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COMMENTS AND CORRECTIONS

Comments on "Cutset Bounds on the Capacity of MIMO Relay Channels"

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***Abstract*—We comment on the paper "Cutset Bounds on the Capacity of MIMO Relay Channels" by Jeong *et al*. and point out that, unlike what appears from a remark and some other contents by these authors, the matrix distribution for the sum of two complex random Wishart matrices has already been derived by Kumar for the general case of arbitrary covariance matrices and not only for the special case when one of them is assumed proportional to the identity matrix. The latter assumption has been made only for deriving the corresponding eigenvalue distribution. Furthermore, we draw attention to the result that when all covariance matrices are chosen proportional to the identity matrix, then it is possible to obtain exact and closed form expressions for the sum of an arbitrary number of Wishart matrices and not only for two as considered by Jeong *et al*.**

***Index Terms*—Sum of Wishart matrices, eigenvalue statistics, MIMO multiple access channels, MIMO relay channels, Shannon transform.**

In a recent article [1], the authors have used the sum of Wishart matrices in the context of multiple-input-multiple-output (MIMO) relay channels. The purpose of the present comment article is to compare certain results presented in [1] with those in [2]–[4]. In particular, we point out that the matrix distribution of the sum of two complex central Wishart matrices for the general case of arbitrary covariance matrices has already been derived in [2], and not only for the special case when one of the covariance matrices is relaxed to be proportional to identity matrix. Furthermore, we also highlight certain generalizations concerning the sum of arbitrary number of Wishart matrices that have already been provided in [3] and [4].

Sums of Wishart matrices play a key role in multivariate statistics [5]–[9] and, among other things, find applications in the analysis of several modern-day MIMO communication models [1], [3], [4], [10], [11]. While the investigation of the sum of independent Wishart matrices dates back to the work of Tan and Gupta [8], recent availability of exact solutions concerning its eigenvalue statistics has revived the interest in exploring such composite matrix models further [2]–[4], [12], [13]. In [2] one of the present authors has derived the matrix probability density for the sum of two independent central complex Wishart matrices which have different covariance matrices associated with them. Moreover, in the case of one of the covariance matrices proportional to the identity matrix, closed form expressions for eigenvalue densities have also been obtained in [2] and [12]. Several other important results for the sum of two Wishart matrices have been worked out in [13], such as an exact expression for the arbitrary order eigenvalue density correlation function. In [3] and [4], exact solvability has been established for the sum of an arbitrary number of independent Wishart matrices with covariance matrices proportional to the identity matrix. This sum is evidently equivalent to the scalar-weighted sum of independent uncorrelated-Wishart matrices. It has been shown that this problem can be mapped to that of a semicorrelated Wishart matrix, and therefore the existing results [14]–[16] for the latter can be used. The eigenvalue statistics derived therein has been applied to investigate the ergodic capacity of distributed antenna systems [3], and the ergodic sum capacity of MIMO multiple access and MIMO relay channels [4].

Jeong *et al.* [1] refer to the sum of $n \times n$-dimensional Wishart matrices $\mathbf{W}_l$ $(l = 1, \ldots, L)$,

$$\mathbf{W} = \sum_{l=1}^{L} \mathbf{W}_l, \tag{1}$$

as a Hyper Wishart matrix, and provide the corresponding probability density function (PDF) in Theorem 1. Concerning this, we would like to point out that this PDF has already been published with a proof in [4], and another proof has been provided in [13]. While Jeong *et al.* [1] do refer to [2], and mention in the footnote that, "*The distribution of the sum of two complex Wishart matrices has been derived using the Harish-Chandra-Itzykson-Zuber unitary group integral when one of the covariance matrices is proportional to the identity matrix while the second is arbitrary*", we would like to clarify that [2] already gives the matrix distribution when both the covariance matrices are arbitrary; see [2, eq. (10)]. It is only for the corresponding eigenvalue density that one of the covariance matrices has been considered proportional to the identity matrix in [2]. We would also like to emphasize that [1, eq. (3)] can be readily obtained by considering the eigenvalue decomposition of $\mathbf{W}$, and does not lead to any further information unless some assumption is made for the covariance matrices.

