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Nonlinear Behaviors of Pulsating Stars with Convective Zones

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Abstract

We discuss the role of convective zones in the treatment of coupled pulsation and convection. In particular, we have investigated behaviors of the convective depth, η, and of the ratio between the convective and total luminosities, γ_{c}. In deep convection, additional fixed points appear, the values of which increase with the depth of zone, and become larger than unity. A diagram of stability criteria derived from the linearized model has been drawn, which shows stable regions fitting each criterion. Then, various flow patterns have been obtained by means of numerical simulations for sets of γ_{c} and η, and the structure of the mode has been discussed. The results of simulations are shown in tables and figures with a fixed set of parameters other than the set of γ_{c} and η, which show that deep convective zones cause chaotic solutions and long-period oscillations. Some suggestions are made based on these findings in stellar pulsation within the convective zone.

Key words: convection — methods: numerical — stars: pulsations — stars: variables: other

1. Introduction

As long-period variables, such as Mira- and semiregular-type variables, are located in the region of red giants in the HR diagram, these variables are assumed to consist of a deep convective envelope and a rigid core. A simple model of pulsating stars with a convective zone was derived by Stellingwerf (1986), who applied the model to stars with shallow convective zones. Later, Munteanu et al. (2005) showed the limit cycles of the model for suitable sets of control parameters, also assuming shallow convective zones. Recently, Tanaka (2011) numerically studied the model with deep convection, and found chaotic solutions of the model, including the period 3 oscillation as a symbol of chaos.[1] He also obtained long-period oscillations with large amplitudes for the deep and convection-dominant model.

In this work, we advance the study by extending the range of the relevant parameters. We present the result of a detailed stability analysis of the fixed point of the model for an arrangement of parameter values. A direct numerical analysis of model equations was done to obtain the type of oscillatory and chaotic states, and its dependence on the convective depth and the ratio between the convective and total luminosities. Our results reveals the existence of chaotic states within the deep convective zone.

The nonlinear equation of Stellingwerf's model of pulsating stars with convective zone is shown in nondimensional forms (section 2). The fixed points were obtained in the case of representative parameters, and the stability conditions of the linearized equation are discussed (section 3). We include the results of a direct numerical analysis of model equations in section 4, and give our conclusions in the last section.

2. The Basic Equation in Nondimensional Forms

Using Baker's one-zone model of pulsating stars, we consider the convective zone located in between the outer stellar radius, r, and the radius of the rigid core, r_{core}. We define the normalized radius, $X = r/\,r_0$, where r_0 is the outer radius in equilibrium and the depth of the convective zone, $\eta = r_{\mathrm{core}}/r_0$. As derived by Stellingwerf (1986), the model is described in the form of nondimensional differential equations of fourth order as follows:

$$\frac{dX}{d\tau} = Y, \tag{1}$$

$$\frac{dY}{d\tau} = \frac{H}{X^q} - \frac{1}{X^2}, \tag{2}$$

$$\frac{dH}{d\tau} = \zeta X^{m(\Gamma_1 - 1)}(1 - \gamma_{\mathrm{r}} X^b H^{s+4} - \gamma_{\mathrm{c}} X^{-c} U_{\mathrm{c}}^3), \tag{3}$$

$$\frac{dU_{\mathrm{c}}}{d\tau} = \zeta_{\mathrm{c}}(X^{-d} H^{1/2} - U_{\mathrm{c}}), \tag{4}$$

where Y, H, and U_{c} mean the radial velocity of the outer surface, the nonadiabatic pressure, and the convective velocity normalized to the equilibrium value, respectively. In particular, $U_{\mathrm{c}} = u_{\mathrm{c}}/u_{\mathrm{ml0}}$, where u_{ml0} is the mixing-length convective velocity in equilibrium (see Stellingwerf 1986 for details).

The equation of motion, the energy equation, and the equation of convective transport are given by equations (2), (3), and (4), respectively. The time τ is described in units of

[1] In typical nonlinear systems the period-doubling bifurcation leads to chaos via the period 3. Li and Yorke (1975) have proved that "Period three implies chaos."

free-fall time, and the density is

$$\frac{\rho}{\rho_0} = X^{-m}, \tag{5}$$

where m is a geometric parameter, which is approximated to be

$$m = \frac{3}{1-\eta^3}. \tag{6}$$

As introduced in Stellingwerf (1986), the related quantities are $q = m\Gamma_1 - 2$, $d = m(\Gamma_1 - 1)/2$, $c = m - 2$, and $b = 4 + m[n - (s+4)(\Gamma_1 - 1)]$, where Γ_1 is the adiabatic exponent, while n and s are the density and temperature exponents in the Kramers opacity law, respectively.

