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ABSTRACT

We have described a new kind of continued fraction which is referred to as an $\mathcal{F}_{1,2}$ -continued fraction. The $\mathcal{F}_{1,2}$ -continued fraction arises from a subgraph (denoted as $\mathcal{F}_{1,2}$) of the Farey graph. We have given a geometric interpretation of the partial quotients and formulated an algorithm to find $\mathcal{F}_{1,2}$ -continued fraction expansion of a number. We have also studied the analogues of certain properties of regular continued fractions in the context of $\mathcal{F}_{1,2}$ -continued fractions.

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1. Introduction

The action of the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$ on the extended set of rationals $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{x}{y} = \frac{ax + by}{cx + dy},$$

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where $x, y \in \mathbb{Z}$, $(x, y) = 1$ with the usual convention that ∞ is represented by $\frac{x}{0}$ ($x \neq 0$) has been studied, in various contexts. This action is transitive and the stabilizer of ∞ is the subgroup

$$\left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

Jones, Singerman, and Wicks, in [3], define certain graphs based on this action. The diagonal action (given by $g(\alpha, \beta) = (g\alpha, g\beta)$) of the modular group on $\hat{\mathbb{Q}} \times \hat{\mathbb{Q}}$ defines the *suborbital graph* $\mathcal{G}_{u,N}$ for every $N \geq 1$ and $u \in \mathbb{Z}$ with $(u, N) = 1$ as follows: the set of vertices is $\hat{\mathbb{Q}}$ and the edges are given by $\alpha \sim \beta \Leftrightarrow \exists g \in \Gamma$ such that $g(\infty) = \alpha$ and $g(\frac{u}{N}) = \beta$. In fact, $\mathcal{G}_{u,N}$ is self-paired if and only if $u^2 \equiv -1 \pmod{N}$ [3, Corollary 3.4]. Denote by $\mathcal{F}_{u,N}$ the subgraph of $\mathcal{G}_{u,N}$ whose vertex set is

$$\left\{ \frac{x}{y} : x, y \in \mathbb{Z}, y > 0, (x, y) = 1 \text{ and } N|y \right\} \cup \{\infty\}.$$

The vertex set for $\mathcal{F}_{1,2}$ is denoted by \mathcal{X} in the remainder of this note.

The congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : N|c \right\}$$

defines the following equivalence relation on $\hat{\mathbb{Q}}$ by $g_1(\infty) \cong_N g_2(\infty)$ for $g_1, g_2 \in \Gamma$, if $g_1\Gamma_0(N) = g_2\Gamma_0(N)$. If $g_1(\infty) = \frac{r}{s}$ and $g_2(\infty) = \frac{x}{y}$, we have $g_1(\infty) \cong_N g_2(\infty)$ if and only if $N|ry - sx$. In fact, the vertex set of $\mathcal{F}_{u,N}$ is the equivalence class of ∞ in $\hat{\mathbb{Q}}$ for this relation [3].

The graph $\mathcal{F}_{1,1}$ is called the Farey graph [3]. The regular continued fraction of a real number is described in terms of the Farey graph and this is classical [4]. This motivates us to investigate whether there is an analogue of the regular continued fraction which is related to the graph $\mathcal{F}_{1,2}$. Hence we arrive at the following definition.

A finite continued fraction of the form

$$\frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_n}{a_n} \quad (n \geq 0)$$

or an infinite continued fraction of the form

$$\frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_n}{a_n+} \dots,$$

where b is an odd integer, a_1, a_2, \dots are even positive integers, and $\epsilon_1, \epsilon_2, \dots \in \{\pm 1\}$, is called an $\mathcal{F}_{1,2}$ -continued fraction.

The above definition makes these new continued fractions closely related in form to certain well-studied continued fractions known as semi-regular continued fractions (see

Kraaikamp [5]) and in particular to continued fractions with even partial quotients (see Schweiger [7]). The latter is referred to as the E.C.F. henceforth. They therefore satisfy several properties analogous to those of the regular continued fraction. These properties are summarized in Section 2.

On the other hand, the striking fact is the relation of this new continued fraction with the graph $\mathcal{F}_{1,2}$ and its properties (which are also recalled in Section 2). This vital geometric link is explored in great detail in the subsequent sections.

Section 3 devotes itself to studying finite $\mathcal{F}_{1,2}$ -continued fractions and establishes a bijective correspondence between them and the vertices of $\mathcal{F}_{1,2}$, namely elements of \mathcal{X} . Each finite $\mathcal{F}_{1,2}$ -continued fraction is shown to correspond naturally to a path in $\mathcal{F}_{1,2}$ from ∞ to its value. This result should be compared with the result of [3] that continued fractions of the form $c_0 - \frac{1}{c_1 - \frac{1}{c_2 - \dots \frac{1}{c_n}}}$ with $c_i \in \mathbb{Z}$ correspond to shortest paths in the Farey graph from ∞ to its value. We also derive (for the special case of $\mathcal{F}_{1,2}$), the results of Değer, Beşenk and Güler [2] which relate certain infinite paths in suborbital graphs with continued fractions. More generally, the main result of this section may be compared to [1, Theorem 3.1].

In Section 4 we derive an algorithm to obtain the $\mathcal{F}_{1,2}$ -continued fraction associated with any element of \mathcal{X} and relate this to the classical definition of the E.C.F. given in [7].

Section 5 shows that any real number can be expressed as an $\mathcal{F}_{1,2}$ -continued fraction and produces a geometric proof of the fact that an irrational number has a unique $\mathcal{F}_{1,2}$ -expansion. This should be compared with analogous result of Kraaikamp and Lopes [6] regarding the E.C.F. expansion of an irrational number.

Section 6 is an elaborate study of the $\mathcal{F}_{1,2}$ -expansions of elements of $\mathbb{Q} \setminus \mathcal{X}$. These are the only real numbers where the uniqueness of $\mathcal{F}_{1,2}$ -expansion fails and this failure displays several patterns which prove useful in the subsequent analysis.

Section 7 deals with the important topic of the approximation properties of $\mathcal{F}_{1,2}$ -continued fractions of real numbers. Schweiger [8] observed that the E.C.F. expansion has very poor measure theoretic approximation properties and this holds for $\mathcal{F}_{1,2}$ -expansions too. However, the regular continued fraction has many classical best approximation properties, and one can pose analogous questions for the new continued fractions. For instance, a rational number p/q is called a best approximation of $x \in \mathbb{R}$ if for every rational number $p'/q' \neq p/q$ with $0 < q' \leq q$, we have $|qx - p| < |q'x - p'|$. It is a classical theorem that every convergent of the regular continued fraction of x is a best approximation of x and conversely (except in the case that x is a half-integer).

The central theme of [5] is to start with an alternate notion of best approximation which is satisfied by the convergents of the regular continued fraction and study those continued fraction expansions which improve these best approximation properties. Note that though the E.C.F. belongs to the class of continued fractions studied in [5], their convergents are not necessarily best approximations in this alternate notion.

In this light, it is an important achievement for the $\mathcal{F}_{1,2}$ -continued fraction that a suitable modification of the notion of best approximation yields the theorems of Section 7

that any best approximation is an $\mathcal{F}_{1,2}$ -convergent and conversely, in almost all cases. Here the exceptions are once again the rationals which are not in \mathcal{X} .

2. Preliminaries

We summarize the definitions and basic results of semi-regular continued fractions. For more details see [5].

A pair of finite or infinite sequences $\{\epsilon_i\}_{i \geq 1}$ and $\{a_i\}_{i \geq 0}$ with $\epsilon_i \in \{\pm 1\}$, $a_0 \in \mathbb{Z}$ and for $n \geq 1$, $a_n \in \mathbb{N}$ is called a *semi-regular continued fraction* when $\epsilon_{n+1} + a_n \geq 1$ and in the infinite case $a_n \geq 2$ infinitely often. A semi-regular continued fraction, when it is finite, is expressed as

$$a_0 + \frac{\epsilon_1}{a_1 +} \frac{\epsilon_2}{a_2 +} \frac{\epsilon_3}{a_3 +} \cdots \frac{\epsilon_n}{a_n}$$

and when infinite, as

$$a_0 + \frac{\epsilon_1}{a_1 +} \frac{\epsilon_2}{a_2 +} \frac{\epsilon_3}{a_3 +} \cdots \frac{\epsilon_n}{a_n +} \cdots$$

The integers a_i ($i \geq 0$) are called the *partial denominators* of the continued fraction. The integers ϵ_i ($i \geq 1$) are called *partial numerators*. The value of the expression

$$\frac{p_k}{q_k} = a_0 + \frac{\epsilon_1}{a_1 +} \frac{\epsilon_2}{a_2 +} \frac{\epsilon_3}{a_3 +} \cdots \frac{\epsilon_k}{a_k}$$

is called the *k-th convergent* of the continued fraction and the sequence $\{p_k/q_k\}_{k \geq 0}$ is called the *sequence of convergents* of this continued fraction. In fact, the sequence of convergents of a finite continued fraction is a finite sequence. The continued fraction

$$a_i + \frac{\epsilon_{i+1}}{a_{i+1} +} \cdots$$

is called the *tail of the continued fraction at the i-th stage*.

