Kuldip Raj* and Charu Sharma

Essential norm of weighted composition operators

https://doi.org/10.1515/anly-2017-0027
Received June 19, 2017; revised January 25, 2018; accepted May 21, 2018

Abstract: In the present paper we characterize the compact, invertible, Fredholm and closed range weighted composition operators on Cesàro function spaces. We also make an effort to compute the essential norm of weighted composition operators.

Keywords: Weighted composition operator, Fredholm operator, invertible operator, essential norm, isometry, Cesàro function space

MSC 2010: 40C05, 40F05, 40G05, 47B37

1 Introduction and preliminaries

Estaremi [6] studied some classes of weighted conditional-type operators and their spectra and he studied some $*$-classes of weighted conditional expectation-type operators. Mohiuddin, Mursaleen and Alotaibi [11] obtained a necessary and sufficient condition for an almost conservative matrix to define a compact operator. They have also established a necessary and sufficient condition for an operator to be compact for matrix classes. Alotaibi, Mursaleen, Alamri and Mohiuddine [1] studied the compact matrix operators using the Hausdorff measure of non-compactness. Malkowsky and Djolovic [10] characterized several classes of matrix transformation between these spaces and proved that the class of all matrix transformation on these spaces are Banach algebra. Further, they used the concept of Hausdorff measure of non-compactness of matrix operators to characterize compact operators and a sufficient condition for those operators to be Fredholm. Başar and Malkowsky [2] have also characterized the classes of all compact operators between the spaces of strongly summable and bounded sequences by using Hausdorff measure of non-compactness.

Let (X, s, μ) be a σ-finite measure space and let $L^0 = L^0(X)$ be the set of all equivalence classes of real-valued Lebesgue measurable functions defined on X, where $X = [0, 1]$ or $X = [0, \infty)$. Then for $1 \le p < \infty$ the Cesàro function space is denoted by $\operatorname{Ces}_p(X)$ and defined as

$$\operatorname{Ces}_p(X) = \left\{ f \in L^0(X) : \int_X \left(\frac{1}{x} \int_0^x f(t)\, d\mu(t) \right)^p d\mu(x) < \infty \right\}.$$

The Cesàro function space $\operatorname{Ces}_p(X)$ is a Banach space under the norm

$$\|f\| = \left(\int_X \left(\frac{1}{x} \int_0^x f(t)\, d\mu(t) \right)^p d\mu(x) \right)^{\frac{1}{p}}.$$

The Cesàro function spaces $\operatorname{Ces}_p[0, \infty)$ for $1 \le p \le \infty$ were considered by Shiue [15], and Hassard and Hussein [7]. The space $\operatorname{Ces}_\infty[0, 1]$ appeared already in 1948 and is known as the Korenblyum–Krein–Levin space K, see [8].

***Corresponding author: Kuldip Raj,** Department of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J & K, India, e-mail: kuldipraj68@gmail.com
Charu Sharma, Department of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J & K, India, e-mail: charu145.cs@gmail.com

Let $T : X \to X$ be a non-singular measurable transformation, i.e. $\mu T^{-1}(A) = \mu(T^{-1}(A)) = 0$ for each $A \in s$ whenever $\mu(A) = 0$. This condition means that the measure μT^{-1} is absolutely continuous with respect to μ. Let

$$f_0 = \frac{d\mu T^{-1}}{d\mu}$$

be the Radon–Nikodym derivative. In addition, we assume that $T^{-1}(s) \subset s$ so that $(X, T^{-1}(s), \mu)$ is σ-finite. In this paper we assume that $T[0, x] \subseteq [0, x]$ for all $x \in X$. An atom of the measure μ is an element $A \in s$ with $\mu(A) > 0$ such that for each $F \in s$, if $F \subset A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. Let A be an atom. Since μ is σ-finite, it follows that $\mu(A) < \infty$. Also every s-measurable function f on X is constant almost everywhere on A. It is a well-known fact that every sigma finite measure space (X, s, μ) can be decomposed into two disjoint sets X_1 and X_2 such that μ is atomic over X_1 and non-atomic over X_2, that is, X_2 is a countable collection of disjoint atoms [17].

Let $u : X \to \mathbb{C}$ and let $T : X \to X$ be a non-singular measurable transformation. Then a bounded linear transformation $M_{u,T} : \mathrm{Ces}_p(X) \to \mathrm{Ces}_p(X)$ defined by

$$(M_{u,T}f)(x) = (u \circ T)(x)(f \circ T)(x)$$

for every $f \in \mathrm{Ces}_p(X)$ and $x \in X$ is called a weighted composition operator induced by the pair (u, T). If we take $u(x) = I$, the identity operator on X, we write $M_{u,T}$ as C_T and call it a composition operator induced by T. In case $T(x) = x$ for some $x \in X$, we write $M_{u,T}$ as M_u and call it a multiplication operator induced by u.

