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$\mathcal{CPT}$-conserved effective mass Hamiltonians through first and higher order charge operator $\mathcal{C}$ in a supersymmetric framework

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This paper examines the features of a generalized position-dependent mass Hamiltonian H_m in a supersymmetric framework in which the constraints of pseudo-Hermiticity and $\mathcal{CPT}$ are naturally embedded. Different representations of the charge operator are considered that lead to new mass-deformed superpotentials $\mathcal{W}_m(x)$ which are inherently $\mathcal{PT}$-symmetric. The qualitative spectral behavior of H_m is studied and several interesting consequences are noted.

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I. INTRODUCTION

Non-Hermitian systems admitting $\mathcal{PT}$-symmetry (i.e. invariance under a combined action of parity $\mathcal{P}$ and time-reversal $\mathcal{T}$) have been a subject matter of intense interest [1, 2]. $\mathcal{PT}$-symmetry has an interesting implication that the whole class of Schrödinger Hamiltonians coming under its assignment namely, $H = p^2/2m + V(x)$ defined on the real line $x \in \mathbb{R}$, where the potential is typically $V(x) = V^*(-x)$, may possess real or conjugate pairs of energy eigenvalues under certain conditions related to $\mathcal{PT}$ being unbroken (i.e. exact) or spontaneously broken. It has also been realized that the concept of $\mathcal{PT}$-symmetry has its roots in the theory of pseudo-Hermitian operators and that pseudo-Hermiticity serves as one of the plausible necessary and sufficient conditions for the reality of the spectrum [3].

In [4] a set of intertwining relations

$$H\zeta = \zeta H^\dagger \tag{1.1}$$

was studied in which a Hermitian operator ζ was proposed to be expressed as a product of the charge operator $\mathcal{C}$ and parity operator $\mathcal{P}$

$$\zeta = \mathcal{CP} \qquad (\zeta = \zeta^\dagger)\,. \tag{1.2}$$

It is straightforward to see that equations (1.1) and (1.2) together imply the $\mathcal{CPT}$ conservation of the Hamiltonian H, $\mathcal{T}$ being the time reversal operator

$$\mathcal{CPT}H = H\mathcal{CPT}\,. \tag{1.3}$$

Interestingly, it also follows from (1.1) that the operator ζ^{-1}, if it exists, also fulfills the intertwining relations

$$H^\dagger \zeta^{-1} = \zeta^{-1} H \tag{1.4}$$

implying that H is pseudo-Hermitian with respect to ζ^{-1}. This can be verified as follows:

$$< \psi, H\phi >_{\zeta^{-1}} = < \psi, \zeta^{-1} H\phi > = < \psi, H^\dagger \zeta^{-1}\phi > = < H\psi, \zeta^{-1}\phi > = < H\psi, \phi >_{\zeta^{-1}} \;. \tag{1.5}$$

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Differential realizations for $\mathcal{C}$ have been considered in the literature such as $\mathcal{C} = \mathrm{d}\,/\,\mathrm{d}\,x + \mathcal{W}(x)$ for the first-order [4] and $\mathcal{C} = \mathrm{d}^2\,/\,\mathrm{d}\,x^2 + \mathcal{W}(x)\,\mathrm{d}\,/\,\mathrm{d}\,x + \mathcal{U}_0(x)$ for the second-order case [5]. The aims of such models have been to search for closed-form solutions and to work out the solvability criterion of the embedded Hamiltonian.

In this article we intend to investigate these and related aspects of pseudo-Hermiticity and $\mathcal{CPT}$-conservation for extended versions of Schrödinger equation admitting SUSY in a position-dependent (effective) mass (PDM) framework. The 1-D effective mass Hamiltonian $H \to H_m$ obeys (in the atomic unit defined by $\hbar^2 = 2$) in real spatial coordinate [6]:

$$H_m(x)\psi_n(x) \equiv \left(-\partial\left[\frac{1}{m(x)}\partial\right] + \widetilde{V}_{\mathrm{m}}(x)\right)\psi_n(x) = E_n\psi_n(x), \qquad \widetilde{V}_{\mathrm{m}}(x) = V_{\mathrm{m}}(x) + \rho(m)\,, \tag{1.6}$$

where $m(x)$ is a real valued mass function in the presence of a complex potential $\widetilde{V}_{\mathrm{m}}(x)$:

$$V_{\mathrm{m}}(x) = V_{\mathrm{m}}^R(x) + \mathrm{i}V_{\mathrm{m}}^I(x)\,. \tag{1.7}$$

In equation (1.6), the mass-dependent function $\rho(m)$ has the form [7]

$$\rho(m) = \frac{1+b}{2}\frac{m''(x)}{m^2(x)} - c\frac{m'^2(x)}{m^3(x)}, \qquad c = 1 + b + a(a+b+1), \tag{1.8}$$

where a and b are the usual ambiguity parameters [8] typical to the effective-mass models. Position dependence in mass shows up in different areas of physics - semiconductors [9], quantum dots [10] , ^{3}He clusters [11] and many more. A number of papers have been written on the issue of PDM in this rapidly expanding literature [12–34].