Remark 2.1.

- (1) If $\epsilon_n = 1$ and $a_n \in \mathbb{N}$, $n \geq 1$ then we have the regular continued fraction.
- (2) If $\epsilon_n = \pm 1$ and a_n is an even positive integer for $n \geq 1$ with $a_0 \in 2\mathbb{Z}$ then we get an E.C.F. (mentioned in Section 1).
- (3) An $\mathcal{F}_{1,2}$ -continued fraction yields an E.C.F. and vice-versa. Observe that $x = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots$ if and only if $2x - b = \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots$.

The following three results about semi-regular continued fractions are well known.

Proposition 2.2. Suppose $\{\frac{p_n}{q_n}\}_{n \geq 0}$ is the sequence of convergents of a semi-regular continued fraction

$$a_0 + \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \frac{\epsilon_3}{a_3+} \cdots \frac{\epsilon_n}{a_n+} \cdots$$

Then $\{p_n\}$ and $\{q_n\}$ satisfy the following recurrence relations

$$p_{n+1} = a_{n+1}p_n + \epsilon_{n+1}p_{n-1} \text{ and } q_{n+1} = a_{n+1}q_n + \epsilon_{n+1}q_{n-1}$$

where $(p_{-1}, q_{-1}) = (1, 0)$, $(p_0, q_0) = (a_0, 1)$ and $n \geq 0$.

Proposition 2.3. Suppose x and y_n ($n \geq 1$) are real numbers such that for every $n \geq 1$,

$$x = a_0 + \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \frac{\epsilon_3}{a_3+} \cdots \frac{\epsilon_n}{a_n + y_{n+1}},$$

such that

$$a_0 + \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \frac{\epsilon_3}{a_3+} \cdots \frac{\epsilon_n}{a_n+} \cdots$$

is a semi-regular continued fraction having value x . If p_i/q_i is the i -th convergent of the continued fraction then

$$x = \frac{p_n + y_{n+1}p_{n-1}}{q_n + y_{n+1}q_{n-1}}. \tag{2.1}$$

Proposition 2.4. Let $x = a_0 + \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \frac{\epsilon_3}{a_3+} \cdots$ be a finite or infinite semi-regular continued fraction. Then the following hold.

- (1) The sequence $\{q_n\}_{n \geq 1}$ is monotonically increasing if and only if $\epsilon_n + a_n \geq 1$, $n \geq 2$.
- (2) Suppose $y_n = \frac{\epsilon_n}{a_n+} \frac{\epsilon_{n+1}}{a_{n+1}+} \cdots \frac{\epsilon_{n+k}}{a_{n+k}+} \cdots$. Then $\epsilon_n y_n \in [\frac{1}{a_n}, 1]$, $n \geq 0$ and $|q_n x - p_n| \leq |q_{n-1} x - p_{n-1}|$.

For an $\mathcal{F}_{1,2}$ -continued fraction $\frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_n}{a_n+} \cdots$, the expression

$$\frac{p_k}{q_k} = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_k}{a_k}$$

for $k \geq 0$ is called the k -th $\mathcal{F}_{1,2}$ -convergent and the sequence $\{\frac{p_k}{q_k}\}_{k \geq 0}$ is called the sequence of $\mathcal{F}_{1,2}$ -convergents. The expression $\frac{\epsilon_i}{a_i+} \frac{\epsilon_{i+1}}{a_{i+1}+} \cdots$ is referred to as the fin at the i -th stage. Observe, if y_i is the fin at the i -th stage, then $\epsilon_i = \text{sign}(y_i)$.

Theorem 2.5. Suppose $x = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \frac{\epsilon_3}{a_3+} \cdots$ is an $\mathcal{F}_{1,2}$ -continued fraction and $\{\frac{p_i}{q_i}\}_{i=0}^\infty$ is the sequence of $\mathcal{F}_{1,2}$ -convergents of x . Suppose $(p_{-1}, q_{-1}) = (1, 0)$ and

$(p_0, q_0) = (b, 2)$. Let y_i be the fin at the i -th stage of an $\mathcal{F}_{1,2}$ -continued fraction of x . Then

- (1) for $i \geq 0$, $p_{i+1} = a_{i+1}p_i + \epsilon_{i+1}p_{i-1}$ and $q_{i+1} = a_{i+1}q_i + \epsilon_{i+1}q_{i-1}$;
- (2) the sequence $\{q_i\}$ is strictly increasing;
- (3) $\frac{p_i}{q_i} \neq \frac{p_j}{q_j}$ for $i \neq j$;
- (4) for $i \geq 1$, $|y_i| \leq 1$;
- (5) $x = \frac{x_{n+1}p_n + \epsilon_{n+1}p_{n-1}}{x_{n+1}q_n + \epsilon_{n+1}q_{n-1}}$ where $x_i = \frac{1}{|y_i|}$.

Proof. By assumption, $x = \frac{1}{0^+} \frac{2}{b^+} \frac{\epsilon_1}{a_1^+} \dots \frac{\epsilon_n}{a_n} \dots$. Hence, $2x = b + \frac{\epsilon_1}{a_1^+} \dots \frac{\epsilon_n}{a_n} \dots$ is a semi-regular continued fraction. Suppose s_i/t_i is the i -th convergent of $2x$. Then

$$p_i = s_i$$

$$q_i = 2t_i$$

Statement (1) follows from Proposition 2.2. Since $a_i \geq 2$, we have $\epsilon_i + a_i \geq 1$ so that by statement (1), $\{q_i\}$ is strictly increasing. Statement (3) follows from statement (2) and the fact that $(p_i, q_i) = 1$. Statement (4) holds from Proposition 2.4(2). By Proposition 2.3

$$2x = \frac{s_n + y_{n+1}s_{n-1}}{t_n + y_{n+1}t_{n-1}}$$

so that (since $x_{n+1} = \frac{1}{|y_{n+1}|}$)

$$x = \frac{x_{n+1}p_n + \epsilon_{n+1}p_{n-1}}{x_{n+1}q_n + \epsilon_{n+1}q_{n-1}}$$

which is statement (5) of the theorem. \square

We now recall a few results from [3] about the graph theoretic properties of $\mathcal{F}_{1,2}$.

Proposition 2.6. (See [3, Theorem 5.1].) Let $x, y, r, s \in \mathbb{Z}$ with $(x, y) = 1 = (r, s)$, $N|y$ and $N|s$ so that $\frac{x}{y}$ and $\frac{r}{s}$ are vertices of $\mathcal{F}_{u,N}$. Then $\frac{r}{s} \sim \frac{x}{y}$ in $\mathcal{F}_{u,N}$ if and only if either

- (1) $x \equiv ur \pmod N$ and $ry - sx = N$, or
- (2) $x \equiv -ur \pmod N$ and $ry - sx = -N$.

We have noted that $\mathcal{G}_{1,2}$ is self-paired. If a and b are adjacent vertices, we treat the pair of edges between a and b as one edge only.

Proposition 2.7. (See [3, Corollary 5.13].) $\mathcal{G}_{1,2}$ is a forest. In particular, $\mathcal{F}_{1,2}$ is a tree.

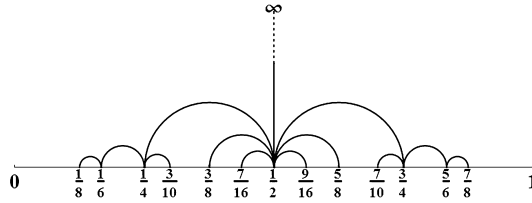


Fig. 1. A few vertices and edges of $\mathcal{F}_{1,2}$ on the interval $(0, 1)$.

Recall that the vertex set of $\mathcal{F}_{1,2}$ is

$$\mathcal{X} = \left\{ \frac{p}{2q} : p, q \in \mathbb{Z}, q > 0, (p, 2q) = 1 \right\} \cup \{\infty\}. \tag{2.2}$$

Since $\mathcal{F}_{1,2}$ is a tree and $\infty \in \mathcal{F}_{1,2}$, we have the following statement.

Corollary 2.8. *There is a unique path (of edges in $\mathcal{F}_{1,2}$) from ∞ to every point in \mathcal{X} .*

Let $x \in \mathcal{X}$. Suppose the path from ∞ to x is given by

$$\infty \sim P_0 \sim P_1 \sim \dots \sim P_k \sim \dots \sim P_n = x. \tag{2.3}$$

For convenience, we replace \sim by an arrow pointing towards x at every stage in (2.3) so that it is expressed as

$$\infty \rightarrow P_0 \rightarrow P_1 \rightarrow \dots \rightarrow P_k \rightarrow \dots \rightarrow P_n = x.$$

By the *distance* between $x, y \in \mathcal{X}$, we mean the number of edges required to join x and y in $\mathcal{F}_{1,2}$. We represent the edges of $\mathcal{F}_{1,2}$ as hyperbolic geodesics in the upper half plane

$$\mathcal{U} = \{z \in \mathbb{C} : \text{Im}(z) > 0\},$$

that is, as Euclidean semicircles or half lines perpendicular to the real line (see [3]). See Fig. 1 where a few edges are displayed for vertices lying in $(0, 1)$.