For more details on composition and weighted composition operators see [4, 5, 9, 13, 14, 16] and the references therein.

Throughout this paper we consider a σ-finite measure space (X, s, μ) and $B(\mathrm{Ces}_p(X))$ denotes the set of all bounded linear operators from $\mathrm{Ces}_p(X)$ into itself, unless stated otherwise.

The main purpose of this paper is to characterize the boundedness, compactness, closed range and Fredholm property of weighted composition operators on Cesàro function spaces. We also make an effort to compute the essential norm of weighted composition operators in the section third of this paper.

2 Weighted composition operators

In this section of the paper we shall investigate the necessary and sufficient condition for a weighted composition operator to be bounded.

Theorem 2.1. *Let $u : X \to \mathbb{C}$ and $T : X \to X$ be two mappings. Then $M_{u,T} : \mathrm{Ces}_p(X) \to \mathrm{Ces}_p(X)$ is a bounded operator if and only if there exists $M > 0$ such that for μ-almost all $x \in X$,*

$$f_0(x)|u(x)| \le M. \tag{2.1}$$

Proof. If condition (2.1) is true, then for every $f \in \mathrm{Ces}_p(X)$, we have

$$\begin{aligned}
\|M_{u,T}f\|^p &= \int_X \left(\frac{1}{x} \int_0^x |((u \circ T)(f \circ T))(t)|\, d\mu(t) \right)^p d\mu(x) \\
&= \int_X \left(\frac{1}{x} \int_0^x |(uf)(t)|\, d\mu T^{-1}(t) \right)^p d\mu(x) \\
&= \int_X \left(\frac{1}{x} \int_0^x f_0(t)|u(t)f(t)|\, d\mu(t) \right)^p d\mu(x) \\
&\le M^p \int_X \left(\frac{1}{x} \int_0^x |f(t)|\, d\mu(t) \right)^p d\mu(x) \\
&= M^p \|f\|^p.
\end{aligned}$$

Therefore,

$$\|M_{u,T}f\| \le M\|f\|$$

for every $f \in \mathrm{Ces}_p(X)$. Hence, $M_{u,T}$ is bounded.

Conversely, if the condition of the theorem is not satisfied, then for every $n \in \mathbb{N}$ there exists a measurable set $\{F_n\}$ of X such that $F_n = \{x \in X : f_0(x)|u(x)| > n\}$ is a set of positive measure. Now consider $f_n = \frac{\chi_{F_n}}{\|\chi_{F_n}\|}$. Then we can easily show that

$$\int_X \left(\frac{1}{x}\int_0^x |f_n(t)|\, d\mu(t)\right)^p d\mu(x) < \infty$$

but

$$\int_X \left(\frac{1}{x}\int_0^x |(M_{u,T}f_n)(t)|\, d\mu(t)\right)^p d\mu(x) = \infty,$$

which contradicts the fact that $M_{u,T}$ is a bounded operator. Hence condition (2.1) must be true. □

Remark 2.2. If (X, s, μ) is an atomic measure space, then every weighted composition transformation $M_{u,T} : \mathrm{Ces}_p(X) \to \mathrm{Ces}_p(X)$ is a bounded.

Proof. Let $\{f_n\}$ be a sequence in $\mathrm{Ces}_p(X)$ such that

$$f_n \to f \quad \text{in } \mathrm{Ces}_p(X) \quad \text{for some } f \in \mathrm{Ces}_p(X). \tag{2.2}$$

Suppose

$$M_{u,T}f_n \to g \quad \text{for some } g \in \mathrm{Ces}_p(X). \tag{2.3}$$

From (2.2) we can select a subsequence $\{f_n'\}$ of the sequence $\{f_n\}$ such that $f_n' \to f$ μ-almost everywhere. Clearly, the sequence $\{M_{u,T}f_n'\}$ converges pointwise μ-almost everywhere to $M_{u,T}f$. From (2.3), we can select a subsequence $\{f_n''\}$ of $\{f_n'\}$ such that $M_{u,T}f_n'' \to g$ almost everywhere. By the uniqueness of limit, it follows that $M_{u,T}f = g$ almost everywhere. This proves that the graph of $M_{u,T}$ is closed. Hence, by the closed graph theorem, $M_{u,T}$ is continuous. □

3 Compactness and essential norm of weighted composition operators

Let $\mathcal{B}$ be a Banach space and let $\mathcal{C}$ be the set of all compact operators in $\mathcal{B}$. For any bounded linear operator L on $\mathcal{B}$, the essential norm of L means the distance from L to $\mathcal{C}$ in the operator norm, namely $\|L\|_e = \inf\{\|L - S\| : S \in \mathcal{C}\}$. Clearly, L is compact if and only if $\|L\|_e = 0$. The essential norm plays an interesting role in the compact problems of concrete operators (see [12]).