Note that the question of boundedness and invertibility of the operator ζ, assuming an explicit representation for it was addressed in [4] for the constant-mass case. In the PDM scenario, the problem is trickier and will be taken up elsewhere.

II. PSEUDO-HERMITICITY AND $\mathcal{CPT}$-SYMMETRY IN A SUPERSYMMETRIC FRAMEWORK

In the framework of supersymmetric (SUSY) quantum mechanics [35–38], an underlying anticommutator K of the supercharges Q and $\overline{Q}$ can be explicitly constructed by specifying the following representation

$$K = \{Q, \overline{Q}\} = \begin{pmatrix} \zeta\zeta^* & 0 \\ 0 & \zeta^*\zeta \end{pmatrix}, \tag{2.1}$$

where Q and $\overline{Q}$ are defined in terms of the operator ζ and its complex conjugate ζ^*

$$Q = \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix}, \quad \overline{Q} = \begin{pmatrix} 0 & 0 \\ \zeta^* & 0 \end{pmatrix}. \tag{2.2}$$

Assuming polynomial expansions

$$\zeta\zeta^* = \sum_{k=0}^{N} l_k\,(H_m)^{N-k}\,, \quad \zeta^*\zeta = \sum_{k=0}^{N} l_k\,(H_m^*)^{N-k}\,, \qquad [l_0 \equiv 1\,, H_m^0 \equiv I_2] \tag{2.3}$$

we get by post-multiplying the first relation and pre-multiplying the second relation above by ζ and subtracting

$$0 = \sum_{k=0}^{N-1}\left[l_k\,(H_m)^{N-k}\,\zeta\right] - \sum_{k=0}^{N-1}\left[l_k\zeta\,(H_m^*)^{N-k}\right]. \tag{2.4}$$

Similarly, by pre-multiplying the first relation and post-multiplying the second relation by ζ^* and subtracting

$$0 = \sum_{k=0}^{N-1}\left[l_k\zeta^*\,(H_m)^{N-k}\right] - \sum_{k=0}^{N-1}\left[l_k\,(H_m^*)^{N-k}\,\zeta^*\right]. \tag{2.5}$$

(2.4) and (2.5) lead to the intertwining relations

$$H_m\zeta = \zeta H_m^*\,, \quad H_m^*\zeta^* = \zeta^* H_m\,. \tag{2.6}$$

At play are also the following constraints

$$\text{Pseudo-hermiticity constraint: } \zeta^\dagger = \zeta \;\Rightarrow\; \mathcal{C}^\dagger[-x] = \mathcal{C}[x]\,, \tag{2.7}$$

$$\mathcal{CPT} \text{ constraint: } \mathcal{CPT}H_m = H_m\mathcal{CPT} \;\Rightarrow\; \mathcal{C}[x]H_m^*[-x] = H_m[x]\mathcal{C}[x]\,, \tag{2.8}$$

$$\text{SUSY constraint: } \zeta\zeta^* = \sum_{k=0}^{N} l_k H_m^{N-k} \;\Rightarrow\; \mathcal{C}[x]\mathcal{C}^*[-x] = \sum_{k=0}^{N} l_k H_m^{N-k}\,. \tag{2.9}$$

Finally, in the context of the N-th order SUSY, a mass-deformed superpotential $\mathcal{W}_m(x)$ can be introduced which is given by the form [28]

$$\mathcal{W}_m(x) = \mathcal{W}(x) - \frac{N}{2}\left[\frac{1}{\sqrt{m^N(x)}}\right]'\,. \tag{2.10}$$

where $\mathcal{W}(x)$ corresponds to the superpotential of the constant mass case. A natural consequence of (2.10) is that unlike $\mathcal{W}(x)$ as in the constant-mass case $\mathcal{W}_m(x)$ turns out to be $\mathcal{PT}$-symmetric from the pseudo-Hermiticity constraint (2.7) as will be revealed below.