Lemma 2.9. (See [3, Corollary 4.2].) *No two edges of $\mathcal{F}_{1,2}$ cross in \mathcal{U} .*

Corollary 2.10. *The set of vertices of a connected component of the subgraph of $\mathcal{F}_{1,2}$ obtained by deleting ∞ is given by $(k, k + 1) \cap \mathcal{X}$ for some $k \in \mathbb{Z}$.*

Proof. Let $b = 2k + 1$ so that $b/2$ is the midpoint of the interval $(k, k + 1)$. Set for each $n \in \mathbb{N}$, $P_n(+) = \frac{b}{2} + \frac{n}{2(n+1)}$ and $P_n(-) = \frac{b}{2} - \frac{n}{2(n+1)}$. Then observe that $\frac{b}{2}$ is connected to $P_n(\pm)$ for every $n \in \mathbb{N}$ and the maximum Euclidean distance can be travelled in n steps from x in $\mathcal{X} \setminus \{\infty\}$ by following $\frac{b}{2} \rightarrow P_1(+) \rightarrow \dots \rightarrow P_n(+)$ or

$\frac{b}{2} \rightarrow P_1(-) \rightarrow \dots \rightarrow P_n(-)$. So the (Euclidean) diameter of the connected component is $\lim_{n \rightarrow \infty} 2 \cdot \frac{n}{2(n+1)} = 1$. Thus the connected component containing $b/2$ is contained in $(k, k + 1) \cap \mathcal{X}$.

On the other hand, if $x \in (k, k + 1) \cap \mathcal{X}$, there exists $n \in \mathbb{N}$ such that $P_n(-) < x < P_n(+)$. By Lemma 2.9, x must be connected to ∞ via $b/2$. Thus, the result follows. \square

From the proof of Corollary 2.10, we also obtain the following result.

Corollary 2.11. *If $x \in \mathcal{X} \setminus \{\infty\}$, then x is connected to ∞ through $\frac{2|x|+1}{2}$.*

3. Finite $\mathcal{F}_{1,2}$ -continued fractions

Theorem 3.1.

- (A) *The path in $\mathcal{F}_{1,2}$ from ∞ to $x \in \mathcal{X}$ defines a finite $\mathcal{F}_{1,2}$ -continued fraction of x .*
- (B) *The value of every finite $\mathcal{F}_{1,2}$ -continued fraction belongs to \mathcal{X} and the continued fraction defines a path in $\mathcal{F}_{1,2}$ from ∞ to its value with the convergents as the vertices.*

Proof. To prove statement (A), let $x \in \mathcal{X}$. Then by Corollary 2.8, there is a path from ∞ to x in $\mathcal{F}_{1,2}$. Suppose

$$\infty \rightarrow P_0 \rightarrow P_1 \rightarrow \dots \rightarrow P_k \rightarrow \dots \rightarrow P_n = x,$$

where $P_k = \frac{p_k}{2q_k}$ for some $p_k \in \mathbb{Z}$, $q_k \in \mathbb{N}$ with $(p_k, 2q_k) = 1$. By Corollary 2.11, $P_0 = \frac{b}{2}$ where $b = 2\lfloor x \rfloor + 1$. We complete the proof by induction on the distance of x from ∞ .

By induction hypothesis, any vertex $P_i = \frac{p_i}{2q_i}$ on the path having distance $i + 1$ ($1 \leq i \leq k$) from ∞ is represented by the continued fraction

$$\frac{p_i}{2q_i} = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_i}{a_i}.$$

Since $P_k \rightarrow P_{k+1}$ and $P_{k-1} \rightarrow P_k$, by Proposition 2.6, we have $2p_{k+1}q_k - 2q_{k+1}p_k = 2e_k$ and $2p_kq_{k-1} - 2q_kp_{k-1} = 2e_{k-1}$ where $e_i = \pm 1$ for each i . Hence,

$$p_{k+1}q_k - q_{k+1}p_k = e_k \tag{3.1}$$

$$p_kq_{k-1} - q_kp_{k-1} = e_{k-1} \tag{3.2}$$

so that $p_{k+1}q_k \equiv -e_{k-1}e_kp_{k-1}q_k \pmod{p_k}$. Since p_k and q_k are coprime, we have $p_{k+1} \equiv -e_{k-1}e_kp_{k-1} \pmod{p_k}$ and hence

$$p_{k+1} = a_{k+1}p_k - e_{k-1}e_kp_{k-1}$$

for some $a_{k+1} \in \mathbb{Z}$. Substituting this in (3.1) and using (3.2), we get

$$\begin{aligned}
 & (a_{k+1}p_kq_k + e_{k-1}e_k(e_{k-1} - p_kq_{k-1}) - q_{k+1}p_k = e_k \\
 \Rightarrow & (a_{k+1}q_k - e_{k-1}e_kq_{k-1})p_k - q_{k+1}p_k = 0 \\
 \Rightarrow & q_{k+1} = a_{k+1}q_k - e_{k-1}e_kq_{k-1} \quad (\text{since } p_k \neq 0).
 \end{aligned}$$

Now, by setting $\epsilon_{k+1} = -e_{k-1}e_k$, we have

$$\begin{aligned}
 p_{k+1} &= a_{k+1}p_k + \epsilon_{k+1}p_{k-1} \\
 q_{k+1} &= a_{k+1}q_k + \epsilon_{k+1}q_{k-1}
 \end{aligned}$$

Since p_{k+1} and $2q_{k+1}$ satisfy the same recurrence relation with the initial condition $(p_{-1}, q_{-1}) = (1, 0)$ and $(p_0, 2q_0) = (b, 2)$, we have

$$P_{k+1} = \frac{p_{k+1}}{2q_{k+1}} = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_k}{a_k+} \frac{\epsilon_{k+1}}{a_{k+1}}.$$

Since p_{k-1} , p_k and p_{k+1} are odd integers and $p_{k+1} = a_{k+1}p_k + \epsilon_{k+1}p_{k-1}$, we have a_{k+1} is an even integer so that the path from ∞ to x defines a finite $\mathcal{F}_{1,2}$ -continued fraction of x given by

$$x = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_n}{a_n}.$$

To prove statement (B), suppose

$$\frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_n}{a_n}$$

is an $\mathcal{F}_{1,2}$ -continued fraction. Let $P_0 = \frac{b}{2}$ and let $P_i = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_i}{a_i}$, for each $1 \leq i \leq n$. By using induction on i , we will show that $P_i \in \mathcal{X}$ for each $0 \leq i \leq n$. It is easy to see that $P_0, P_1 \in \mathcal{X}$. Since P_i is a rational number, it is s_i/t_i for $s_i, t_i \in \mathbb{Z}$ with $(s_i, t_i) = 1$ and $t_i > 0$.

Then $s_k = a_k s_{k-1} + \epsilon_k s_{k-2}$ and $t_k = a_k t_{k-1} + \epsilon_k t_{k-2}$ for $k \geq 2$. By induction hypothesis, $P_0, P_1, \dots, P_{k-1} \in \mathcal{X}$ so that s_i is odd and t_i is even with $t_{i-1} < t_i$ for each $0 \leq i \leq k-1$. Therefore using the above relations, s_k is odd and t_k is even so that $x = P_n \in \mathcal{X}$ and $t_{k-1} < t_k$ (since $a_k \geq 2 > |\epsilon_k|$).

Now, by induction hypothesis, $P_{i-1} \rightarrow P_i$, for $1 \leq i \leq k-1$. Therefore,

$$\begin{aligned}
 s_k t_{k-1} - t_k s_{k-1} &= (a_k s_{k-1} + \epsilon_k s_{k-2})t_{k-1} - (a_k t_{k-1} + \epsilon_k t_{k-2})s_{k-1} \\
 &= \epsilon_k (s_{k-1}t_{k-2} - t_{k-1}s_{k-2}) \\
 &= \pm 2
 \end{aligned}$$

so that $P_{k-1} \rightarrow P_k$. So the given $\mathcal{F}_{1,2}$ -continued fraction defines the following path from ∞ to the value (namely, x) of the given continued fraction

$$\infty \rightarrow P_0 \rightarrow P_1 \rightarrow \dots \rightarrow P_k \rightarrow \dots \rightarrow P_n = x. \quad \square$$

Corollary 3.2. *The directions assigned to the edges of a path in $\mathcal{F}_{1,2}$ from ∞ to x satisfy the following property for $a/b, c/d \in \mathcal{X}$:*

$$\frac{a}{b} \rightarrow \frac{c}{d} \text{ if and only if } \frac{a}{b} \sim \frac{c}{d} \text{ and } b < d.$$

Proof. By Theorem 3.1, a/b and c/d are consecutive convergents of x and if a/b is k -th convergent then c/d is $(k + 1)$ -th convergent. Therefore, the corollary follows from Theorem 2.5(2). \square

To interpret the parameters of an $\mathcal{F}_{1,2}$ -continued fraction geometrically, we introduce the following definitions. Let $Q \in \mathcal{X}$ be such that it is at least two units away from ∞ (that is, at least two edges are required to join Q to ∞). Suppose the unique path from ∞ to Q is given by

$$\infty \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_n = Q$$

where $n \geq 1$. Then Q is called *direction retaining* if either $P_{n-2} < P_{n-1} < Q$ or $P_{n-2} > P_{n-1} > Q$ where $P_{-1} = \infty$. If Q is not direction retaining, we call it *direction changing*.