We are concerned with the case that L is a weighted composition operator $M_{u,T}$ on $\mathrm{Ces}_p(X)$. In [3], Chen has showed that uC_φ is compact on $L^p(\Sigma)$ if and only if for any $\varepsilon > 0$, the set

$$\{x \in X : J(x) \ge \varepsilon\} \tag{3.1}$$

consists of finitely many atoms. From this point of view, we compute the essential norm of $M_{u,T}$.

In this section of the paper we first characterize the compactness and then compute the essential norm of weighted composition operators on Cesàro function spaces.

Theorem 3.1. *Suppose $M_{u,T} \in B(\mathrm{Ces}_p(X))$. Then $M_{u,T}$ is a compact operator if and only if $uf_0 = 0$ a.e.*

Proof. We first assume that $M_{u,T}$ is compact. For if $uf_0 \ne 0$, there exists a constant $K > 0$ such that the set $E = \{x \in X : u(x) > \frac{1}{K}\} \cap \{x \in X : f_0(x) > \frac{1}{K}\}$ has positive measure. Since μ is non-atomic, there exists a measurable set E_n such that $E_{n+1} \subseteq E_n \subset E$, $\mu(E_n) = \frac{\alpha}{2^n}$ for some $\alpha > 0$. Let $e_n = \frac{1}{\mu(E_n)}\chi_{E_n}$. Then clearly, $\|e_n\| = 1$ so that the sequence $\{e_n\}$ is bounded in $\mathrm{Ces}_p(X)$. On the other hand for any $m, n \in \mathbb{N}$, let $m = 2n$. Then $E_m \subset E_n$

and

$$\begin{aligned}\|M_{u,T}e_n - M_{u,T}e_m\|^p &= \int_X \left(\frac{1}{x}\int_0^x |(M_{u,T}e_n - M_{u,T}e_m)(t)|\,d\mu(t)\right)^p d\mu(x)\\ &= \int_X \left(\frac{1}{x}\int_0^x f_0(t)(|u|\chi_{E_n} - |u|\chi_{E_m})(t)|\,d\mu(t)\right)^p d\mu(x)\\ &\geq \left(\frac{1}{K^2}\right)^p \|\chi_{E_n} - \chi_{E_m}\|^p.\end{aligned}$$

Therefore,

$$\|M_{u,T}e_n - M_{u,T}e_m\| > \frac{\varepsilon}{K^2} \quad \text{for some } \varepsilon > 0,$$

which shows that $\{M_{u,T}e_n\}$ does not contain a convergent subsequence. Therefore $M_{u,T}$ is not compact which contradicts the hypothesis. Hence $uf_0 = 0$ a.e.

Conversely, if $uf_0 = 0$, then $M_{u,T} = 0$ and so it is compact. □

Theorem 3.2. *Let $1 < p < \infty$ and let $M_{u,T} : \mathrm{Ces}_p(X) \to \mathrm{Ces}_p(X)$ be a bounded weighted composition operator. Then the essential norm of $M_{u,T}$ is given by*

$$\|M_{u,T}\|_e = \inf\{r > 0 : G_r \text{ consists of finitely many atoms}\}, \tag{3.2}$$

where $G_r = \{x \in X : \sqrt[p]{J(x)} \geq r\}$ and $\sqrt[p]{J} = uf_0$ considering the case $\|M_{u,T}\|_e = 0$ in (3.2), we know that (3.1) is necessary and sufficient condition for $M_{u,T}$ to be compact.

Proof. Denote the right-hand side of (3.2) by α. We first show that $\|M_{u,T}\|_e \geq \alpha$. If $\alpha = 0$, there is nothing to prove, and so we assume that $\alpha > 0$. Take $\varepsilon > 0$ arbitrarily. The definition of α implies that $F = G_{\alpha-\frac{\varepsilon}{2}}$ either contains a non-atomic subset or has infinitely many atoms. If F contains a non-atomic subset, then there are measurable sets F_n, $n \in \mathbb{N}$, such that $F_{n+1} \subseteq F_n \subset F$, $0 < \mu(F_n) < \frac{1}{n}$. Define $f_n = \mu(F_n)^{\frac{-1}{p}}\chi_{F_n}$. Then $\|f_n\| = 1$ for all $n \in \mathbb{N}$. We claim that $f_n \to 0$ weakly. For this we show that