III. FIRST ORDER CHARGE OPERATOR

The first order representation of the charge operator $\mathcal{C}$ in a PDM scheme is given by

$$\mathcal{C} = \frac{1}{\sqrt{m(x)}}\frac{\mathrm{d}}{\mathrm{d}\,x} + \mathcal{W}(x)\,. \tag{3.1}$$

From (2.3) we have for $N = 1$ the projections

$$\zeta\zeta^* = H_m + l_1\,, \quad \zeta^*\zeta = H_m^* + l_1\,. \tag{3.2}$$

Imposing the pseudo-hermiticity restriction (2.7), we have the solutions:

$$m(x) = m(-x)\,, \quad \mathcal{W}(x) = \mathcal{W}^*(-x) - \frac{1}{2}\frac{m'(x)}{m^{3/2}(x)}\,. \tag{3.3}$$

It is evident from (2.10) and (3.3) that

$$\mathcal{W}_m(x) = \mathcal{PT}\,\mathcal{W}_m(x)\,. \tag{3.4}$$

implying $\mathcal{W}_m(x)$ to be $\mathcal{PT}$-symmetric and the mass function to be parity-invariant. As remarked earlier, $\mathcal{W}(x)$ ceases to be $\mathcal{PT}$-symmetric. A $\mathcal{PT}$-symmetric $\mathcal{W}_m(x)$ can be implemented by choosing for $\mathcal{W}(x)$ the form say, $\mathcal{W}(x) = \exp(i\alpha x) + h(x)$ where $\alpha \in \mathbb{R}$ and a non-$\mathcal{PT}$ $h(x)$ can be confronted with a suitable parity-invariant mass function leaving $\mathcal{W}_m(x)$ to be $\mathcal{PT}$-symmetric. The following concrete example is one we have in mind

$$\mathcal{W}(x) = \exp(i\alpha x) - \sin(x), \quad m(x) = \frac{1}{4}\sec^2(x) \quad \Rightarrow \mathcal{W}_m(x) = \exp(i\alpha x) \quad 0 < x < \frac{\pi}{2} \tag{3.5}$$

where $\mathcal{W}_m(x)$ is a periodic potential.

Turning to the $\mathcal{CPT}$-constraint (2.8) and using (3.1) we get two relations. While comparison of the ∂-term yields the difference

$$\widetilde{V}_{\mathrm{m}}(x) - \mathcal{PT}\widetilde{V}_{\mathrm{m}}(x) = 2\frac{\mathcal{W}_m'(x)}{\sqrt{m(x)}}\,, \tag{3.6}$$

the remaining part results in a first-order differential equation which can be readily integrated to provide for $\widetilde{V}_{\mathrm{m}}(x)$ the expression

$$\widetilde{V}_{\mathrm{m}}(x) = \mathcal{W}_m^2(x) + \frac{\mathcal{W}_m'(x)}{\sqrt{m(x)}} + \frac{1}{4}\frac{m''}{m^2} - \frac{7}{16}\frac{m'^2}{m^3} + \Lambda\,, \tag{3.7}$$

where Λ is an arbitrary constant of integration.

A non-trivial form for $\widetilde{V}_{\rm m}(x)$ is also obtained on employing the SUSY constraint (2.9) for $N=1$ namely

$$\mathcal{C}[x]\mathcal{C}^*[-x] = H_m + l_1 \,. \tag{3.8}$$

A comparison between (3.7) and (3.8) fixes $\Lambda = -l_1$. (3.8) is our final form of $\widetilde{V}_{\rm m}(x)$ for the $N=1$ case. Note that the underlying $\mathcal{CPT}$-invariance has the implication

$$H_m \mathcal{CPT}\psi(x) = \mathcal{CPT}H_m\psi(x) = \mathcal{CPT}E\psi(x) = E^*\mathcal{CPT}\psi(x). \tag{3.9}$$

Thus if (ψ, E) is an eigenpair of a $\mathcal{CPT}$-invariant PDM Hamiltonian H_m, then $(\mathcal{CPT}\psi, E^*)$ must form another eigenpair provided $\mathcal{CPT}\psi \neq 0$. Thus $\mathcal{CPT}$-invariance of ψ leads to the corresponding Hamiltonian having real eigenvalues.