Since the parameter γ_c is the ratio between the convective luminosity and the total one in equilibrium, $\gamma_c + \gamma_r = 1$, where γ_r is the ratio between the radiative luminosity and the total one in equilibrium. Moreover, ζ_c represents the convective efficiency, which is the ratio between the free-fall time and the convective timescale, while ζ is the ratio of the free-fall time to the thermal timescale. For the sake of simplicity, we hereafter fix the values of the stellar structure at $n = 1$, $s = 3$, and $\Gamma_1 = 1.1$. Then, the behaviors of model depend on a set of main control parameters: m (or η), γ_c, ζ, and ζ_c.

3. Fixed Points and Stability Conditions

Equations (1)–(4) have fixed points when their right-hand sides equal zero (see Tanaka 2011 for details). One of the fixed points on the X axis appears as $X_f = 0$. The equation of fixed points of X is given by

$$1 - (1-\gamma_c)X^b\left(X^{q-2}\right)^{s+4} - \gamma_c X^{-3d-c+3(q-2)/2} = 0, \tag{7}$$

solutions of which are numerically obtained. Since equation (7) is independent of ζ and ζ_c, the fixed point is determined by η (or m) and γ_c. If $\gamma_c \neq 0$ and 1, typical values of fixed points, except $X_f = 0$ and 1, are given in table 1 for the set of η and γ_c where m is also given for convenience.

In the case of $n = 1$, $s = 3$, and $\Gamma_1 = 1.1$, equation (7) gives one solution, $X_f = 1$, for $m > 8$, as shown in Tanaka (2011), where only the case of $\gamma_c = 0.9$ is given. Two fixed points appear when m is less than 8, and one of them has the value of unity. The value of the other fixed point increases with the depth, and becomes larger than unity for a given γ_c. With the increase of γ_c, the fixed points that are larger than unity appear at the shallow convective zones.

It appears that there are three fixed points in the case of deep convective zone; that is, $X_f = 0$, $X_f = 1$, and $X_f < 1$ or > 1.

We next consider the stability conditions, using linearized equations and the equation of the eigenvalue. Stellingwerf (1986) derived the linearized forms of equations (1)–(4). For instance, equation (1) is put in a form,

$$X = 1 + x\, e^{\sigma\tau}, \tag{8}$$

where x is assumed to be small, and σ is an eigenvalue. The equation of the eigenvalue is written in the form

$$\sigma^4 + A_1\sigma^3 + A_2\sigma^2 + A_3\sigma + A_4 = 0, \tag{9}$$

where A_i ($i = 1$ to 4) are complicated and lengthy functions of the parameters shown in Stellingwerf (1986).

Table 1. Fixed points for η (or m) and γ_c except $X_f = 0$ and 1.

η	$m \setminus \gamma_c$	0.2	0.4	0.5	0.6	0.8
0.9	11.07	—	—	—	—	—
0.85	7.77	6.E−7	3.E−4	2.E−3	0.01	0.13
0.8	6.14	0.17	0.37	0.47	0.57	0.78
0.75	5.18	0.31	0.52	0.61	0.69	0.86
0.7	4.56	0.39	0.58	0.66	0.74	0.91
0.65	4.13	0.43	0.62	0.70	0.78	0.97
0.6	3.82	0.46	0.65	0.74	0.83	1.05
0.55	3.59	0.48	0.69	0.79	0.89	1.18
0.5	3.42	0.51	0.74	0.86	—	1.39
0.45	3.30	0.55	0.83	—	1.17	1.77
0.4	3.20	0.61	—	1.20	1.48	2.52
0.35	3.13	0.73	1.27	1.63	2.13	4.37