The edges (which are semicircles) emanating from a vertex Q in a given direction (left or right) are ordered in the following way. Suppose the path from ∞ to Q is as in the previous paragraph. Suppose that in the given direction, the farthest (in Euclidean sense) vertex adjacent to Q but different from P_{n-1} is Q_1 . Then the edge joining Q to Q_1 is called the *first semicircle emanating from Q* in the given direction. The edge joining Q to the farthest vertex in the same direction different from P_{n-1} and Q_1 is called the *second semicircle emanating from Q* in that direction. For any $k \geq 1$, we define inductively the *k -th semicircle emanating from Q* in the given direction.

Proposition 3.3. *Let $x \in \mathcal{X} \setminus \{\infty\}$ be such that it is not a half integer. Suppose, for $n \geq 1$,*

$$\infty \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_n = x$$

and

$$x = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_n}{a_n}.$$

Then

- (1) $\epsilon_1 = -1$ if and only if $x < \frac{b}{2}$ (and hence, $P_1 < P_0 = \frac{b}{2}$);
- (2) for $i \geq 1$, $\epsilon_i = -1$ if and only if P_i is direction retaining;
- (3) for $i \geq 1$, P_i is on the k -th semicircle in the direction of P_i emanating from P_{i-1} if and only if $a_i = 2k$.

Proof. Suppose $P_i = \frac{p_i}{q_i}$. Then $\frac{p_i}{q_i} = \frac{b}{2} + \frac{\epsilon_1}{2a_1}$ implies $P_1 = P_0 + \frac{\epsilon_1}{2a_1}$. The statement (1) follows. Observe that

$$\frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} = -\epsilon_{i+1} \frac{q_{i-1}}{q_{i+1}} \left(\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} \right) \tag{3.3}$$

Statement (2) now follows from (3.3). Next, suppose $P'_i = \frac{p'_i}{q'_i}$ is another vertex such that $P_{i-1} \rightarrow P'_i$ with P_i and P'_i lying in the same side of P_{i-1} . Suppose the i -th partial quotient of P'_i is ϵ_i/a'_i . Then

$$\begin{aligned} P_i - P'_i &= \frac{\epsilon_i q_{i-1}}{q'_i} \left(\frac{p_{i-1}}{q_{i-1}} - \frac{p_{i-2}}{q_{i-2}} \right) (a_i - a'_i) \\ &= \frac{q_{i-1}}{q'_i} (P_i - P_{i-1})(a'_i - a_i) \quad (\text{using (3.3)}) \end{aligned} \tag{3.4}$$

Statement (3) follows from (3.4) by considering all possibilities for a'_i . \square

The following corollary to Proposition 3.3 tells that the “farthest” points (real numbers) connected to ∞ through a given point in \mathcal{X} are in fact rational numbers. As mentioned in the introduction, these results are related to those of [2, Section 4].

Corollary 3.4. *Let $\{\alpha_i\}_{i \geq 1}$ be a sequence of elements in $\mathcal{X} \setminus \{\infty\}$ such that α_{i+1} lies on the first semicircle emanating from α_i for $i \geq 1$ and α_i is direction retaining for $i \geq 2$. Then $\{\alpha_i\}$ converges to a rational number.*

Proof. Note that the given conditions ensure that the path

$$\alpha_1 \rightarrow \alpha_2 \rightarrow \dots \rightarrow \alpha_i \rightarrow \dots$$

is the farthest possible path from α_1 in the direction given by α_2 . Let

$$\alpha_1 = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_n}{a_n+}.$$

Then Proposition 3.3 ensures that the limit of $\{\alpha_i\}$ is given by

$$\frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_n}{a_n+} \frac{\pm 1}{2+} \frac{-1}{2+} \frac{-1}{2+} \frac{-1}{2+} \dots$$

with ± 1 coming from the choice of α_2 . Therefore the corollary holds as

$$1 = \frac{1}{2+} \frac{-1}{2+} \frac{-1}{2+} \dots. \quad \square$$

4. An algorithm to find the $\mathcal{F}_{1,2}$ -continued fraction

In the previous section we saw that any $x \in \mathcal{X}$ has a unique (finite) $\mathcal{F}_{1,2}$ -continued fraction. Here we derive an algorithm to find this $\mathcal{F}_{1,2}$ -continued fraction for a given x in \mathcal{X} .

Theorem 4.1. *Given any $x \in \mathcal{X}$, the $\mathcal{F}_{1,2}$ -continued fraction expansion*

$$x = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_n}{a_n},$$

is obtained as follows: $b = 2[x] + 1$ and for $(1 \leq i \leq n)$, setting $y_1 = 2x - b$,

- (1) $a_i = 2 \lfloor \frac{1}{2} \left(1 + \frac{1}{|y_i|} \right) \rfloor$,
- (2) $\epsilon_i = \text{sign}(y_i)$,
- (3) $y_{i+1} = \frac{1}{|y_i|} - a_i$.

In fact, n is the smallest non-negative integer for which $y_{n+1} = 0$.

Proof. Let y_i be the fin at the i -th stage of the $\mathcal{F}_{1,2}$ -expansion of x , namely

$$y_i = \frac{\epsilon_i}{a_i+} \frac{\epsilon_{i+1}}{a_{i+1}+} \dots \frac{\epsilon_n}{a_n}.$$

Then $b = 2[x] + 1$ (by Corollary 2.11) and $y_1 = 2x - b$. For $1 \leq i \leq n$, we also have

$$y_i = \frac{\epsilon_i}{a_i + y_{i+1}}. \tag{4.1}$$

By Proposition 2.5(4), $|y_{i+1}| \leq 1$ and a_i is a positive even integer. This means that the denominator in (4.1) is positive and hence $\epsilon_i = \text{sign}(y_i)$ and a_i is the nearest even integer to $1/|y_i|$. This gives us steps (1), (2) and (3) of the algorithm.

The last claim is also clear from the fact that the denominator in (4.1) is non-zero for $1 \leq i \leq n$ and $y_n = \frac{\epsilon_n}{a_n}$ by definition, giving $y_{n+1} = 0$. \square

Remark 4.2. The fundamental step in the above algorithm is the process of writing $x = |y_i| \in [0, 1]$ in the form

$$x = \frac{1}{2k + \epsilon T(x)}, \tag{4.2}$$

where $2k = a_i$ is a positive even integer, $\epsilon = \pm 1$ and $T(x) = |y_{i+1}| \in [0, 1]$.

For a precise definition of the map $T : [0, 1] \rightarrow [0, 1]$, we subdivide $[0, 1]$ into disjoint intervals of the form $B(+1, k) = (\frac{1}{2k}, \frac{1}{2k-1}]$ and $B(-1, k) = (\frac{1}{2k+1}, \frac{1}{2k}]$ for all integers $k \geq 1$. Now, we define the map

$$T(x) = \epsilon \left(\frac{1}{x} - 2k \right), \quad \text{where } \epsilon = \pm 1, \quad x \in B(\epsilon, k). \tag{4.3}$$

Essentially, $2k$ is the nearest even integer to $\frac{1}{x}$ and $T(x) = |\frac{1}{x} - 2k|$.

This map T has been studied classically and the iteration of (4.2) results in the E.C.F. expansion of y_1 (Schweiger [7]).

5. $\mathcal{F}_{1,2}$ -continued fractions of real numbers

Theorem 5.1. *Every real number has an $\mathcal{F}_{1,2}$ -continued fraction.*

Proof. In the concluding remarks of the previous section, we had seen that the $\mathcal{F}_{1,2}$ -expansion of $x \in \mathcal{X}$ is obtained by repeated iteration of the map T starting from $y_1 = 2x - b$ with $b = 2\lfloor x \rfloor + 1$. We claim that this repeated iteration (infinitely many times, if necessary) produces an $\mathcal{F}_{1,2}$ -expansion for *any* real number x .

In other words, while it is clear that the n -th iteration of T on $y_1 = 2x - b$ yields the relation (for $n \geq 1$)

$$x = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_n}{a_n + y_{n+1}}, \tag{5.1}$$

we need to show that the (infinite) continued fraction

$$\frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_n}{a_n+} \cdots$$

converges to x .