$$\int_X \left(\frac{1}{x}\int_0^x |(f_n g)(t)|\,d\mu(t)\right) d\mu(x) \to 0 \quad \text{for all } g \in \mathrm{Ces}_q(X),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Let $A \subseteq F$ with $0 < \mu(A) < \infty$ and $g = \chi_A$. Then

$$\begin{aligned}\int_X \left(\frac{1}{x}\int_0^x |(f_n g)(t)|\,d\mu(t)\right) d\mu(x) &= \int_X \left(\frac{1}{x}\int_0^x |\mu(F_n)^{\frac{-1}{p}}\chi_{A\cap F_n}|\,d\mu(t)\right) d\mu(x)\\ &= \int_{A\cap F_n} \left(\frac{1}{x}\int_0^x |\mu(F_n)^{\frac{-1}{p}}|\,d\mu(t)\right) d\mu(x)\\ &= \mu(F_n)^{\frac{-1}{p}}\mu(A\cap F_n)\\ &\leq \left(\frac{1}{n}\right)^{1-\frac{1}{p}} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Since simple functions are dense in $\mathrm{Ces}_q(X)$, it is thus proved that f_n converges to 0 weakly. Now assume that F consists finitely many atoms. Let $\{F_n\}_{n=1}^{\infty}$ be disjoint atoms in F. Again put f_n as above. It is easy to see that for $A \subseteq F$ with $0 < \mu(A) < \infty$ we have $\mu(A \cap F_n) = 0$, for sufficiently large n. So in both cases

$$\int_X \left(\frac{1}{x}\int_0^x |(f_n g)(t)|\,d\mu(t)\right) d\mu(x) \to 0.$$

Now take a compact operator L on $\mathrm{Ces}_p(X)$ such that

$$\|M_{u,T} - L\| \leq \|M_{u,T}\|_e + \frac{\varepsilon}{2}.$$

Then we have

$$\begin{aligned}
\|M_{u,T}\|_e &> \|M_{u,T} - L\| - \frac{\varepsilon}{2} \\
&\geq \|M_{u,T} f_n - L f_n\| - \frac{\varepsilon}{2} \\
&\geq \left(\int_X \left(\frac{1}{x} \int_0^x |((u \circ T)(f_n \circ T))(t)| \, d\mu(t) \right)^p d\mu(x) \right)^{\frac{1}{p}} - \|Lf_n\| - \frac{\varepsilon}{2} \\
&= \left(\int_X \left(\frac{1}{x} \int_0^x |(u(t) f_n(t))| \, d\mu T^{-1}(t) \right)^p d\mu(x) \right)^{\frac{1}{p}} - \|Lf_n\| - \frac{\varepsilon}{2} \\
&\geq \left(\int_X \left(\frac{1}{x} \int_0^x f_0(t) |(u(t) f_n(t))| \, d\mu(t) \right)^p d\mu(x) \right)^{\frac{1}{p}} - \|Lf_n\| - \frac{\varepsilon}{2} \\
&\geq \left(\int_X \left(\frac{1}{x} \int_0^x |J^{\frac{1}{p}} f_n(t)| \, d\mu(t) \right)^p d\mu(x) \right)^{\frac{1}{p}} - \|Lf_n\| - \frac{\varepsilon}{2} \\
&\geq \left(\alpha - \frac{\varepsilon}{2} \right) - \|Lf_n\| - \frac{\varepsilon}{2}
\end{aligned}$$

for all $n \in \mathbb{N}$. Since compact operator maps weekly convergent sequences into norm convergent ones, it follows that $\|Lf_n\| \to 0$. Hence

$$\|M_{u,T}\|_e \geq \alpha - \varepsilon.$$

Since ε is arbitrary, we obtain $\|M_{u,T}\|_e \geq \alpha$.

For the opposite inequality, take ε arbitrary. Put $K = G_{\alpha+\varepsilon}$ and $v = \chi_K u$. The definition of α implies that K consist of finitely many atoms. So we can write $K = \{K_1, K_1, \dots, K_m\}$, where $K_1, K_1, \dots, K_m$ are distinct. We have

$$M_{v,T} f(X) = \sum_{i=1}^{m} v(K_i) f(T(K_i)) \quad \text{for all } f \in \mathrm{Ces}_p(X)$$

and hence $M_{v,T}$ has finite rank. Noting that $M_{v,T}$ is compact operator, we obtain