The stability conditions of Hurwitz (1964) that all roots of equation (9) should have negative real parts are given as follows:

$$Hu1 = A_1 > 0, \tag{10}$$

$$Hu2 = A_1A_2 - A_3 > 0, \tag{11}$$

$$Hu3 = A_1(A_2A_3 - A_1A_4) - A_3^2 > 0, \tag{12}$$

$$Hu4 = A_4 > 0. \tag{13}$$

Hu4 and *Hu3* will be called the convective- and pulsational-stability criteria, respectively, while *Hu2* is slightly different from *Hu2* of Stellingwerf (1986). As in the case of $\zeta = \zeta_c = 0$, only A_2 out of four coefficients is not zero; then,

$$Hu5 = A_2 > 0 \tag{14}$$

gives the dynamical condition according to Baker (1966), which is adopted instead of *Hu1*.

The boundary curves of $Hu2 = 0$ etc. are drawn on the (γ_c, η) plane for a set of (ζ_c, ζ) under given values of n and others. Figure 1 shows the boundaries of *Hu3* (full lines) and *Hu4* (dashed line) for a set of (ζ_c, ζ) = (1, 1). *Hu2* and *Hu*5 are positive in this case. The stable region of pulsation, that is, $Hu3 > 0$ is the region between the full curves, while that of convection is the left side of the dashed curve. In regions A and D, $Hu3 > 0$, while $Hu4 > 0$ in A and $Hu4 < 0$ in D. Region B represents the region of $Hu3 < 0$ and $Hu4 > 0$. Regions C and E are unstable in both *Hu3* and *Hu4*.

4. Numerical Simulations and Types of Solutions

The analysis in the linear approximation gives the set of parameters for which solutions of the Stellingwerf equation can be stable. Since the behavior of numerical solutions in the unstable region reveals the role of stability criteria, we describe the results for (ζ_c, ζ) = (1, 1). The initial values are given as $(X, Y, H, U_c)_0 = (1.1, 0, 0.9, 1.0)$. For $\eta = 0.7$, the solution of $\gamma_c = 0.6$, which is located in region A, converges to the fixed point $X_f = 1$ [type (a)], while at $\gamma_c = 0.5$ and 0.4 in region B, limit cycles are obtained [type (b)]. We have divergent solutions that oscillate and increase the amplitude to infinity for $\gamma_c \leq 0.3$ [type (c)]. These types (a), (b), and (c) are classified by Munteanu et al. (2005).

For $\gamma_c = 0.4$, limit cycles appear in the range of $0.45 \le \eta \le 0.85$, but convergent solutions appear at larger η. The periods of the limit cycle in time τ range from ~ 3 for $\eta = 0.85$ to ~ 20 for $\eta = 0.45$. For $\eta \leqq 0.4$, the solutions contract to the origin, $X_f = 0$ [type (d)]. It is clear that the limit cycles around $X_f = 1$ are realized just inside of the unstable region B, where *Hu3* < 0 and *Hu4* > 0. In the unstable region C for *Hu3* and *Hu4*, only type (d) is obtained.

However, in region A, not only type (a) but also type (dF) can be seen at, e.g., $(\gamma_c, \eta) = (0.5, 0.5)$, the trajectory in the (X, Y) plane of which passes near the fixed point less than unity, and contracts to the origin (the left panel of figure 2). In region D, where the pulsation is stable and convectively unstable, we have solutions similar to types (a) and (b). The former converges to a fixed point larger than unity [type (aF), see the right panel in figure 2], and the latter becomes limit cycles around fixed points [type (bF)]. Type (bF) has long