To prove this, we note that (5.1) implies that (2.1) holds (see Theorem 2.5) and hence we have

$$\left| x - \frac{p_n}{q_n} \right| = \frac{2|y_{n+1}|}{q_n(q_n + y_{n+1}q_{n-1})}.$$

The right side above converges to 0 because q_n are monotonically increasing integers (Theorem 2.5) and $|y_n| \leq 1$ as $|y_n| = T^{(n-1)}(|y_1|)$. This completes the proof of the theorem. \square

Next, we explore whether every real number has a unique $\mathcal{F}_{1,2}$ -continued fraction. In fact, if x is a rational number not in \mathcal{X} , there are exactly two $\mathcal{F}_{1,2}$ -continued fractions of x and we discuss this in Section 6. On the other hand, an irrational number has a unique $\mathcal{F}_{1,2}$ -continued fraction expansion and this is our next result.

Proposition 5.2. *Every irrational number has a unique infinite $\mathcal{F}_{1,2}$ -continued fraction.*

Proof. Suppose y_1 is the fin at the first stage of an $\mathcal{F}_{1,2}$ -continued fraction of x so that

$$x = \frac{1}{0+} \frac{2}{b + y_1}.$$

Since x is not an integer, $|y_1| < 1$. Hence $2x - 1 < b < 2x + 1$ so that b has only one possible value.

Let there be two distinct $\mathcal{F}_{1,2}$ -continued fractions of a given irrational x . Suppose $\{P_k\}_{k \geq 0}$ and $\{P'_k\}_{k \geq 0}$ are the sequences of $\mathcal{F}_{1,2}$ -convergents corresponding to these $\mathcal{F}_{1,2}$ -continued fractions. We have shown in the above discussion that $P_0 = P'_0$. Suppose N is such that $P_i = P'_i$ for every $i \leq N$ but $P_{N+1} \neq P'_{N+1}$.

Without loss of generality, let $P_{N+1} < P'_{N+1}$. Since no edges cross each other in the graph $\mathcal{F}_{1,2}$, we must have $P_{N+1} < x < P'_{N+1}$. Suppose for $i \in \mathbb{N}$, $\alpha_i, \beta_i \in \mathcal{X}$ are such that $\alpha_1 = P_{N+1}$ and $\beta_1 = P'_{N+1}$ with $\alpha_{i+1} > \alpha_i$ and $\beta_{i+1} < \beta_i$ so that α_{i+1} and β_{i+1} are the farthest (in Euclidean sense) points from α_i and β_i respectively, towards x connected by an edge. Suppose α and β are the limits of $\{\alpha_i\}$ and $\{\beta_i\}$ respectively, then $\alpha \leq x \leq \beta$. By Corollary 3.4, α and β are both rational so that $\alpha < x < \beta$. Observe, $P_{N+i} \leq \alpha_i < \alpha < x$ and $P'_{N+i} \geq \beta_i > \beta > x$. Hence, $\{P_i\}_{i \geq 0}$ and $\{P'_i\}_{i \geq 0}$ do not converge to x and this contradicts the fact that they are sequences of $\mathcal{F}_{1,2}$ -convergents of x . \square

6. Infinite $\mathcal{F}_{1,2}$ -continued fractions of rational numbers

We have already seen that every real $x \notin \mathbb{Q} \setminus \mathcal{X}$ has a unique $\mathcal{F}_{1,2}$ -expansion and hence unique fins at every stage. In this section, on the other hand, we will show that if $x \in \mathbb{Q} \setminus \mathcal{X}$, it has exactly two $\mathcal{F}_{1,2}$ -expansions both of which are eventually constant. The fin at the i -th stage is unique for all but finitely many values of i and when x has more than one fin at the i -th stage, it has exactly two fins at that stage.

Proposition 6.1. *Suppose $x \in \mathbb{R}$ has an eventually constant $\mathcal{F}_{1,2}$ -continued fraction. Then $x \in \mathbb{Q}$ if and only if all but finitely many partial numerators are -1 and all but finitely many partial denominators are 2 .*

Proof. Suppose $x = \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots \frac{\epsilon_m}{a_m+} \dots$ is such that $a_i = a_{i+1}$ and $\epsilon_i = \epsilon_{i+1}$ for each $i \geq m$. Then for some $y \in \mathbb{R}$ (a fin), we have $\frac{\epsilon_m}{a_m+y} = y$ so that $y^2 + a_m y - \epsilon_m = 0$. Thus, y is rational if and only if $a_m = 2$ and $\epsilon_m = -1$. Since x is rational if and only if y is rational, the result follows. \square

Proposition 6.2. *Suppose $x \in \mathbb{Q} \setminus \mathcal{X}$. Then*

- (1) *A fin of x at any stage is a quotient of two odd integers.*
- (2) *Fins of x are ± 1 for all but finitely many. For a fixed $\mathcal{F}_{1,2}$ -continued fraction of x with fins y_i , we define $N = \max(\{0\} \cup \{i \geq 1 | y_i \neq \pm 1\})$. If $N \geq 1$, then y_N is the reciprocal of an odd integer.*

- (3) The i -th fin y_i of x is unique (independent of the $\mathcal{F}_{1,2}$ -continued fraction) for $i \geq 1$ except for y_{N+1} (N as in statement (2)), which takes two possible values. Here N also depends only on x and not on the $\mathcal{F}_{1,2}$ -continued fraction chosen.
- (4) There are at least two $\mathcal{F}_{1,2}$ -continued fraction expansions of x . In fact, they are both infinite and eventually constant.

Proof. Since $y_1 = 2x - b$ where b is odd, y_1 is the quotient of odd integers. Statement (1) follows by induction since we have $y_{i+1} = \frac{1}{|y_i|} - a_i$ (by the definition of fin). Suppose $y_i = \frac{r_i}{s_i} \neq \pm 1$. Again using the relation between successive fins, we get $|s_i| > |r_i| = |s_{i+1}|$ so that the absolute values of both numerator and denominator are decreasing. Hence, for some $i \geq 1$, $y_i = \pm 1$. Let N be as in statement (2), then $y_{N+1} = \pm 1$ which implies $y_{N+k} = -1$ for $k \geq 2$. In particular, if $N \geq 1$, y_N is a reciprocal of an odd integer.

To establish statement (3), we proceed by induction. If $N = 0$ then $y_1 = \pm 1$ and we have nothing to prove. If $N \geq 1$ we have $|y_1| = |2x - b| < 1$ and so there is a unique choice of odd integer b at this stage. Hence y_1 is unique. In fact, for every stage $i < N$, there is a unique choice of even integer a_i such that $|y_{i+1}| = |\frac{1}{|y_i|} - a_i| < 1$, which makes y_{i+1} unique. Since $y_{N+1} = \pm 1$ and the succeeding fins are -1 , we get N to be independent of the particular $\mathcal{F}_{1,2}$ -continued fraction.

Next we observe that if $\frac{1}{y_N} = d$ is an odd positive integer, there are two nearest even integers so that $d = 2\lfloor \frac{d+1}{2} \rfloor - 1 = 2\lfloor \frac{d-1}{2} \rfloor + 1$ and $1 = \frac{1}{2+} \frac{-1}{2+} \frac{-1}{2+} \dots, -1 = \frac{-1}{2+} \frac{-1}{2+} \frac{-1}{2+} \dots$. Thus, statement (4) follows. \square

Corollary 6.3. A fin of $x \in \mathbb{R}$ is ± 1 if and only if $x \in \mathbb{Q} \setminus \mathcal{X}$.

Proof. Suppose a fin of x is ± 1 . Then clearly $x \in \mathbb{Q}$. Since $1 = \frac{1}{2+} \frac{-1}{2+} \frac{-1}{2+} \dots$ and $-1 = \frac{-1}{2+} \frac{-1}{2+} \frac{-1}{2+} \dots$, x has an infinite $\mathcal{F}_{1,2}$ -continued fraction, so that $x \in \mathbb{Q} \setminus \mathcal{X}$. The converse is statement (2) of Proposition 6.2. \square

Lemma 6.4. Let for each $i \in \mathbb{N}$, $\epsilon_i = \pm 1$ and $a_i \in 2\mathbb{Z}$. Then the following hold.

- (1) If $1 = \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots$, then $\epsilon_1 = 1, \epsilon_{i+1} = -1$ and $a_i = 2$ for every $i \geq 1$.
- (2) If $-1 = \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots$, then $\epsilon_i = -1$ and $a_i = 2$ for every $i \geq 1$.