From the first relation of (2.3) and (1.6), the ground state ψ_0 in the $N=1$ case has to obey

$$\zeta^*\psi_0(x) = 0 \Rightarrow \left[\frac{1}{\sqrt{m(x)}}\partial + \mathcal{W}^*(x)\right]\psi_0(-x) = 0\,, \tag{3.10}$$

with the lowest eigenvalue $-l_1$. From (3.10) we find

$$\psi_0(x) = \mathcal{N}_0 m^{1/4}(x)\exp\left[\int^x \sqrt{m(y)}\mathcal{W}_m(y)\,\mathrm{d}\,y\right]\,, \tag{3.11}$$

$\mathcal{N}_0$ being the normalization constant. Note that $\psi_0(x)$ is non-$\mathcal{PT}$-symmetric.

IV. SECOND ORDER CHARGE OPERATOR

We now look at the following mass-dependent second order representation of the charge operator $\mathcal{C}$

$$\mathcal{C} = \frac{1}{m(x)}\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + \mathcal{W}(x)\frac{\mathrm{d}}{\mathrm{d}\,x} + \mathcal{U}_0(x)\,, \tag{4.1}$$

accompanied by the $N=2$ SUSY representations

$$\zeta\zeta^* = H_m^2 + l_1 H_m + l_2 I_2\,. \tag{4.2}$$

as follows from (2.3). For the literature on $N=2$ SUSY in the constant-mass case we refer to the readers [39–44].

Employing the pseudo-Hermiticity requirement (2.7) gives the following solutions

$$m(-x) = m(x)\,, \qquad \mathcal{W}_m(x) = \mathcal{PT}\,\mathcal{W}_m(x) \tag{4.3}$$

which are similar to the $N=1$ case i.e. $\mathcal{W}_m(x)$ is $\mathcal{PT}$-symmetric and the mass function $m(x)$ is parity-invariant. Note that according to (2.10), $\mathcal{W}_m(x)$ is related to the constant-mass superpotential $\mathcal{W}(x)$ by

$$\mathcal{W}_m(x) = \mathcal{W}(x) - \left(\frac{1}{m}\right)'\,. \tag{4.4}$$

As an illustrative example we can take this time

$$\mathcal{W}(x) = \exp(i\alpha x) - \sin(x), \quad m(x) = \sec(x) \qquad \alpha\in\mathbb{R}, \quad 0 < x < \frac{\pi}{2} \tag{4.5}$$

leading again to a periodic $\mathcal{PT}$-symmetric $\mathcal{W}_m(x) = \exp(i\alpha x)$.

Apart from (4.3), the pseudo-Hermiticity condition also furnishes another relation namely

$$\triangle\,\mathcal{U}_0(x) \equiv \mathcal{U}_0(x) - \mathcal{PT}\,\mathcal{U}_0(x) = \mathcal{W}_m'(x)\,, \tag{4.6}$$

which reflects the non-$\mathcal{PT}$-symmetric character of the function $\mathcal{U}_0(x)$ present in (4.1).

Next, consideration of the $\mathcal{CPT}$ requirement (2.8) furnishes

$$\triangle\widetilde{V}_{\rm m}(x) \equiv \widetilde{V}_{\rm m}(x) - \mathcal{PT}\widetilde{V}_{\rm m}(x) = 2\mathcal{W}_m'(x) + \frac{m'}{m}\mathcal{W}_m(x) \tag{4.7}$$

which is slightly different in form from the $N=1$ result (3.7). In (4.7) $\widetilde{V}_{\rm m}(x)$ is restricted by

$$\widetilde{V}_{\rm m}(x) = \triangle\widetilde{V}_{\rm m}(x) + f(x) - \mathcal{U}_0(x) + \Lambda \tag{4.8}$$

where

$$f(x) = \frac{1}{2}\left[m\mathcal{W}_m^2(x) - \frac{m'}{m}\mathcal{W}_m(x) - \mathcal{W}_m'(x)\right] \tag{4.9}$$

on making use of (4.3) and (4.6). The constant Λ appears in (4.8) through the process of integration and is left arbitrary at this stage.