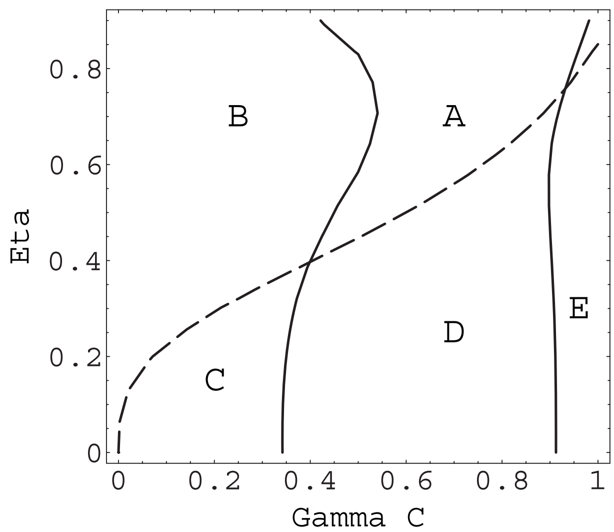

Fig. 1. Boundaries of *Hu3* (full lines) and *Hu4* (dashed lines) for $(\zeta_c, \zeta) = (1, 1)$ in the (γ_c, η) plane. The stable region (*Hu3* > 0 and *Hu4* > 0) is labeled as A. B represents the region of *Hu3* < 0 and *Hu4* > 0. In C and E, *Hu3* < 0 and *Hu4* < 0. In D, *Hu3* > 0 and *Hu4* < 0.

Table 2. Types of solutions with $(\zeta_c, \zeta) = (1, 1)$ for η and γ_c.

$\eta \setminus \gamma_c$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
0.9	c	c	a	a	a	a	a	a	a
0.85	c	c	b	a	a	a	a	a	dF
0.8	c	c	b	a	a	a	a	dF	dF
0.75	c	c	b	b	a	a	a	dF	dF
0.7	c	c	b	b	a	a	dF	dF	aF
0.65	c	b	b	b	a	a	dF	aF	aF
0.6	c	b	b	a	a	dF	dF	aF	aF
0.55	c	b	b	a	dF	dF	aF	aF	bF
0.5	d	d	b	dF	dF	aF	aF	bF	bF
0.45	d	d	b	d	aF	bF	bF	bF	cF
0.4	d	d	d	d	d	d	bF	cF	cF
0.35	d	d	d	d	d	d	cF	cF	cF

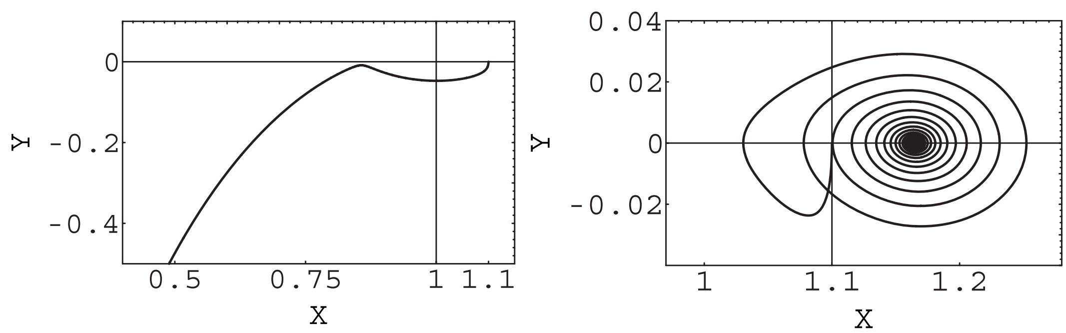

Fig. 2. Solutions in the (X, Y) plane for $(\zeta_c, \zeta) = (1, 1)$. Left panel: Type (dF) solution contracts to the origin after bypassing the fixed point less than unity for $(\gamma_c, \eta) = (0.5, 0.5)$. Right panel: Type (aF) solution for $(\gamma_c, \eta) = (0.7, 0.5)$ converges to the fixed point greater than unity.

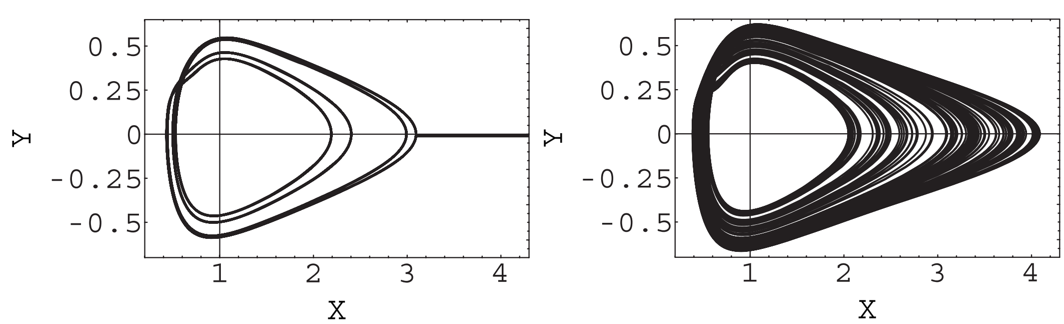

Fig. 3. Left panel: The solution of period 4 in period doubling in the (X, Y) plane for $(\gamma_c, \eta) = (0.25, 0.55)$ and $(\zeta_c, \zeta) = (1, 1)$. Right panel: A chaotic solution for the same but $(\gamma_c, \eta) = (0.23, 0.55)$.

periods and large amplitudes around the fixed points $X_{\mathrm{f}} > 1$, which is similar to solutions pointed out by Tanaka (2011).