Proof. Suppose $-1 = \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \dots$ where ϵ_i and a_i are as stated in the lemma. Then,

$$-1 = \frac{1}{0+} \frac{2}{-1+} \frac{\epsilon_1}{a_1+} \dots \tag{6.1}$$

defines an $\mathcal{F}_{1,2}$ -continued fraction. By Theorem 3.1, there is a unique path from ∞ to each convergent P_n via the convergents P_i with $0 \leq i < n$. Observe $P_0 = -\frac{1}{2}$. Note that $a_i = 2$ and $\epsilon_i = -1$ for each $i \in \mathbb{N}$ is a solution of Eq. (6.1). In this case, P_{i+1} lies in the first circle emanating from P_i (so that it is the farthest vertex towards left of P_i) and the path is direction retaining. Since no two edges intersect (Lemma 2.9), it is not possible to

have one more sequence of $\mathcal{F}_{1,2}$ -convergents of -1 via $-1/2$. This proves statement (2). Statement (1) can be proved using a similar argument. \square

Theorem 6.5. *For every $x \in \mathbb{Q} \setminus \mathcal{X}$, there are exactly two $\mathcal{F}_{1,2}$ -continued fraction expansions of x . In fact, there exist a unique pair of positive odd integers b, b' , a unique non-negative integer n , a unique pair of finite sequences $\{a_1, a_2, \dots, a_n\}$ and $\{a'_1, a'_2, \dots, a'_n\}$ of positive even integers and a unique sequence $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ of ± 1 such that*

$$\begin{aligned} x &= \frac{1}{0+} \frac{2}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_n}{a_n+} \frac{-1}{2+} \frac{-1}{2+} \frac{-1}{2+} \cdots \\ &= \frac{1}{0+} \frac{2}{b'+} \frac{\epsilon_1}{a'_1+} \frac{\epsilon_2}{a'_2+} \cdots \frac{\epsilon_n}{a'_n+} \frac{1}{2+} \frac{-1}{2+} \frac{-1}{2+} \cdots \end{aligned}$$

where $n = 0$ if and only if $x \in \mathbb{Z}$, and in this case, $b = b' + 2$; and when $n \geq 1$, $b = b'$, $a_i = a'_i$ ($1 \leq i \leq n - 1$) and $a_n = a'_n + 2$.

Proof. In view of Proposition 6.2(4), it is enough to show that there are at most two $\mathcal{F}_{1,2}$ -continued fractions for every element in $\mathbb{Q} \setminus \mathcal{X}$.

Let $x \in \mathbb{Q} \setminus \mathcal{X}$ and let x be an integer. Suppose y_1 is the fin at the first stage of an $\mathcal{F}_{1,2}$ -continued fraction of x so that

$$x = \frac{1}{0+} \frac{2}{b + y_1}. \tag{6.2}$$

Since $|y_1| \leq 1$ and $y_1 = 2x - b$ (which is an integer), we have $y_1 = \pm 1$ so that $b = 2x \pm 1$. Therefore, there are exactly two choices of b and their difference is 2. We conclude by using Lemma 6.4 that x has exactly two $\mathcal{F}_{1,2}$ -continued fractions with $n = 0$.

Next, let $x \in \mathbb{Q} \setminus \mathcal{X}$ but $x \notin \mathbb{Z}$. Again, suppose b and y_1 are as in (6.2). Since $2x - b$ is not an integer, $y_1 \neq \pm 1$. Thus $|y_1| < 1$ and b has only one choice, namely, the odd integer nearest to x . For every $\mathcal{F}_{1,2}$ -continued fraction of x , we have a solution of the following equation:

$$y_1 = \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_n}{a_n + y_{n+1}} \tag{6.3}$$

where $n \geq 1$, a_1, a_2, \dots, a_n are positive even integers, $\epsilon_i = \pm 1$ ($1 \leq i \leq n$) and $0 < |y_{n+1}| \leq 1$. Suppose in Eq. (6.3), $n \geq 1$ is such that there are unique solutions for a_1, a_2, \dots, a_{n-1} and $\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}$ but $(a_n, \epsilon_n, y_{n+1})$ has more than one choices. Our assumption implies that y_n is unique, hence so is ϵ_n . Let α_n, α'_n be two possible values of a_n and let P_n and P'_n be the corresponding convergents. Since no two edges of $\mathcal{F}_{1,2}$ cross (Lemma 2.9), x lies between P_n and P'_n . Again by Lemma 2.9, P_n and P'_n lie in consecutive circles emanating from P_{n-1} in the same direction. Thus, difference between α_n and α'_n is exactly 2. In particular, a_n has at most two possible values satisfying Eq. (6.3). Assume that there are two distinct possible values of a_n and denote

by β_{n+1} (and β'_{n+1}) the value of y_{n+1} corresponding to α_n (respectively, α'_n). Using Proposition 4.1(3), $|\beta_{n+1} - \beta'_{n+1}| = 2$ so that $\beta_{n+1} = \pm 1$, $\beta'_{n+1} = \mp 1$ respectively. By Lemma 6.4, the continued fractions of x will be obtained by using $1 = \frac{1}{2+} \frac{-1}{2+} \frac{-1}{2+} \cdots$ and $-1 = \frac{-1}{2+} \frac{-1}{2+} \frac{-1}{2+} \cdots$. Thus, we have shown that there are exactly two $\mathcal{F}_{1,2}$ -continued fractions for every element in $\mathbb{Q} \setminus \mathcal{X}$. \square

Example 1. For $x = 2/7$, we have, $b = b' = 1$, $n = 2$, $a_1 = a'_1 = 2$, $a_2 = 4$, $a'_2 = 2$, $\epsilon_1 = -1$ and $\epsilon_2 = 1$. The corresponding fins are $\{-\frac{3}{7}, \frac{1}{3}, -1, -1, -1 \dots\}$ and $\{-\frac{3}{7}, \frac{1}{3}, 1, -1, -1 \dots\}$ respectively. The corresponding convergents are

$$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{5}{18}, \frac{9}{32}, \frac{13}{46} \cdots \right\} \text{ and } \left\{ \frac{1}{2}, \frac{1}{4}, \frac{3}{10}, \frac{7}{24}, \frac{11}{38}, \frac{15}{52} \cdots \right\}$$

respectively.

Corollary 6.6. *There are only finitely many common $\mathcal{F}_{1,2}$ -convergents of the two $\mathcal{F}_{1,2}$ -continued fraction expansions of x .*

The next result characterizes the number N defined in Proposition 6.2 by the two sequences of $\mathcal{F}_{1,2}$ -convergents of $x \in \mathbb{Q} \setminus \mathcal{X}$.

Corollary 6.7. *Suppose $x \in \mathbb{Q} \setminus \mathcal{X}$ and $\frac{p_k}{q_k} \in \mathcal{X}$ is the k -th convergent of one of the $\mathcal{F}_{1,2}$ -continued fractions of x . Then $\frac{p_k}{q_k}$ appears in both sequences of $\mathcal{F}_{1,2}$ -convergents of x if and only if $k \leq N - 1$, where N is as defined in Proposition 6.2.*

We call the $\mathcal{F}_{1,2}$ -continued fraction of $x \in \mathbb{Q} \setminus \mathcal{X}$ obtained by using the smallest even integer greater than $1/|y_N|$ if $N \geq 1$ (see Theorem 6.2(2) for y_N), the *first $\mathcal{F}_{1,2}$ -continued fraction* of x . If $N = 0$, x is an integer and there are two choices for b (namely, $2x \pm 1$). In this case, the $\mathcal{F}_{1,2}$ -continued fraction obtained by setting $b = 2x + 1$, will be called the *first $\mathcal{F}_{1,2}$ -continued fraction*. The other $\mathcal{F}_{1,2}$ -continued fraction of $x \in \mathbb{Q} \setminus \mathcal{X}$ is called the *second $\mathcal{F}_{1,2}$ -continued fraction* of x .

Here we record a couple of lemmata which are useful in proving several results afterwards.

Lemma 6.8. *Suppose -1 is the fin at the $(i + 1)$ -th stage of an $\mathcal{F}_{1,2}$ -continued fraction of x . Then for every $k \geq 0$, $q_{i+k} - q_{i+k-1} = 2s$, where $x = \frac{r}{s}$ with $(r, s) = 1$.*

Proof. Suppose $x = r/s$ with $(r, s) = 1$. Use statement (5) of Theorem 2.5 for $n = i$ to get $x = \frac{p_i - p_{i-1}}{q_i - q_{i-1}}$ (since the $(i + 1)$ -th fin is -1). Since $(r, s) = 1$, we have $q_i - q_{i-1} = 2ls$ for $l \in \mathbb{N}$. In both the cases, we can show that $|q_{i-1}x - p_{i-1}| = \frac{2}{q_i - q_{i-1}}$. Thus $\frac{|q_{i-1}r - p_{i-1}s|}{s} = \frac{2}{2ls}$ which implies $l = 1$ (since $|q_{i-1}r - p_{i-1}s|$ is an integer). As $q_{i+k} = 2q_{i+k-1} - q_{i+k-2}$ (recall $a_{i+k} = 2$ and $\epsilon_{i+k} = -1$) for every $k \geq 0$, the lemma follows. \square

Lemma 6.9. Let $\{\frac{p_i}{q_i}\}_{i \geq 0}$ be a sequence of $\mathcal{F}_{1,2}$ -convergents of $x \in \mathbb{R}$. Let $p/q \in \mathcal{X}$ which is not in this sequence of convergents. There exists a unique solution of the following system of equations in α, β

$$\begin{pmatrix} p \\ q \end{pmatrix} = \alpha \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} + \beta \begin{pmatrix} p_n \\ q_n \end{pmatrix}. \tag{6.4}$$

In fact, the solution $(\alpha, \beta) \in \mathbb{Z}^2$ and if $q_n \leq q \leq q_{n+1}$, then $|\beta| \geq 2$.