In addition to (4.7) and (4.8), the non-$\mathcal{PT}$-function $\mathcal{U}_0(x)$ has to satisfy the differential equation

$$\left[\frac{\mathcal{U}_0'(x)}{m(x)}\right]' - \mathcal{U}_0(x)\triangle\widetilde{V}_{\rm m}(x) + \mathcal{W}_m(x)\left\{\mathcal{PT}\,\widetilde{V}_{\rm m}(x)\right\}' + \left[\frac{\left\{\mathcal{PT}\,\widetilde{V}_{\rm m}(x)\right\}'}{m(x)}\right]' = 0\,. \tag{4.10}$$

Substitution of (4.8) into (4.10) converts it to the form

$$\mathcal{U}_0'(x)\mathcal{W}_m(x) + \mathcal{U}_0(x)\left[2\mathcal{W}_m'(x) + \frac{m'}{m}\mathcal{W}_m(x)\right] = \frac{f''(x)}{m} + f'(x)\left[\mathcal{W}_m(x) - \frac{m'}{m^2}\right], \tag{4.11}$$

which may be solved to arrive at

$$\mathcal{U}_0(x) = \frac{f'(x)}{m(x)\mathcal{W}_m(x)} + \frac{f^2(x)}{m(x)\mathcal{W}_m^2(x)} + \frac{\Theta}{m\mathcal{W}_m^2(x)}\,. \tag{4.12}$$

Θ being an arbitrary constant of integration.

We now attend to the SUSY constraint (2.9). Here we need to compare the five coefficients of $\partial^\ell\,, \ell = 0,1,2,3,4$. While the first two produce solutions similar to (4.3), the last three respectively yields the following three relations:

$$2\frac{1}{m(x)}\widetilde{V}_{\rm m}(x) + \frac{1}{m(x)}\left[\mathcal{U}_0(x) + \mathcal{PT}\,\mathcal{U}_0(x)\right] - 2\frac{1}{m(x)}\mathcal{W}_m'(x) = \mathcal{W}_m^2(x) - \left(\frac{1}{m(x)}\right)'\mathcal{W}_m(x) - l_1\frac{1}{m(x)}\,, \tag{4.13}$$

$$\frac{1}{m(x)}\left[\triangle\mathcal{U}_0(x)\right]' + \mathcal{W}_m(x)\triangle\mathcal{U}_0(x) = \frac{1}{m(x)}\mathcal{W}_m''(x) + \left[\mathcal{W}_m^2(x)/2\right]'\,, \tag{4.14}$$

$$\left[\frac{1}{m(x)}\left\{\mathcal{PT}\,\mathcal{U}_0(x)\right\}'\right]' + \mathcal{W}_m(x)\left\{\mathcal{PT}\,\mathcal{U}_0(x)\right\}' + \mathcal{U}_0(x)\mathcal{PT}\,\mathcal{U}_0(x) = \widetilde{V}_{\rm m}^2(x) + l_1\widetilde{V}_{\rm m}(x) + l_2 - \left[\frac{1}{m(x)}\widetilde{V}_{\rm m}'(x)\right]'. \tag{4.15}$$

To tackle the set of equations (4.13)–(4.15), we observe that the second equation here can be integrated out entirely to have

$$\triangle\mathcal{U}_0(x) = \mathcal{W}_m'(x) + \mathrm{C}\exp\left[-\int^x m(y)\mathcal{W}_m(y)\,\mathrm{d}\,y\right], \tag{4.16}$$

where C is a constant of integration. But C has to be set equal to zero to be consistent with (4.6). So we are left with (4.6) only. Incorporating it along with (4.7),(4.9) and (4.13) $\widetilde{V}_{\rm m}(x)$ reads

$$\widetilde{V}_{\rm m}(x) = \triangle\widetilde{V}_{\rm m}(x) + f(x) - \mathcal{U}_0(x) - l_1/2\,, \tag{4.17}$$

Then looking at (4.8) prompts us to identify $\Lambda = -\frac{l_1}{2}$ and recast $\widetilde{V}_{\rm m}(x)$ as

$$\widetilde{V}_{\rm m}(x) = \frac{3}{2}\mathcal{W}_m'(x) + \frac{m'}{2m}\mathcal{W}_m(x) + \frac{m}{2}\mathcal{W}_m^2(x) - \mathcal{U}_0(x) - \frac{l_1}{2}\,. \tag{4.18}$$