$$\begin{aligned}
\|M_{u,T} - M_{v,T}\|^p &= \|(1 - \chi_K) M_{u,T}\|^p \\
&= \sup_{\|f\| \leq 1} \|\chi_{X \setminus K} M_{u,T} f\|^p \\
&= \sup_{\|f\| \leq 1} \int_{X \setminus K} \left(\frac{1}{x} \int_0^x |((u \circ T)(f \circ T))(t)| \, d\mu(t) \right)^p d\mu(x) \\
&= \sup_{\|f\| \leq 1} \int_{X \setminus K} \left(\frac{1}{x} \int_0^x |(uf)(t)| \, d\mu T^{-1}(t) \right)^p d\mu(x) \\
&= \sup_{\|f\| \leq 1} \int_{X \setminus K} \left(\frac{1}{x} \int_0^x f_0(t) |u(t) f(t)| \, d\mu(t) \right)^p d\mu(x) \\
&\leq \sup_{\|f\| \leq 1} \int_{X \setminus K} \left(\frac{1}{x} \int_0^x J^{\frac{1}{p}} |f(t)| \, d\mu(t) \right)^p d\mu(x) \\
&\leq (\alpha + \varepsilon) \sup_{\|f\| \leq 1} \int_{X \setminus K} \left(\frac{1}{x} \int_0^x |f(t)| \, d\mu(t) \right)^p d\mu(x) \\
&\leq \alpha + \varepsilon.
\end{aligned}$$

Since ε is arbitrary so that $\|M_{u,T}\| \leq \alpha$, this completes the proof of the theorem. □

4 Fredholm and invertible weighted composition operators

In this section we first establish a condition for the weighted composition operators to have closed range and then we make use of it to characterize the Fredholm weighted composition operators. We also make an effort to characterize invertible and isometric weighted composition operators on Cesàro function spaces.

The Hölder inequality for Cesàro measurable function spaces is as follows: if $f \in \mathrm{Ces}_p(X)$ and $g \in \mathrm{Ces}_q(X)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left|\int fg\,d\mu\right| \le \|f\|_p \|g\|_q.$$

We find that every $g \in \mathrm{Ces}_q(X)$ gives rise to a bounded linear functional $Fg \in (\mathrm{Ces}_p(X))^*$ which is defined as

$$Fg(f) = \int fg\,d\mu \quad \text{for every } f \in \mathrm{Ces}_p(X).$$

Theorem 4.1. *Let $M_{u,T} \in B(\mathrm{Ces}_p(X))$. Then $M_{u,T}$ has closed range if and only if there exists a constant $\delta > 0$ such that*

$$f_0(x)|u(x)| \ge \delta$$

for μ-almost all $x \in S = \operatorname{supp} u \cap T(X)$.

Proof. Assume that the condition is true. Then for every $f \in \mathrm{Ces}_p(X \setminus S)$ and $\delta > 0$, we have

$$\begin{aligned}
\|M_{u,T}f\|^p &= \int\limits_X \left(\frac{1}{x}\int\limits_0^x |((u \circ T)(f \circ T))(t)|\,d\mu(t)\right)^p d\mu(x) \\
&= \int\limits_X \left(\frac{1}{x}\int\limits_0^x |(uf)(t)|\,d\mu T^{-1}(t)\right)^p d\mu(x) \\
&= \int\limits_X \left(\frac{1}{x}\int\limits_0^x f_0(t)|u(t)f(t)|\,d\mu(t)\right)^p d\mu(x) \\
&\ge \delta^p \int\limits_X \left(\frac{1}{x}\int\limits_0^x |f(t)|\,d\mu(t)\right)^p d\mu(x) \\
&= \delta^p \|f\|^p.
\end{aligned}$$

Therefore

$$\|M_{u,T}f\| \ge \delta\|f\|$$

for all $f \in \mathrm{Ces}_p(X)$. So that $M_{u,T}$ has closed range, since $\ker M_{u,T} = \mathrm{Ces}_p(X \setminus S)$.

Conversely, suppose that $M_{u,T}$ has closed range. Then there exists a constant $\delta > 0$ such that

$$\|M_{u,T}f\| \ge \delta\|f\| \quad \text{for all } f \in \mathrm{Ces}_p(X). \tag{4.1}$$