Proof. Since the determinant of the coefficient matrix of Eqs. (6.4) is $p_{n+1}q_n - q_n + 1p_n = \pm 2 \neq 0$, it has a unique solution. In fact, $\alpha = \pm(pq_n - qp_n)/2$ and $\beta = \pm(p_{n+1}q - q_{n+1}p)/2$. In particular, they are integers.

Next, suppose $q_n \leq q \leq q_{n+1}$. To prove that $|\beta| \geq 2$, we require to show that $p/q \approx p_{n+1}/q_{n+1}$. In fact, if $p/q \sim p_{n+1}/q_{n+1}$, then $q < q_{n+1}$ and hence the unique path from ∞ to p_{n+1}/q_{n+1} is via p/q . This contradicts our assumption that p/q is not a convergent. \square

We conclude this section by describing the relation between the two sequences of $\mathcal{F}_{1,2}$ -convergents of any $x \in \mathbb{Q} \setminus \mathcal{X}$. As a consequence, we also obtain a simple criterion for an element of \mathcal{X} to be an $\mathcal{F}_{1,2}$ -convergent of x .

Proposition 6.10. Let $x = \frac{r}{s} \in \mathbb{Q} \setminus \mathcal{X}$. Suppose $\{\frac{p_k}{q_k}\}_{k \geq 0}$ and $\{\frac{p'_k}{q'_k}\}_{k \geq 0}$ are the sequences of the first and second $\mathcal{F}_{1,2}$ -convergents of x . Let N be the smallest number such that $\frac{p_N}{q_N} \neq \frac{p'_N}{q'_N}$. Then, for every $k \geq 0$,

- (1) $N = 0 \Rightarrow q_k = q'_k = 2k + 2$;
- (2) $N \geq 1 \Rightarrow q'_{N+k} < q_{N+k} < q'_{N+k+1}$;
- (3) $|q_{N+k-1}x - p_{N+k-1}| = |q'_{N+k-1}x - p'_{N+k-1}| = \frac{1}{s}$;
- (4) $q'_{N+k} = (2k+1)q_{N+k} - (2k+2)q_{N+k-1}$ and $p'_{N+k} = (2k+1)p_{N+k} - (2k+2)p_{N+k-1}$.

Further, any $p/q \in \mathcal{X}$ with $q > q_{N-1}$ is an $\mathcal{F}_{1,2}$ -convergent of x if and only if $|qr - ps| = 1$.

Proof. If $N = 0$, then x is an integer. Hence statement (1) follows because $q_0 = q'_0 = 2$ by definition and the difference between consecutive denominators is $2s = 2$ from Lemma 6.8.

Denote the partial denominators and partial numerators of the first $\mathcal{F}_{1,2}$ -continued fraction of x by a_i and ϵ_i respectively. Similarly, denote the partial denominators and partial numerators of the second $\mathcal{F}_{1,2}$ -continued fraction of x by a'_i and ϵ'_i respectively. If $N \geq 1$ and $d = \frac{1}{|y_N|}$, then $a_N = 2\lfloor \frac{d+1}{2} \rfloor$, $a'_N = 2\lfloor \frac{d-1}{2} \rfloor$ and $\epsilon_N = \epsilon'_N$. Since $q_{N-1} = q'_{N-1}$, we have, by Theorem 2.5(1), $q_N = q'_N + 2q_{N-1}$ so that $q_N > q'_N$. By Lemma 6.8, the difference of successive elements of $\{q_{N+k}\}_{k \geq 0}$ and $\{q'_{N+k}\}_{k \geq 0}$ is the same constant and this establishes (2).

Statement (3) was verified in the proof of Lemma 6.8, and statement (4) can be verified by induction on k in a way similar to the proof of statement (2).

Suppose $p/q \in \mathcal{X}$ is such that $|qr - ps| = 1$. If $N = 0$, then statement (1) and (4) give that p/q is an $\mathcal{F}_{1,2}$ -convergent of x . Let $N \geq 1$ and assume that p/q is not a convergent of the first $\mathcal{F}_{1,2}$ -continued fraction of x . This implies that $q_{N+k-1} < q \leq q_{N+k}$ for some $k \geq 0$ (using statement (3) and $s \geq 3$). By Lemma 6.9, $q = \beta q_{N+k-1} + \alpha q_{N+k}$ and $p = \beta p_{N+k-1} + \alpha p_{N+k}$ for $\alpha, \beta \in \mathbb{Z}$ with $|\beta| \geq 2$.

Hence $1 = |qr - ps| = |\beta + \alpha|$. One can verify that $\beta + \alpha = -1$ with $\beta < 0$, is the only possibility given the bounds on q . Thus, $q = \alpha q_{N+k} - (\alpha + 1)q_{N+k-1} = 2s\alpha - q_{N+k-1}$. Hence there has to be a unique value of α that satisfies the bounds on q . Note that the case $\alpha = 2k + 1$ corresponds to q'_{N+k} which satisfies the given bounds by statement (2). Thus, $q = q'_{N+k}$ so that p/q is an $\mathcal{F}_{1,2}$ -convergent of x .

The converse follows from (3). \square

7. Best approximations and convergents

A rational number p/q is called a best approximation of $x \in \mathbb{R}$ if for every rational p'/q' different from p/q with $0 < q' \leq q$, we have $|qx - p| < |q'x - p'|$. A rational number $p/q \in \mathcal{X}$ is called a *best approximation of x by an element of \mathcal{X}* , if for every $p'/q' \in \mathcal{X}$ different from p/q with $0 < q' \leq q$, we have $|qx - p| < |q'x - p'|$.

Recall when $x \notin \mathbb{Q} \setminus \mathcal{X}$, there is a unique $\mathcal{F}_{1,2}$ -continued fraction of x so that the sequence of $\mathcal{F}_{1,2}$ -convergents is well defined. Further, when $x \in \mathcal{X}$, the sequence of convergents is finite. But when $x \in \mathbb{Q} \setminus \mathcal{X}$, there are two distinct $\mathcal{F}_{1,2}$ -continued fractions giving two sequences of $\mathcal{F}_{1,2}$ -convergents of x . Recall that there are only finitely many common $\mathcal{F}_{1,2}$ -convergents (of x) corresponding to the two $\mathcal{F}_{1,2}$ -continued fractions (see Corollary 6.6).

Theorem 7.1. *Suppose $x \in \mathbb{R}$. Then*

- (1) *If $x \notin \mathbb{Q} \setminus \mathcal{X}$, every $\mathcal{F}_{1,2}$ -convergent of x is a best approximation of x by an element of \mathcal{X} .*
- (2) *Suppose $x \in \mathbb{Q} \setminus \mathcal{X}$. An $\mathcal{F}_{1,2}$ -convergent is a best approximation of x by an element of \mathcal{X} if and only if it is a member of both the sequences of $\mathcal{F}_{1,2}$ -convergents of x .*

The second statement of this theorem is illustrated in the following example.

Example 2. Recall (refer Example 1 after Theorem 6.5) that the two sequences of $\mathcal{F}_{1,2}$ -convergents of $2/7$ are

$$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{5}{18}, \frac{9}{32}, \frac{13}{46}, \dots \right\} \text{ and } \left\{ \frac{1}{2}, \frac{1}{4}, \frac{3}{10}, \frac{7}{24}, \frac{11}{38}, \dots \right\}.$$

The common convergents are $1/2$ and $1/4$. It is not difficult to see that each of them is a best approximation of $2/7$ by an element of \mathcal{X} . Suppose $p/q \in \mathcal{X}$ with $q > 4$.

Then $|q \cdot 2/7 - p| = 1/7|2q - 7p| \geq 1/7 = |4 \cdot 2/7 - 1|$. Therefore, p/q cannot be a best approximation of $2/7$ by an element of \mathcal{X} . Thus $1/2$ and $1/4$ are the only best approximations of $2/7$ by an element of \mathcal{X} .

Proof of Theorem 7.1. Suppose $x \notin \mathbb{Q} \setminus \mathcal{X}$. Let $\{ \frac{p_k}{q_k} \}_{k=0}^M$ be the sequence of $\mathcal{F}_{1,2}$ -convergents. Here, M is finite if $x \in \mathcal{X}$, else $M = \infty$. By Theorem 3.1, $\frac{p_0}{q_0} = \frac{b}{2}$, where $b = 2[x] + 1$. This is clearly a best approximation of x by an element of \mathcal{X} .

Let $n \geq 0$ be an integer with the restriction that $n < M - 1$ if M is finite. Assume that, for $0 \leq k \leq n$, p_k/q_k is a best approximation of x by an element of \mathcal{X} . Now we show that p_{n+1}/q_{n+1} is a best approximation of x by an element of \mathcal{X} . Note that the case of M finite and $n = M - 1$ is clear as $q_M x - p_M = 0$.

