+1} \tilde{p} \frac{g}{f}. \quad (18)$$

Equation (18) stands as the Hamiltonian for the generalized Liénard equation (3). Numerous models follow from it depending on the specific choices of the functional ratio g/f . The latter in turn acquires its form from the knowledge of f by solving (11). However, we will be interested here in a quadratic representation of g/f namely $g/f = ax^2 + b$, a and b are constants, to make a connection to [5] transparent. This is also clear from the choices of f and g provided in (4). We thus see that the two solutions of η furnished in (12) produce the following Hamiltonians for the Liénard oscillator (1):

$$H(x, \tilde{p}, \eta = -3) = \frac{x^2}{2(3a\tilde{p})^{-1}} + \frac{3}{2}b \left(\sqrt{\tilde{p}} - \frac{1}{3b}\right)^2 - \frac{1}{6b}. \quad (19)$$

$$H(x, \tilde{p}, \eta = -3/2) = (3a\tilde{p})x^2 + 3b\tilde{p} + \frac{2}{\tilde{p}}. \quad (20)$$

In both the above equations the run of \tilde{p} is restricted to $0 < \tilde{p} < \infty$. We observe that (19) corresponds to the non-standard scenario studied in [5] that reflects the harmonic oscillator problem. On the other hand, (20) is a candidate for another legitimate Hamiltonian associated with the Liénard oscillator (1) which also shares with (19) a similar feature of interchange of the roles of variables x and \tilde{p} . Indeed it has the form

$$H = \frac{x^2}{2m(\tilde{p})} + U(\tilde{p}) \quad (21)$$

where from (20) we readily identify the mass and potential function to be

$$m(\tilde{p}) = (6a\tilde{p})^{-1} \quad \text{and} \quad U(\tilde{p}) = 3b\tilde{p} + \frac{2}{\tilde{p}}. \quad (22)$$

Such a system supports the following periodic solution of \tilde{p} corresponding to $x(t)$ in (2)

$$\tilde{p} = \frac{[1 - \frac{kA}{3\omega} \cos(\omega t + \delta)]}{\sqrt{\frac{3\omega^2}{2k} - \frac{kA^2}{6}}}, \quad (23)$$

with $-1 < kA/3\omega < 1$. The trajectory is confined to the upper-half of the $(x - \tilde{p})$ and has the form

$$\left[1 + \left(\frac{kx}{3\omega}\right)^2\right] \tilde{p}^2 - 2\tilde{p} + \left[1 - \left(\frac{kA}{3\omega}\right)^2\right] = 0, \quad |A| < \frac{3\omega}{k}, \quad (24)$$

where we have set $\bar{p} := \tilde{p}\sqrt{\frac{3\omega^2}{2k} - \frac{kA^2}{6}}$ and $|x| \leq A/\sqrt{1 - (\frac{kA}{3\omega})^2}$.

In Sec.IV, we consider the Schrodinger equation in the presence of the momentum-dependent mass function and potential given respectively by (22). We shall see that we run into an isotonic potential [5] in a momentum-dependent mass background.

4 The Schrödinger equation with a momentum dependent mass

In this section we will be specifically concerned with the quantized version of (21) and seek a solution of the corresponding Schrödinger equation having momentum-dependent mass and the potential function given in (22) in contrast to the coordinate-dependent mass situation that has been well studied in the literature [15, 16, 17, 18, 19] in the configuration space. In fact taking cue from such investigations we begin this section with a von Roos type of decomposition [20] for the generic Hamiltonian in the momentum space

$$H(\hat{x}, \hat{p}) = \frac{1}{4} \left[m^\alpha(\hat{p}) \hat{x} m^\beta(\hat{p}) \hat{x} m^\gamma(\hat{p}) + m^\gamma(\hat{p}) \hat{x} m^\beta(\hat{p}) \hat{x} m^\alpha(\hat{p}) \right] + U(\hat{p}). \quad (25)$$

Here α, β and γ are the so called ambiguity parameters which must satisfy the constraint $\alpha + \beta + \gamma = -1$ to ensure dimensional consistency. Following the standard procedure we apply the following quantization rule on \hat{x} and \hat{p} in the momentum space, noting that their corresponding operator representations reverse their roles apart from a change in sign compared to what we normally encounter in the configuration space of standard quantum mechanics:

$$\hat{x} \rightarrow i(\eta + 1) \frac{\partial}{\partial \tilde{p}}, \quad \hat{p} \rightarrow \tilde{p}. \quad (26)$$

Inserting this representation into the Schrödinger equation, $H\psi(\tilde{p}) = E\psi(\tilde{p})$, leads us to the differential equation

$$\begin{aligned} -\frac{(\eta + 1)^2}{2m(\tilde{p})} \left[\psi''(\tilde{p}) - \frac{m'(\tilde{p})}{m(\tilde{p})} \psi'(\tilde{p}) + \frac{\beta + 1}{2} \left(2\frac{m'^2(\tilde{p})}{m^2(\tilde{p})} - \frac{m''(\tilde{p})}{m(\tilde{p})} \right) \psi(\tilde{p}) + \alpha(\alpha + \beta + 1) \frac{m'^2(\tilde{p})}{m^2(\tilde{p})} \psi(\tilde{p}) \right] \\ + U(\tilde{p})\psi(\tilde{p}) = E\psi(\tilde{p}). \end{aligned} \quad (27)$$

