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Asymptotic Stability Analysis Applied in Two and Three-Dimensional Discrete Systems to Control Chaos

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Abstract
Asymptotic stability analysis applied to stabilize unstable fixed points and to control chaotic motions in two and three-dimensional discrete dynamical systems. A new set of parameter values obtained which stabilizes an unstable fixed point and control the chaotic evolution to regularity. The output of the considered model and that of the adjustable system continuously compared by a typical feedback and the difference used by the adaptation mechanism to modify the parameters. Suitable numerical simulation which are used thoroughly discussed and parameter values are adjusted. The findings are significant and interesting. This strategy has some advantages over many other chaos control methods in discrete systems but, however it can be applied within some limitations.

Keywords- Asymptotic stability, Control parameter, Chaos, Lyapunov exponents.

1. Introduction

Orbits of dynamical system, originating nearby an unstable fixed point, may remain unstable throughout the evolution of the system. The evolutionary motion starting from such positions often turn to be chaotic. To obtain regular motion, one has to stabilize the initial point by changing values of certain parameters of the system. For this, there should be a well-defined procedures and asymptotic stability method can be considered as one of the most effective one. This technique proved to be useful, as it could regulate the chaos to achieve desirable results.

Chaos management refers to the technique of modifying and regularizing the chaotic motion displayed in nonlinear systems. Several interesting reports on chaos control, (Auerbach et al., 1992; Braiman et al., 1995; Carroll & Pecora, 1993; Garfinkel et al., 1992; Litak et al., 2007; Ott et al., 1990; Pecora & Carroll, 1990; Pyragas, 1992; Shinbrot et al., 1993) have shown that If properly applied to chaotic and dynamic systems, management of disorder can be of great benefit. Chaos control techniques like OGY and feedback control method with a good effect in a relative short period are being used in many nonlinear dynamical systems (Bilal Ajaz et al., 2020; Wang et al.,

2020) plays an important role in transforming unusual behavior into regularity on various engineering devices such as contact signals, chemical reactions etc.

The area of chaos reduction appears as an emerging study, as it includes work in all knowledge areas. Due to the severe tolerance and complexity of chaotic dynamics there is a big challenge in the control of chaos. Since almost all systems exhibiting chaotic behavior are of a nonlinear form, and there is no standard way to describe such behavior, the tools of chaos management for different chaotic systems are different.

The present article is based on a method of asymptotic stability,(Erjaee, 2002; Litak et al., 2007; Sandeep Reddy & Ghosal, 2016; Schuster, 1999),on chaos control, and applied to some two and three dimensional maps.

2. Description of the Method

Dynamics of the actual map X_{n+1} and that of the desired map Y_{n+1} can be explained by following mapping:

$$X_{n+1} = F(x_n, p) \tag{1}$$

$$Y_{n+1} = F(y_n, p^*) \tag{2}$$

Also, the neighborhood dynamics of X_{n+1} and Y_{n+1} can be represented by the relation:

$$X_{n+1} = A_R X_n + B_R p\,,$$
$$Y_{n+1} = A_D Y_n + B_D p^*.$$

Matrices A_R, A_D, B_R, B_D can be obtained from the following:

$$A_R = D_{x_n} F(X_n, p) \qquad A_D = D_{y_n} F(Y_n, p^*)$$
$$B_R = D_p F(X_n, p) \qquad B_D = D_{p^*} F(Y_n, p^*)$$
$$X_{n+1} = \begin{pmatrix} \mathbf{x_n} \\ \mathbf{y_n} \end{pmatrix} Y_{n+1} = \begin{pmatrix} \mathbf{x^*_n} \\ \mathbf{y^*_n} \end{pmatrix}.$$

Let a, b are two parameters of the system and (x^u, y^u) be any unstable fixed point of above system for given values of a and b. Then, our objective is to obtain two new suitable values for a and b so that this unstable point becomes stable. For this, we need the Jacobian matrices defined by

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}, \quad J^* = \begin{pmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial b} \end{pmatrix}$$

The control input parameter matrix p^* can be given by

$$p^* = C_R X_n + C_M p - C_D Y_n \tag{3}$$

Then, using (1), (2) & (3), one obtains the following error equation:

$$e_{n+1} = (A_R - B_D C_R)\, e_n + \{A_R - A_D + B_D(C_D - C_R)\} Y_n + (B_R - B_D C_M)\, p \tag{4}$$

where $e_n = X_n - Y_n$.

Note that in equation (3) and (4) the coefficient matrices C_R, C_D and C_M are to be determined so that if the error vector $e_n = X_n - Y_n$ is initialized as, $e_0 = 0$ then it will be zero for all n future times. For asymptotic stability, we must have $e_n \to 0$ as $n \to \infty$ then equation (4) implies

$$A_R - A_D + B_D(C_D - C_R) = 0 \text{ this implies } B_D(C_D - C_R) = A_D - A_R \tag{5}$$

$$\text{and } B_R - B_D C_M = 0 \quad \Rightarrow \quad B_D C_M = B_R \tag{6}$$

The necessary and sufficient condition for $e_n \to 0$ as $n \to \infty$ is

$$A_R - B_D C_R = -I \tag{7}$$

From these, one can obtain matrices C_M, C_D, C_R and then control parameter matrix p^* from equation (3). A necessary and sufficient condition for the existence of matrices C_M, C_D, C_R that satisfies above equations (5) is given by:

$$Rank\ (B_D) = Rank\ (B_D, A_D - A_R) = Rank\ (B_D, B_R).$$

3. Dynamic Models

(i) Prey-Predator Map

First **,** we have considered Prey-Predator map, (Elsadany, 2012), written as

$$\begin{aligned} X_{n+1} &= a\, x_n (1 - x_n) - b\, x_n\, y_n \\ Y_{n+1} &= c\, y_n (1 - y_n) + b\, x_n y_n \end{aligned} \tag{8}$$