In Corollary 1 [1], the authors provide the joint PDF of eigenvalues for $L = 2$ case when both the covariance matrices are proportional to the identity matrix. Actually, from the works [3], [4] it is evident that if the covariance matrices are taken proportional to the identity matrix, then it is possible to obtain the joint PDF of eigenvalues for the sum of an arbitrary number (L) of Wishart matrices, as discussed below. In fact, the corresponding marginal density of a generic eigenvalue has already been provided in [3] and [4].

Consider the covariance matrices associated with the Wishart matrices in (1) to be

$$\boldsymbol{\Sigma}_l = \sigma_l \boldsymbol{I}_n, \quad l = 1, ..., L, \tag{2}$$

where σ_l are positive scalars, and $\boldsymbol{I}_n$ is the n-dimensional identity matrix. Moreover, suppose the degrees of freedom of the Wishart matrices $\mathbf{W}_l$ in (1) are $m_1, ..., m_L$, respectively. Then, as shown in [4], we can write $\mathbf{W} = \mathbf{G}^\dagger \mathbf{G}$, where $\mathbf{G}$ is an $m \times n$ matrix with $m = m_1 + \cdots m_L$ and is described by the PDF

$$\mathcal{P}(\mathbf{G}) \propto e^{-\mathrm{tr}\mathbf{G}^\dagger \boldsymbol{\Sigma}^{-1}\mathbf{G}}, \tag{3}$$

with $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1 \boldsymbol{I}_{m_1}, ..., \sigma_L \boldsymbol{I}_{m_L})$. Evidently, the $m \times m$-dimensional matrix $\widetilde{\mathbf{W}} = \mathbf{G}\mathbf{G}^\dagger$ is complex central Wishart distributed, i.e., $\widetilde{\mathbf{W}} \sim \mathcal{CW}_m(n, \boldsymbol{\Sigma})$. Consequently, one can use the existing results for semicorrelated Wishart matrices; see e.g. [14]–[16]. We can have the following two possibilities:

A. $m \leq n$

In this case $\mathbf{W}$ and $\widetilde{\mathbf{W}}$ share the nonzero eigenvalues $\lambda_1, .., \lambda_m$, which are described by the joint PDF

$$\begin{aligned} P(\lambda_1, \ldots, \lambda_m) &= \frac{(-1)^{m(m-1)/2}}{m!} \Delta(\boldsymbol{\lambda}) \prod_{i=1}^{m} \frac{\lambda_i^{n-m}}{\Gamma(n-m+i)} \\ &\times \prod_{l=1}^{L} \sigma_l^{-nm_l} \cdot \frac{\det\left[[(-\lambda_j)^{k-1} e^{-\sigma_l^{-1}\lambda_j}]_{\substack{j=1,...,m \\ k=1,...,m_l}} \right]_{l=1,...,L}}{\det\left[[\frac{\Gamma(j)\sigma_l^{k-j}}{\Gamma(j-k+1)}]_{\substack{j=1,...,m \\ k=1,...,m_l}} \right]_{l=1,...,L}}. \end{aligned} \tag{4}$$

Here, $\Delta(\boldsymbol{\lambda}) = \prod_{j>k}(\lambda_j - \lambda_k)$ is the Vandermonde determinant, $\Gamma(\cdot)$ is the Gamma function [17], and $\det[[f_{j,k,l}]_{\substack{j=1,...,m \\ k=1,...,m_l}}]_{l=1,...,L}$ denotes

$$\det[[f_{j,k,1}]_{\substack{j=1,...,m \\ k=1,...,m_1}} \cdots [f_{j,k,L}]_{\substack{j=1,...,m \\ k=1,...,m_L}}].$$

In (4), as well as the equations below, $1/\Gamma(k)$ should be taken as 0 if k happens to be a non-positive integer. In addition to the eigenvalues $\lambda_1, .., \lambda_m$, $\mathbf{W}$ possesses $n - m$ zero eigenvalues, and a full PDF incorporating these can be written by introducing Dirac delta functions in (4).