In region E the solutions of types (a), (aF), (bF), (cF), and (dF) can be obtained at $\gamma_{\mathrm{c}} = 0.95$. Solutions of type (cF) are similar to type (c), but oscillate around $X_{\mathrm{f}} > 1$ and expand gradually or explosively. Finally, chaotic solutions [type (bC)] are found between (γ_{c}, η) = (0.3, 0.55) and (0.2, 0.55), for instance. As γ_{c} decreases for a constant $\eta = 0.55$, period doubling occurs, as can be seen in figure 3. It is also noted that amplitudes become large along with a decrease of γ_{c}, such as $0.5 < X < 3$ or 4, as shown in the figure.

Thus, we have the types of solutions of the Stellingwerf equation as type (a) and its variation, type (aF), which is in pulsational stability and converges to fixed points. The limit cycle type (b) has two variations that are a chaotic oscillation, type (bC), and a large-amplitude one, type (bF), around fixed points greater than unity. Types (c) and (cF) diverge both gradually and explosively, respectively. Types (d) and (dF) contract to the origin gradually and rapidly, respectively. Table 2 illustrates the type for the set of parameters (γ_{c}, η). The column of $\gamma_{\mathrm{c}} = 0.95$ corresponds to region E.

5. Discussion and Conclusion

We studied the effect of convective zones and its dependence on γ_{c} and η. The behaviors of numerical solutions of the Stellingwerf equation were illustrated and classified by the types of patterns on the (γ_{c}, η) plane with a fixed set of (ζ_{c}, ζ) = (1, 1). Although Stellingwerf (1986) fixed η to be equal to 0.88 ($m = 10$), and Munteanu et al. (2005) also studied the case of $\eta = 0.75$ and others, the effects of deep convective zones were shown by Tanaka (2011) as being chaotic motions and long-period pulsations for rather extreme values of ζ_{c} and ζ. We express in this letter that all types of solutions depend on γ_{c} and η, due to the stability conditions for other given parameters.

The major results obtained can be summarized as follows:

1. For values of $\gamma_{\mathrm{c}} = 0$ and 1, the fixed points obtained are zero and unity. For other values, fixed points other than zero and one exist, and they increase in value with the depth of the convective zone η or with γ_{c}. Thus, for deep convective zones and large γ_{c}, values of the fixed points exceed unity. These characteristics of fixed points are independent of (ζ_{c}, ζ).
2. The pulsational- and convective-stability conditions among the four Hurwitz conditions are more relevant in the case of (ζ_{c}, ζ) = (1, 1). These conditions divide the (γ_{c}, η) plane into six regions. The nature of the dynamics in each region has been studied numerically.
3. The types of solutions obtained numerically are classified as follows:
   (a) Type (a), which converges to a fixed point at unity, and type (aF), which spirals into fixed points other than zero or unity.
   (b) Types (d) and (dF), which contract to a fixed point at zero both gradually and rapidly, respectively.
   (c) Type (b), which forms limit cycles around a fixed point at unity, and type (bF), limit cycles of long periods with large amplitudes around a fixed point greater than unity.
   (d) Type (bC), which is in chaotic oscillation that arises from period doubling of type (b) oscillations as γ_{c} or η is varied.
   (e) Types (c) and (cF), which diverge both gradually and explosively, respectively.

In the present work, the values of (ζ_{c}, ζ) considered are (1, 1). However, since the boundaries of the stability curves change with (ζ_{c}, ζ), a further and detailed study in the case of large ζ will be interesting. The dependence of various types of solutions on the initial value is also relevant. Moreover, a study on the long-period and chaotic oscillations obtained theoretically from the Stellingwerf model with the observed data of long-period and other variable stars is planned for an extended analysis concerning the convective theory of pulsating stars.

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