Let

$$H_n = \left\{x \in S : \frac{1}{(n+1)^2} \le f_0(x)|u(x)| = \left(\frac{1}{n^2}\right)\right\}$$

and $H = \{n : \mu(H_n) > 0\}$. Let $f = \sum_{n\in H} \frac{1}{\mu(H_n)} M_{u,T}\chi_{H_n}$. Then

$$\begin{aligned}
\int\limits_X \left(\frac{1}{x}\int\limits_0^x |f(t)|\,d\mu(t)\right)^p d\mu(x) &= \int\limits_X \left(\frac{1}{x}\int\limits_0^x \sum_{n\in H} \left|\left(\frac{1}{\mu(H_n)} M_{u,T}\chi_{H_n}\right)(t)\right| d\mu(t)\right)^p d\mu(x) \\
&\le \sum_{n\in H} \int\limits_X \left(\frac{1}{x}\int\limits_0^x f_0(t)\frac{1}{\mu(H_n)}|u(t)|\,d\mu(t)\right)^p d\mu(x) \\
&< \sum_{n\in H} \frac{1}{n^2} < \infty.
\end{aligned}$$

Now, for

$$g = \sum_{n\in H} \frac{1}{\mu(H_n)} \chi_{H_n},$$

we have

$$f = M_{u,T}\Big[\sum_{n\in H} \frac{1}{\mu(H_n)} \chi_{H_n}\Big] = M_{u,T} g$$

In view of inequality (4.1),

$$\|M_{u,T} g\| \geq \delta \|g\|$$

but

$$\int_X \Big(\frac{1}{x}\int_0^x |g(t)|\, d\mu(t)\Big)^p \, d\mu(x) = \infty,$$

which is a contradiction. Hence H must be finite. In other words, there exists n_0 such that $\mu(H_n) = 0$ for all $n \geq n_0$, i.e.

$$f_0(x)|u(x)| \geq \Big[\frac{1}{n^2}\Big] = \delta \quad \text{(say)}. \qquad \square$$

Theorem 4.2. *Suppose $M_{u,T} \in B(\mathrm{Ces}_p(X))$. Then $M_{u,T}$ is an injection if and only if $u \circ T \neq 0$ and T is surjective.*

Proof. Suppose $M_{u,T} f = 0$ so that $(u \circ T)(f \circ T) = 0$ for μ-almost all x. Since $u \circ T \neq 0$, by hypothesis $f \circ T = 0$ and thus $f(T(x)) = 0$ for μ-almost all x. Hence $f = 0$. Thus $M_{u,T}$ is an injection operator.

Conversely, suppose that $M_{u,T}$ is injective. If T is not surjective, then there exists a positive measurable set F such that $\chi_F \in \mathrm{Ces}_p(X)$. Clearly $M_{u,T}\chi_F = 0$, so that $M_{u,T}$ has a non-trivial kernel which is a contradiction. Hence T must be surjective. Further suppose that there exists a measurable set $E = \{x \in X : |u \circ T(x)| = 0\}$ such that $\mu(E) > 0$. Suppose that $T^{-1}(F) \subset E$ is such that $\chi_F \in \mathrm{Ces}_p(X)$. For given $\varepsilon > 0$ we have

$$\|M_{u,T} f\|^p = \int_X \Big(\frac{1}{x}\int_0^x |((u \circ T)(\chi_F \circ T))(t)|\, d\mu(t)\Big)^p \, d\mu(x) = 0.$$

Hence $M_{u,T}$ has a non-trivial kernel, which is again a contradiction. Hence $u \circ T \neq 0$ a.e. $\square$

Theorem 4.3. *Suppose $M_{u,T} \in B(\mathrm{Ces}_p(X))$. Then $M_{u,T}$ has dense range if and only if $u \circ T \neq 0$ a.e. and $T^{-1}(s) = s$ a.e.*

Proof. Let $M_{u,T}$ have dense range and $E = \{x : |u \circ T(x)| = 0\}$. If possible, suppose $\mu(E) > 0$. Let F be a measurable subset of E such that $0 < \mu(F) < 1$. Consider

$$\langle \chi_F, M_{u,T} f\rangle = \int_F u(T(x)) f(T(x))\, d\mu(x) = 0,$$

which shows that $\chi_F \in (\mathrm{ran}\, M_{u,T} f)^{\perp}$ so that $\mathrm{ran}\, M_{u,T}$ is not dense in $\mathrm{Ces}_p(X)$. Hence $\mu(E) = 0$. Thus, $u \circ T \neq 0$ a.e. Next we show that $T^{-1}(s) = s$. Let $F \in s$ be such that $0 < \mu(F) < \infty$. Since $\mathrm{ran}\, M_{u,T}$ is dense, we can find a sequence $\{f_n\}$ in $\mathrm{Ces}_p(X)$ such that $M_{u,T} f_n \to \chi_F$ as $n \to \infty$ or $(uf_n) \circ T \to \chi_F$ as $n \to \infty$ so that F is $T^{-1}(s)$ measurable. Hence, $T^{-1}(s) = s$ a.e. Thus, we have proved that if $M_{u,T}$ has dense range, then $u \circ T \neq 0$ a.e. and $T^{-1}(s) = s$.