B. $m > n$

In this case there are n nonzero eigenvalues $\lambda_1, ..., \lambda_n$ eigenvalues, shared by both $\mathbf{W}$ and $\widetilde{\mathbf{W}}$. The corresponding joint PDF is

$$\begin{aligned} &P(\lambda_1, \ldots, \lambda_n) \\ &\quad = (-1)^{n(n-1)/2} \Delta(\boldsymbol{\lambda}) \frac{\prod_{l=1}^{L} \sigma_l^{-nm_l}}{\prod_{i=1}^{n} \Gamma(i+1)} \\ &\quad \times \frac{\det\begin{bmatrix} [(-\lambda_j)^{k-1} e^{-\sigma_l^{-1}\lambda_j}]_{\substack{j=1,...,n \\ k=1,...,m_l}} \\ [\frac{\Gamma(j)\sigma_l^{k-j}}{\Gamma(j-k+1)}]_{\substack{j=1,...,m-n \\ k=1,...,m_l}} \end{bmatrix}_{l=1,..,L}}{\det\left[[\frac{\Gamma(j)\sigma_l^{k-j}}{\Gamma(j-k+1)}]_{\substack{j=1,...,m \\ k=1,...,m_l}} \right]_{l=1,..,L}}. \end{aligned} \tag{5}$$

Additionally, $\widetilde{\mathbf{W}}$ possesses $m - n$ zero eigenvalues.

In either case, i.e., for $m \leq n$ or $m > n$, the marginal PDF describing a generic nonzero eigenvalue is given by [3], [4]

$$\begin{aligned} &p(\lambda) \\ &= -\nu^{-1} \det{}^{-1}\left[\left[\frac{\Gamma(j)\sigma_l^{k-j}}{\Gamma(j-k+1)} \right]_{\substack{j=1,...,m \\ k=1,...,m_l}} \right]_{l=1,...,L} \\ &\times \det\begin{bmatrix} 0 & [\frac{\Gamma(k)e^{-\lambda/\sigma_l}}{\sigma_l^{n-k+1}} \mathcal{L}_{k-1}^{(n-k+1)}(\frac{\lambda}{\sigma_l})]_{k=1,...,m_l} \\ [\frac{\lambda^{n-j}}{\Gamma(n-j+1)}]_{j=1,...,m} & [\frac{\Gamma(j)}{\Gamma(j-k+1)}\sigma_l^{k-j}]_{\substack{j=1,...,m \\ k=1,...,m_l}} \end{bmatrix}_{l=1,...,L}, \end{aligned} \tag{6}$$

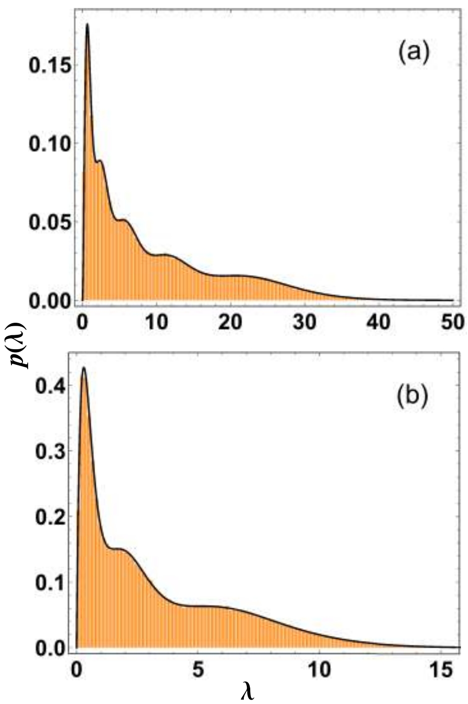

FIGURE 1. Comparison between the analytical marginal density (solid line) and simulation result (histogram) for (a) $n > m$ case: $n = 7, L = 3, m_1 = 2, m_2 = 1, m_3 = 2, \sigma_1 = 2, \sigma_2 = 6/5, \sigma_3 = 3/4$, and (b) for $n < m$ case: $n = 3, L = 2, m_1 = 2, m_2 = 2, \sigma_1 = 1/2, \sigma_2 = 7/6$.