We now focus on the remaining SUSY constraint (4.15). This can be converted to a second-order differential equation

$$\left[\frac{1}{m(x)}\left\{\mathcal{U}_0(x) + \widetilde{V}_{\rm m}^*(-x)\right\}'\right]' - \mathcal{U}_0'(x)\mathcal{W}_m(x) + \left[\,\mathcal{U}_0(x) - \mathcal{W}_m'(x)\,\right]\mathcal{U}_0(x) = \left[\widetilde{V}_{\rm m}^*(-x) + \frac{l_1}{2}\right]^2 + \left(l_2 - \frac{l_1^2}{4}\right), \tag{4.19}$$

by applying the $\mathcal{PT}$-operator on both sides and rearranging. Note that the action of $\mathcal{PT}$ on any function $g(x)$ is to be understood in the usual sense: $\mathcal{PT}\, g(x) = g^*(-x)\,, \mathcal{PT}\, g'(x) = -{g^*}'(-x)$ and so on. The nonlinear term $\mathcal{U}_0^2$ in (4.19) is redundant and can be eliminated in the following way. Using the relation $\widetilde{V}_{\rm m}^*(-x) \equiv \mathcal{PT}\,\widetilde{V}_{\rm m}(x) = \widetilde{V}_{\rm m}(x) - \triangle\widetilde{V}_{\rm m}(x)$, (4.18) results in

$$\widetilde{V}_{\rm m}^*(-x) + \mathcal{U}_0(x) = f(x) - \frac{l_1}{2}\,. \tag{4.20}$$

Employing (4.20), (4.19) can be reduced to the first order form

$$\mathcal{W}_m(x)\mathcal{U}_0'(x) + [\,\mathcal{W}_m'(x) - 2f(x)\,]\,\mathcal{U}_0(x) = \left[\frac{1}{m(x)}(x)f'(x)\right]' - f^2(x) + \left[\frac{l_1^2}{4} - l_2\right]. \tag{4.21}$$

Equation (4.21) which essentially results from the SUSY constraint (4.15) is consistent with the $\mathcal{CPT}$ equation (4.11) for $\mathcal{U}_0(x)$ given by (4.12) should we identify $\Theta = \left[l_2 - \frac{l_1^2}{4}\right]$. In terms of $\delta = +\sqrt{l_1^2 - 4l_2}$ we express $\mathcal{U}_0(x)$ as

$$\begin{aligned}\mathcal{U}_0(x) \;=\; & \frac{m(x)\mathcal{W}_m^2(x)}{4} + \frac{\mathcal{W}_m'(x)}{2} - \frac{\mathcal{W}_m''(x)}{2m(x)\mathcal{W}_m(x)} + \frac{1}{m(x)}\left(\frac{\mathcal{W}_m'(x)}{2\mathcal{W}_m(x)}\right)^2 \\ & + \frac{3}{4}\frac{m'^2(x)}{m^3(x)} - \frac{m''(x)}{2m^2(x)} - \frac{1}{m(x)}\left(\frac{\delta}{2\mathcal{W}_m(x)}\right)^2 .\end{aligned} \tag{4.22}$$

As a specific example we can go for the choice (4.5) which would give

$$\begin{aligned}\mathcal{U}_0(x) \;=\; & \frac{1}{4}\sec(x)\exp(2i\alpha x) - \frac{\delta^2}{4}\cos(x)\exp(-2i\alpha x) + \frac{i\alpha}{2}\exp(i\alpha x) \\ & + \frac{\alpha^2}{4}\cos(x) + \frac{1}{4}\sin^2(x)\sec(x) - \frac{1}{2}\sec(x).\end{aligned} \tag{4.23}$$

Evidently $\mathcal{U}_0(x)$ is non-$\mathcal{PT}$-symmetric.

Let us now analyze the solution of the zero-mode equation

$$\zeta^*\psi(x) = 0 \Rightarrow \left[\frac{1}{m}\partial^2 + \mathcal{W}^*(x)\partial + \mathcal{U}_0^*(x)\right]\psi(-x) = 0\,. \tag{4.24}$$

Two linearly independent solutions of zero-mode equation (4.24) may be expressed in the following compact form (see for details [28]):

$$\psi_j(x) = \mathcal{N}_j\sqrt{m(x)\mathcal{W}_m(x)}\exp\left[\int^x F_j(y)\,{\rm d}\,y\right], \tag{4.25}$$

where

$$F_j(x) = \frac{m(x)\mathcal{W}_m^2(x) + (-1)^j\delta}{2\mathcal{W}_m(x)}\,, \quad j = 1, 2. \tag{4.26}$$

These solutions will correspond the ground and first excited states of H_m, a feature known in the quadratic SUSY algebra.

Now it follows from the quadratic SUSY algebra (4.2) that the lowest eigenvalues of H_m are roots of the following quadratic equation

$$E^2 + l_1E + l_2 = 0 \;\Rightarrow E_0 = -\frac{l_1+\delta}{2}\,, E_1 = -\frac{l_1-\delta}{2}\,. \tag{4.27}$$

It is clear that the lowest two eigenvalues $E_{0,1}$ will be purely real if and only if the SUSY constants l_1, l_2 satisfy following inequality

$$l_1^2 \geq 4l_2\,. \tag{4.28}$$

It may be pointed out that the condition $l_1^2 \geq 4l_2$ was identified with the reducibility of the second-order SUSY construction [40] in the context of Hermitian QM. In non-Hermitian QM, we have shown that the same condition is related with the reality of the spectra.