Conversely, suppose that $u \circ T \neq 0$ a.e. and $T^{-1}(s) = s$. For $m \in \mathbb{N}$, let $E_m = \{x : |u \circ T(x)| > \frac{1}{m}\}$. Let $F \in s$ be such that $0 < \mu(F) < \infty$. Set $F_m = F \cap E_m$ and $H_m = \chi_{F_m}$. Let $g_m = \chi_{F_m} \circ T^{-1}$ (as T is bijective) and take $G_m = \mathrm{supp}\, g_m$ so that clearly $G_m \subset \mathrm{supp}\, g_m$ and $T^{-1}(G_m) = F_m$. Now

$$0 = \mu\Big\{x : \chi_{F_n} |u \circ T(x)| < \frac{1}{m}\Big\} = \mu T^{-1}\Big\{x : g_m(x) < \frac{1}{m}\Big\}.$$

Since $0 < f_0 < M$ on G_m, this implies that

$$\mu\Big\{x : g_m(x) < \frac{1}{m}\Big\} = 0.$$

Let $G_{m,n}$ be an increasing sequence of measurable sets of finite measure such that $\bigcup_{n=1}^{\infty} G_{m,n} = G_m$. Let $h_{m,n} = \chi_{G_{m,n}}|G_m$. Then

$$\begin{aligned}\|h_{m,n} \circ T - H_m\|^p &= \int_X \left(\frac{1}{x} \int_0^x |(h_{m,n} \circ T - H_m)(t)|\, d\mu(t) \right)^p d\mu(x) \\ &= \int_X \left(\frac{1}{x} \int_0^x |(\chi_{G_{m,n}} - \chi_{G_m})(t)|\, d\mu T^{-1}(t) \right)^p d\mu(x) \\ &= \int_X \left(\frac{1}{x} \int_0^x f_0(t)|(\chi_{G_{m,n}} - \chi_{G_m})(t)|\, d\mu(t) \right)^p d\mu(x) \\ &\leq M^p \|\chi_{G_{m,n}} - \chi_{G_m}\|^p.\end{aligned}$$

Hence

$$\|h_{m,n} \circ T - H_m\| \leq M\|\chi_{G_{m,n}} - \chi_{G_m}\| \to 0 \quad \text{as } n \to \infty.$$

Thus

$$M_{u,T}h_{m,n} \to \chi_{F_m} \quad \text{as } n \to \infty.$$

So $\chi_{F_m} \in \operatorname{ran} M_{u,T}$ and hence $\chi_F \in \operatorname{ran} M_{u,T}$. This completes the proof. □

Corollary 4.4 (to Theorem 4.1). *Let $M_{u,T} \in B(\mathrm{Ces}_p(X))$. Then $M_{u,T}$ is bounded away from zero if and only if there exists $\delta > 0$ such that*

$$f_0(x)|u(x)| > \delta$$

for μ-almost all $x \in X$.

Theorem 4.5. *Let $M_{u,T} \in B(\mathrm{Ces}_p(X))$. Then $M_{u,T} : \mathrm{Ces}_p(X) \to \mathrm{Ces}_p(X)$ is invertible if and only if*
(i) $u \neq 0$ *a.e.,*
(ii) *T is invertible,*
(iii) $f_0(x)|u(x)| > \delta$ *for μ-almost all $x \in X$.*

Proof. We first suppose that conditions (i)–(iii) are satisfied. Then by Corollary 4.4 and Theorem 4.3, $M_{u,T}$ is bounded away from zero and has dense range. Hence $M_{u,T}$ is invertible.

Conversely, suppose that $M_{u,T}$ is invertible. Then clearly $u(x) \neq 0$ and $T^{-1}(s) = s$. Also in view of Corollary 4.4,

$$f_0(x)|u(x)| > \delta \tag{4.2}$$

for μ-almost all $x \in X$ and for some $\delta > 0$. Since $T^{-1}(s) = s$, it follows that T is injective. Again from (4.2), $f_0 \neq 0$ a.e., so T is surjective. Thus T is invertible. □

Theorem 4.6. *Let $M_{u,T} \in B(\mathrm{Ces}_p(X))$. Then $M_{u,T}$ is Fredholm if and only if $M_{u,T}$ is invertible.*