where $\nu = \min(m, n)$, and $\mathcal{L}_j^{(k)}(\cdot)$ represent the associated Laguerre polynomials [17]. With the aid of (4) and (5), one can also write down marginal densities (correlation functions) of higher orders. In Fig. 1, we show the comparison between the analytical marginal density of eigenvalues as predicted by (6), and numerical simulation involving 50000 matrices for two sets of parameter values, as indicated in the caption. We can see an excellent agreement in both cases.

The knowledge of the marginal density enables us to compute the Shannon transform, which is given by [18]

$$\mathcal{S}(\rho) = \int_0^\infty \ln(1+\rho\lambda)\, p(\lambda)\, d\lambda. \tag{7}$$

With the aid of result (6), we obtain the following closed form expression for the Shannon transform:

$$\mathcal{S}(\rho) = \nu^{-1}\det{}^{-1}\left[\left[\frac{\Gamma(j)\,\sigma_l^{k-j}}{\Gamma(j-k+1)}\right]_{\substack{j=1,\dots,m\\ k=1,\dots,m_l}}\right]_{l=1,\dots,L} \times \sum_{\mu=1}^{\nu} \det\left[[\psi_{j,k}^{(\mu)}(\sigma_l)]_{\substack{j=1,\dots,m\\ k=1,\dots,m_l}}\right]_{l=1,\dots,L}. \tag{8}$$

Here, $\psi_{j,k}^{(\mu)}(\sigma)$ are given by

$$\psi_{j,k}^{(\mu)}(\sigma) = \begin{cases} \dfrac{\sigma^{k-1}\rho^{j-1}}{\Gamma(n-j+1)} G_{3,4}^{3,2}\left({0, j-1; j \atop j-1, j-1, n; k-1} \middle| \dfrac{1}{\sigma\rho}\right), & j=\mu \\ \dfrac{\Gamma(j)\,\sigma^{k-j}}{\Gamma(j-k+1)}, & j\neq\mu, \end{cases} \tag{9}$$

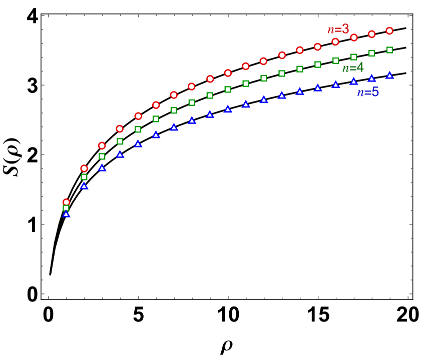

FIGURE 2. Comparison between the analytical predictions (solid lines) and simulation results (symbols) for the Shannon transform. Parameter values used are $n = 3, 4, 5$ and $L = 3$, $m_1 = 1$, $m_2 = 1$, $m_3 = 2$, $\sigma_1 = 3/2$, $\sigma_2 = 1$, $\sigma_3 = 2/3$.

with $G_{3,4}^{3,2}(\cdot)$ being a Meijer G-function [19]. The derivation involved is similar to that of the mean channel capacity, as provided in [3] and [4]. We show a comparison of the above analytical result with numerical simulation for three n values in Fig. 2. The Shannon transform values depicted in the figure have been obtained by averaging over the values calculated for 50000 matrices used in the simulation. Once again, the agreement is perfect.

Finally, we would like to point out that the application of sum of Wishart matrices to the MIMO relay channel, as discussed in [1], has also been considered in [4].

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