V. N-TH ORDER CHARGE OPERATOR

The charge conjugate operator $\mathcal{C}$ may be represented as N-th order differential operator with N coefficient functions

$$\mathcal{C} = \frac{1}{\sqrt{m^N}}\partial^N + \mathcal{W}(x)\partial^{N-1} + \sum_{j=0}^{N-2} \mathcal{U}_j(x)\partial^j\,, \quad N = 1, 2, \ldots . \tag{5.1}$$

Some of the previous results are possible to generalize. Firstly, the pseudo-Hermiticity constraint (2.7) need to be compared order by order from both sides for N-th order representation (5.1) of $\mathcal{C}$. To do this, we note that the contributions from the adjoint operation on the term $g(x)\partial^\ell$ may be computed using the Libneitz rule as follows

$$\left[g(x)\partial^\ell\right]^\dagger = (-1)^\ell\partial^\ell\left[g(x)\right] = (-1)^\ell\sum_{r=0}^{\ell}\left[{}^\ell\mathrm{C}_r\partial^r\{g(x)\}\partial^{\ell-r}\right] . \tag{5.2}$$

Then order by order comparison gives the following restrictions on the coefficient functions in the charge operator $\mathcal{C}$ given by (5.1)

$$\left.\begin{array}{lll} \ell = N: & \mathcal{P}\left[m(x)\right] = m(x)\,, & \text{(mass is parity-invariant)} \\ \ell = N-1: & \mathcal{PT}\left[\mathcal{W}_m(x)\right] = \mathcal{W}_m(x)\,. & \text{(superpotential is } \mathcal{PT}\text{-invariant)} \end{array}\right\}, \quad N = 1, 2, 3, \ldots . \tag{5.3}$$

One may compare the general result derived above with the corresponding results for $N = 1$ [see (3.3) and (3.4)] and for $N = 2$ [see (4.3)]. Note that the second condition means as usual that the $\mathrm{Re}\,\mathcal{W}_m(x)$ is an even function while its imaginary part $\mathrm{Im}\,\mathcal{W}_m(x)$ is an odd function.

In contrast to the superpotential $\mathcal{W}_m$, the functions $\mathcal{U}_j, j = 0$ to $N-2$, are not $\mathcal{PT}$-symmetric. For instance, for $\ell = N-2$ and $\ell = N-3$ we have

$$\mathcal{U}_{N-2}(x) - \mathcal{PT}\left[\mathcal{U}_{N-2}(x)\right] = (N-1)\mathcal{W}'_m(x)\,, \quad N = 2, 3, \ldots \tag{5.4}$$

$$\mathcal{U}_{N-3}(x) - \mathcal{PT}\left[\mathcal{U}_{N-3}(x)\right] = (N-2)\left[-\frac{N-1}{2}\left\{\frac{N}{6}\left(\frac{1}{\sqrt{m^N}}\right)''' + \mathcal{W}''_m\right\} + \mathcal{U}'_{N-2}(x)\right], \quad N = 3, 4, \ldots , \tag{5.5}$$

and so on. More generally,

$$\triangle\mathcal{U}_{N-s}(x) = \mathcal{U}_{N-s}(x) - \mathcal{PT}\mathcal{U}_{N-s}(x)\,{}^N\mathrm{C}_s\partial^s\left(\frac{1}{\sqrt{m^N}}\right) + {}^{N-1}\mathrm{C}_{s-1}\partial^{s-1}\left[\mathcal{PT}\mathcal{W}(x)\right] + \sum_{j=1}^{s-2}{}^{N-s+j}\mathrm{C}_j\partial^j\left[\mathcal{PT}\{\mathcal{U}_{N-s+j}(x)\}\right] , \tag{5.6}$$

From the results, it is clear that the pseudo-Hermiticity constraints measure the amount of $\mathcal{PT}$-asymmetry in the coefficient functions. In particular, the measure is zero for first coefficient $m(x)$ and mass-deformed superpotential $\mathcal{W}_m(x)$.