Proof. Suppose that $M_{u,T}$ is Fredholm. Then $\ker M_{u,T}$ and $\mathrm{Ces}_p(X) \setminus \operatorname{ran} M_{u,T}$ are finite dimensional and $\operatorname{ran} M_{u,T}$ is closed. If $\ker M_{u,T}$ is finite dimensional, then $\ker M_{u,T} = 0$ otherwise it will be infinite dimensional. Therefore by Theorem 4.2, $u \circ T \neq 0$ and T is surjective so that $u \neq 0$. Suppose $\operatorname{codim} \operatorname{ran} M_{u,T} < \infty$. We claim that $M_{u,T}$ is onto. For if $\operatorname{ran} M_{u,T}$ is not dense, then it is a proper closed subspace of $\mathrm{Ces}_p(X)$ and so for $f \in \mathrm{Ces}_p(X) \setminus \operatorname{ran} M_{u,T}$, there exists a continuous linear functional $g^* \in \mathrm{Ces}_p^*(X)$ such that

$$g^*(M_{u,T}f) = \int (M_{u,T}f)g^*\, d\mu = 0 \quad \text{and} \quad (g^*f) = \int fg^*\, d\mu = 1,$$

from the later inequality $\mathrm{Re}(fg^*) = 1$. Hence the set $E_\delta = \{x \in X : \mathrm{Re}(fg^*)(x) \geq \delta\}$ has positive measure for some $\delta > 0$. From the non-atomicness of μ we can find a sequence $\{E_n\}$ of measurable subsets of E_δ with $0 < \mu(E_n) < \infty$ such that $E_n \cap E_m = \emptyset$ for $m \neq n$. Let $g_n^* = \chi_{E_n} g^*$. Clearly $g_n^* \in \mathrm{Ces}_p^*(X)$ and is non-zero. Now for each $f \in \mathrm{Ces}_p(X)$,

$$(M_{u,T}^* g_n^*)(f) = g_n^*(M_{u,T}f)$$

and

$$\int_X \left(\frac{1}{x}\int_0^x \chi_{E_n} g^* M_{u,T} f d\mu\right)^p d\mu = \int_X \left(\frac{1}{x}\int_0^x (M_{u,T} f \chi_{E_n}) g^*\, d\mu\right)^p d\mu = 0.$$

Thus, $M_{u,T}^* g_n^* = 0$ for each $n = 1, 2, \ldots$. This proves that $\ker M_{u,T}^*$ and hence $\mathrm{Ces}_p(X) \setminus \mathrm{ran}\, M_{u,T}$ is infinite dimensional, which is a contradiction. Hence $M_{u,T}$ has dense range. Now $M_{u,T}$ is bounded away from zero and hence dense range. Therefore, $M_{u,T}$ is invertible. The converse of the theorem is obvious. □

Theorem 4.7. *Let $M_{u,T} \in B(\mathrm{Ces}_p(X))$. Then $M_{u,T}$ is an isometry if and only if $f_0 = |u| = 1$ a.e.*

Proof. Suppose $f_0 = |u| = 1$ a.e. For $f \in \mathrm{Ces}_p(X)$, consider

$$\begin{aligned}
\|M_{u,T} f\|^p &= \int_X \left(\frac{1}{x}\int_0^x |((u\circ T)(f\circ T))(t)|\, d\mu t\right)^p d\mu(x) \\
&= \int_X \left(\frac{1}{x}\int_0^x |(uf)(t)|\, d\mu T^{-1}(t)\right)^p d\mu(x) \\
&= \int_X \left(\frac{1}{x}\int_0^x f_0(t)|uf(t)|\, d\mu(t)\right)^p d\mu(x) \\
&\le \int_X \left(\frac{1}{x}\int_0^x |f(t)|\, d\mu(t)\right)^p d\mu(x) \\
&= \|f\|^p.
\end{aligned}$$

Thus $M_{u,T}$ is an isometry.

Conversely, suppose that $M_{u,T}$ is an isometry. Then

$$\begin{aligned}
\|M_{u,T} f\|^p = \|f\|^p &\implies \int_X \left(\frac{1}{x}\int_0^x |((u\circ T)(f\circ T))(t)|\, d\mu t\right)^p d\mu(x) = \int_X \left(\frac{1}{x}\int_0^x |f(t)|\, d\mu(t)\right)^p d\mu(x) \\
&\implies \int_X \left(\frac{1}{x}\int_0^x f_0(t)|uf(t)|\, d\mu(t)\right)^p d\mu(x) = \int_X \left(\frac{1}{x}\int_0^x |f(t)|\, d\mu(t)\right)^p d\mu(x).
\end{aligned}$$

The relation holds if and only if $f_0 = |u| = 1$ a.e., hence the result. □

Acknowledgment: We would like to express our sincere thanks of the reviewer for their kind remarks which improved the presentation of the paper.

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