For any $p/q \in \mathcal{X}$ different from p_{n+1}/q_{n+1} with $0 < q \leq q_n$, then $|qx - p| > |q_n x - p_n| \geq |q_{n+1} x - p_{n+1}|$. Next if $q_n < q \leq q_{n+1}$, we use Theorem 2.5(5) to get

$$|q_{n+1} x - p_{n+1}| = \frac{2}{x_{n+2} q_{n+1} + \epsilon_{n+2} q_n},$$

$$|qx - p| = \frac{|x_{n+2}(p_{n+1}q - q_{n+1}p) + \epsilon_{n+2}(p_n q - q_n p)|}{x_{n+2} q_{n+1} + \epsilon_{n+2} q_n}. \tag{7.1}$$

Now, we will show that numerator in (7.1) is greater than 2. By Lemma 6.9, $p = \beta p_n + \alpha p_{n+1}$, $q = \beta q_n + \alpha q_{n+1}$ for some $\alpha, \beta \in \mathbb{Z}$ with $|\beta| \geq 2$. Thus,

$$|qx - p| = \frac{2|\beta x_{n+2} - \alpha \epsilon_{n+2}|}{x_{n+2} q_{n+1} + \epsilon_{n+2} q_n}.$$

The proof of part (1) will be complete if we show that

$$|\beta x_{n+2} - \alpha \epsilon_{n+2}| > 1. \tag{7.2}$$

Case 1. Suppose $\beta \geq 2$. Since $q \leq q_{n+1}$, we have $(\alpha - 1)q_{n+1} \leq -\beta q_n < 0$. Since $\alpha \in \mathbb{Z}$, we have $\alpha \leq 0$. Again, since $q_n < \beta q_n + \alpha q_{n+1}$, we have $\frac{-\alpha}{\beta - 1} < \frac{q_n}{q_{n+1}}$. Hence, $\alpha > 1 - \beta$ (since $q_n/q_{n+1} < 1$). Thus we have shown $1 - \beta < \alpha \leq 0$. Using these bounds and the fact that $x_{n+2} > 1$ (by Corollary 6.3), inequality (7.2) follows.

Case 2. Suppose $\beta \leq -2$. Since $q > 0$, $\alpha \geq 1$. Since $q \leq q_{n+1}$, $\frac{-\beta}{\alpha - 1} \geq \frac{q_{n+1}}{q_n}$ so that $\alpha \leq -\beta$ (since $q_{n+1}/q_n > 1$). Since $p = \beta p_n + \alpha p_{n+1}$ is odd, $\alpha \neq -\beta$ and hence $\alpha \leq -\beta - 1$. Thus, we have shown that $1 \leq \alpha \leq -\beta - 1$ which implies inequality (7.2) (since $x_{n+2} > 1$).

For the proof of part (2), let $x \in \mathbb{Q} \setminus \mathcal{X}$ so that it has two $\mathcal{F}_{1,2}$ -continued fractions. Suppose p_k/q_k appears in both the sequences of $\mathcal{F}_{1,2}$ -convergents. Then, by Corollary 6.7, we have $x_{k+1} > 1$. Hence the result follows from the computation of the first part.

Conversely, suppose p_n/q_n occurs in only one of the sequences of $\mathcal{F}_{1,2}$ -convergents. Then $|q_n x - p_n| = |q_{N-1} x - p_{N-1}| = |q_N x - p_N|$ (by Proposition 6.10). If $N \geq 1$, this

immediately gives that p_n/q_n is not a best approximation (since $0 < q_{N-1} < q_n$). In the case $N = 0$, we can use the same reasoning to get that p_n/q_n is not a best approximation when $2 = q_N < q_n$. The remaining case $q_0 = 2$ ($N = 0, q_N = q_n$) is completed by noting that x is an integer and $|2x - (2x - 1)| = |2x - (2x + 1)|$ which implies that neither of these are best approximations. \square

Lemma 7.2. *Suppose $x \in \mathcal{X}$. Suppose $\{\frac{p_i}{q_i}\}_{i=0}^M$ is the sequence of $\mathcal{F}_{1,2}$ -convergents of x . Let $p/q \in \mathcal{X}$ be a best approximation of x by an element of \mathcal{X} . Then $q \leq q_M$.*

Proof. Suppose $q > q_M$. Observe that $x = \frac{p_M}{q_M}$. Hence $|qx - p| \geq |q_Mx - p_M| = 0$ so that p/q is not a best approximation of x by an element of \mathcal{X} . \square

Lemma 7.3. *Suppose $x \in \mathbb{Q} \setminus \mathcal{X}$ and N is as in Proposition 6.10. If p/q is a best approximation of x by an element of \mathcal{X} , then $N \geq 1$ and $q \leq q_{N-1}$.*

Proof. Suppose $p/q \in \mathcal{X}$ and set $x = r/s$ so that s is odd. By Proposition 6.10(3), $|q_{N-1}x - p_{N-1}| \leq \frac{|qr-ps|}{s} = |qx - p|$ (since $|qr - ps| \geq 1$). Thus, if $N \geq 1$ and $q > q_{N-1}$, p/q is not a best approximation of x by an element of \mathcal{X} .

If $N = 0$, we replace $N - 1$ by N in the above steps to conclude that $q \leq q_0 = 2$. However, in this case, x is an integer and $1 = |2x - (2x - 1)| = |2x - (2x + 1)|$ is the smallest possible value for $|qx - p|$, and hence p/q cannot be a best approximation by an element of \mathcal{X} . \square

Theorem 7.4. *Every best approximation of a real number x by an element of \mathcal{X} is an $\mathcal{F}_{1,2}$ -convergent of x .*

Proof. Let $\{\frac{p_i}{q_i}\}_{i \geq 0}$ be a sequence of $\mathcal{F}_{1,2}$ -convergents of x . Let $p/q \in \mathcal{X}$ be a best approximation of x by an element of \mathcal{X} . Then for some $n \geq 0, q_n \leq q < q_{n+1}$. If $x \in \mathcal{X}$, by Lemma 7.2, $q \leq q_M$ with $x = p_M/q_M$. Observe that if $q = q_M$ then $p = p_M$ is the only possibility and the theorem holds. Hence we can assume that $q < q_M$ and hence $n < M$ and $x_{n+1} > 1$. For $x \notin \mathcal{X}$, we note that $x_i \neq 1 \forall i \geq 1$ unless $x \in \mathbb{Q} \setminus \mathcal{X}$ and if $x \in \mathbb{Q} \setminus \mathcal{X}$, then by Lemma 7.3, $N \geq 1$ and $n < N$ so that $x_{n+1} \neq 1$ (Proposition 6.2(2)). Thus, we can assume that $x_{n+1} > 1$ in all cases.

The proof of the theorem is by contradiction. Suppose that p/q is not an $\mathcal{F}_{1,2}$ -convergent of x . By using Lemma 6.9, we obtain $q = \beta q_{n-1} + \alpha q_n, p = \beta p_{n-1} + \alpha p_n$ for $\alpha, \beta \in \mathbb{Z}$ with $|\beta| \geq 2$ and

$$|qx - p| = \frac{2|\beta x_{n+1} - \epsilon_{n+1}\alpha|}{x_{n+1}q_n + \epsilon_{n+1}q_{n-1}}.$$

We will show that the numerator is strictly bigger than 2 which contradicts that $\frac{p}{q}$ is a best approximation of x by an element of \mathcal{X} .

Case 1. Suppose $\beta \geq 2$. Then $q_n \leq q = \beta q_{n-1} + \alpha q_n < q_{n+1} = \epsilon_{n+1} q_{n-1} + a_{n+1} q_n$. Hence $1 - \beta < \alpha \leq a_{n+1} - 1$ (since $q_{n-1} > 0$). Using $a_{n+1} = 2 \lfloor \frac{x_{n+1} + 1}{2} \rfloor \leq x_{n+1} + 1$, we have $1 - \beta < \alpha \leq x_{n+1}$. These bounds on α imply $\beta x_{n+1} - \epsilon_{n+1} \alpha > 1$.

Case 2. Suppose $\beta \leq -2$. Then $0 < q = \beta q_{n-1} + \alpha q_n < \epsilon_{n+1} q_{n-1} + a_{n+1} q_n$. Hence $1 \leq \alpha \leq \epsilon_{n+1} - \beta + a_{n+1} - 1$ (since $\frac{q_{n-1}}{q_n} < 1$). Since $p = \beta p_{n-1} + \alpha p_n$ is odd, we have $\alpha \neq \epsilon_{n+1} - \beta + a_{n+1} - 1$ so that $\alpha \leq \epsilon_{n+1} - \beta + a_{n+1} - 2$. These bounds on α imply $\beta x_{n+1} - \epsilon_{n+1} \alpha < -1$. \square

Conflict of interest statement

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We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed.

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