Further simplification can be achieved by making the following scaling transformation, $y = \sqrt{4\tilde{p}/3a}$, using the mass function in (22) which causes (27) to reduce to

$$-(\eta + 1)^2 \left[\frac{d^2\psi}{dy^2} + \frac{1}{y} \frac{d\psi}{dy} + \frac{4\alpha(\alpha + \beta + 1)}{y^2} \psi \right] = (E - U(y))\psi, \quad (28)$$

and implies

$$\frac{d^2\psi}{dy^2} + \frac{1}{y} \frac{d\psi}{dy} + \frac{4\alpha(\alpha + \beta + 1)}{y^2} \psi + \tilde{E}\psi - \tilde{U}\psi = 0, \quad (29)$$

where $\tilde{E} = E/(\eta + 1)^2$ and $\tilde{U} = U/(\eta + 1)^2$. Note that an explicit reference to the mass function is absent in (29) as it has been scaled out. Employing a similarity transformation [21]

$$\psi(y) = \frac{\phi(y)}{\sqrt{y}}, \quad (30)$$

to get rid of the linear derivative term, (29) becomes (in the momentum space)

$$\frac{d^2\phi}{dy^2} + \left[\frac{4\alpha(\alpha + \beta + 1) + \frac{1}{4}}{y^2} + \tilde{E} - \tilde{U} \right] \phi = 0, \quad 0 < y < \infty. \quad (31)$$

Setting $\epsilon = -4/(\alpha(\alpha + \beta + 1))$ we have therefore

$$\frac{d^2\phi}{dy^2} + \left(\tilde{E} - \frac{\epsilon - \frac{1}{4}}{y^2} - \tilde{U} \right) \phi = 0, \quad (32)$$

which has the standard structure of the Schrödinger equation for a particle of unit mass confined to an effective potential (in momentum space)

$$\tilde{U}_{eff}(y) = \tilde{U}(y) + \frac{\epsilon - \frac{1}{4}}{y^2}, \quad 0 < y < \infty. \quad (33)$$

From (4) we easily deduce the following values of the constants a and b : $a = k/9$ and $b = \omega^2/k$. When these are inserted into (22) we have for $\eta = -3/2$ bearing in mind the transformation $\tilde{p} = 3ay^2/4$ and the fact that $\tilde{U} = U/(\eta + 1)^2$

$$\tilde{U}(y) = \omega^2 y^2 + \frac{96}{ky^2}, \quad 0 < y < \infty. \quad (34)$$

Consequently (32) becomes

$$-\frac{d^2\phi}{dy^2} + \left[\frac{\ell(\ell + 1)}{y^2} + \omega^2 y^2 \right] \phi = \tilde{E}\phi, \quad 0 < y < \infty \quad (35)$$

where we have set $\ell(\ell + 1) = \epsilon - \frac{1}{4} + 96k^{-1} > -1/4$ with ℓ being a real number. The term in square brackets clearly indicates the isotonic nature of the potential. Introducing the change of variable $y = \rho/\sqrt{\omega}$ and defining $\Lambda = \tilde{E}/\omega$ we find that (35) may be written as

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell + 1)}{\rho^2} + \Lambda - \rho^2 \right] \phi(\rho) = 0 \quad (36)$$

which under the following change of the dependent variable, $\phi(\rho) = e^{-\rho^2/2}v(\rho)$, is transformed to the equation

$$\left[\frac{d^2}{d\rho^2} - 2\rho \frac{d}{d\rho} - \frac{\ell(\ell+1)}{\rho^2} + \Lambda - 1 \right] v(\rho) = 0. \quad (37)$$

A further transformation given by, $v(\rho) = \rho^{\ell+1}\chi(\rho)$, together with a change in the independent variable, $\zeta = \rho^2$, allows us to cast the equation into the confluent hypergeometric form, *viz*

$$\zeta \frac{d^2\chi}{d\zeta^2} + \left[\left(\ell + \frac{3}{2} \right) - \zeta \right] \frac{d\chi}{d\zeta} - \left[\frac{1}{2} \left(\ell + \frac{3}{2} \right) - \frac{\Lambda}{4} \right] \chi = 0, \quad (38)$$

which has well behaved solutions in the neighbourhood of $\zeta = 0$ given by

$$\chi(\zeta) = \text{const. } {}_1F_1 \left(\frac{1}{2} \left(\ell + \frac{3}{2} \right) - \frac{\Lambda}{4}; \ell + \frac{3}{2}; \zeta \right). \quad (39)$$

Polynomial solutions of the above series results upon imposing the condition

$$\frac{1}{2} \left(\ell + \frac{3}{2} \right) - \frac{\Lambda}{4} = -n, \quad n = 0, 1, \dots, \quad (40)$$

and leads to the equispaced energy eigenvalues

$$\tilde{E}_n = \left[n + \frac{1}{2} \left(\ell + \frac{3}{2} \right) \right] \omega, \quad n = 0, 1, \dots \quad (41)$$

for a fixed $\ell = -\frac{1}{2} \pm \sqrt{96k^{-1} + \epsilon}$. The solution of (31) then is expressible in the form

$$\phi_n(y) = N_n e^{-\frac{1}{2}y^2} y^{\ell+1} {}_1F_1(-n; \ell + \frac{3}{2}; y), \quad 0 < y < \infty, \quad (42)$$

where N_n represents the normalization constant.

5 Summary

In this article we have considered the nonlinear Liénard oscillator and, adopting the JLM approach, solved completely for the governing dynamical system. We have found that depending upon the choice of the coupling parameters the Liénard system has a bi-Hamiltonian character and that for both the forms the roles of the coordinate variable and momentum are transposed. This causes the mass function and the potential to be explicitly momentum-dependent. Furthermore, while one Hamiltonian is harmonic oscillator like, the other one speaks of an isotonic potential. While the former has been recently solved in terms of Hermite polynomials, here we give the complete solution of the latter in terms of a confluent hypergeometric function.

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