Next comparing the coefficients of each derivative ∂^ℓ for $\ell = 0, 1, \ldots N+2$ from both sides of the $\mathcal{CPT}$-constraint (2.8), a straightforward calculation shows

$$\triangle\widetilde{V}_{\mathrm{m}}(x) \equiv \widetilde{V}_{\mathrm{m}}(x) - \mathcal{PT}\left[\widetilde{V}_{\mathrm{m}}(x)\right] = \frac{\sqrt{m^N}}{m^2}\left[2m\mathcal{W}'_m(x) + (N-1)m'\mathcal{W}_m(x)\right] , N = 1, 2, 3, \ldots . \tag{5.7}$$

Comparison for $\ell = N-1$ gives a closed expression for the potential due to the integrability of the equation

$$\begin{aligned} N\widetilde{V}_{\mathrm{m}}(x) \;=\; & \frac{\sqrt{m^N}}{m^2}\left[{}^N\mathrm{C}_2 m'\mathcal{W}_m(x) + m\left\{\sqrt{m^N}\mathcal{W}^2_m(x) + (2N-1)\mathcal{W}'_m(x) - 2\mathcal{U}_{N-2}(x)\right\}\right] \\ & + \frac{N(N-2)}{48}\left[4(2N+1)\left(\frac{1}{m}\right)'' + 3N(N-2)m\left(\frac{1}{m}\right)'^2\right] + \Lambda\,, N = 1, 2, 3, \ldots\ , \end{aligned} \tag{5.8}$$

where we set a convention that ${}^N\mathrm{C}_j \equiv 0\,, \mathcal{U}_{N-j} \equiv 0$ for $N < j$. Continuing this comparison up to the term ∂^0, we find that for all order N, only two coefficient functions in the representation of charge operator $\mathcal{C}$ remain independent, which are the mass function $m(x)$ and the superpotential $\mathcal{W}_m$. As for instance, comparing ∂^{N-2} from both sides of (2.8), one obtains for $N \geq 3$

$$\left[\frac{\mathcal{U}'_{N-2}(x)}{m}\right]' - \left[\triangle\widetilde{V}_{\rm m}(x) + {}^{N-1}{\rm C}_2\left(\frac{1}{m}\right)''\right]\mathcal{U}_{N-2}(x) + \frac{2}{m}\mathcal{U}'_{N-3}(x) - (N-3)\left(\frac{1}{m}\right)'\mathcal{U}_{N-3}(x)$$

$$= {}^{N+1}{\rm C}_4\frac{1}{\sqrt{m^N}}\left(\frac{1}{m}\right)^{IV} - {}^{N}{\rm C}_2\frac{(\widetilde{V}^*_{\rm m})''(-x)}{\sqrt{m^N}} + \mathcal{W}(x)\left[{}^{N}{\rm C}_3\left(\frac{1}{m}\right)''' - (N-1)(\widetilde{V}^*_{\rm m})'(-x)\right], \; N = 2, 3, \dots, \quad (5.9)$$

Similar to the first and second order cases, a general nonlinear SUSY algebra can be set up. The energy in such an algebra are zeros of the same N-th degree polynomial

$$E^N + l_1E^{N-1} + l_2E^{N-2} + \cdots + l_{N-2}E^2 + l_{N-1}E + l_N = 0 \quad (5.10)$$

from which we conclude that for an odd-order charge operator, the Hamiltonian H_m possesses at least one real energy eigenvalue.

VI. CONCLUSION

In this article we have studied a generalized PDM Schrödinger equation in a non-Hermitian framework. We have proposed new differential realization for the charge operator and sought for the solvability of the model. Several interesting consequences due to PDM and non-Hemiticity of the Hamitonian are derived. It should be noted that not all the results of the constant-mass non-Hermitian system are carried over to the PDM case. In constant-mass case, we showed that the superpotential $\mathcal{W}(x)$ had to be $\mathcal{PT}$-symmetric to preserve $\mathcal{CPT}$-symmetry and pseudo-hermiticity. In contrast, in the present work we have shown that the superpotential $\mathcal{W}(x)$ loses its $\mathcal{PT}$-symmetric property. Instead a new mass-deformed superpotential $\mathcal{W}_m(x)$ can be defined which turns out to be $\mathcal{PT}$-symmetric. Our work uncovers a new class of potentials $\widetilde{V}_m(x)$ admitting $\mathcal{CPT}$-symmetry in PDM non-Hermitian systems. We have also obtained extension of some of our results to a general N-th order charge operator wherein the mass function remains even and the mass-deformed superpotential $\mathcal{PT}$-symmetric.

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