For a = 3.7, b = 3.5, c = 0.2, there are 4 fixed points:(0, 0), (0, - 4.0), (0.72973, 0) & (0.25712, 0.49961) out of which fixed point (0.25712, 0.49961) is unstable. So, the Orbits originating nearby (0.25712, 0.49961) would also be unstable and unpredictable & may be chaotic. A nearby unstable fixed point, we call a desired initial point as (0.3, 0.5). At this initial point together with above parameter values, time series, attractor and LCE plots are obtained and shown by Figure 1. Clearly the system (8) is showing chaos at (0.3,0.5) with a = 3.7, b = 3.5, c = 0.2.

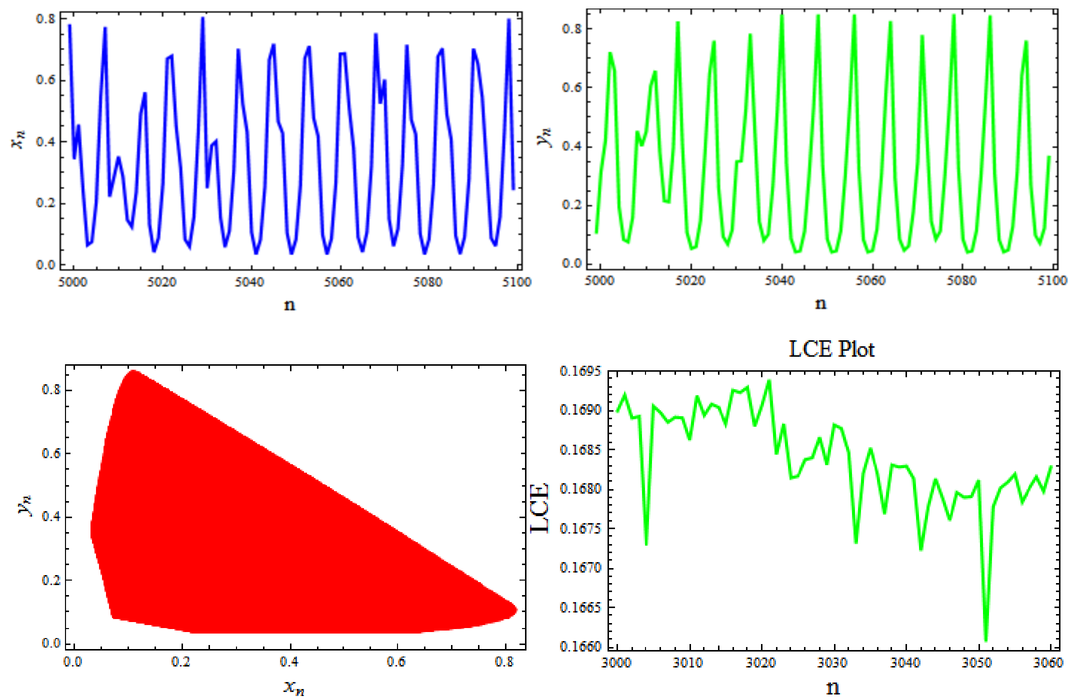

Figure 1. Time series graphs, attractor and LCE plots of the unstable system.

Now, let us apply method of asymptotic stability discussed above for this map (8).

For fixed value c = 0.2, we have the unstable fixed point (0.25712, 0.49961), and a nearby point (0.3, 0.5) and also $p=\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3.7 \\ 3.5 \end{pmatrix}$. We apply the above-mentioned method and obtain:

$$A_R = \begin{bmatrix} 0.048652 & -0.899924 \\ 1.74865 & 0.900078 \end{bmatrix} \qquad A_D = \begin{bmatrix} -0.27 & -1.05 \\ 1.75 & 1.05 \end{bmatrix}$$

$$B_R = \begin{bmatrix} 0.19101 & -0.128462 \\ 0 & 0.128462 \end{bmatrix} \qquad B_D = \begin{bmatrix} 0.21 & -0.15 \\ 0 & 0.15 \end{bmatrix}$$

$$C_M = \begin{bmatrix} 0.90957 & 0 \\ 0 & 0.85641 \end{bmatrix} \qquad C_R = \begin{bmatrix} 3.79669 & -4.76117 \\ 11.6577 & -0.66615 \end{bmatrix}$$

$$C_D = \begin{bmatrix} 2.28571 & -4.7619 \\ 11.6667 & 0.333333 \end{bmatrix} \qquad p^* = \begin{pmatrix} 3.91525 \\ 2.99538 \end{pmatrix}$$

For the case when c = 0.2; new values of a and b; a = 3.91525, b = 2.99538 along with initial point (0.3, 0.5) a phase plot showing regular motion and plot of Lyapunov exponents (LCE) is also showing regularity, given in Figure 2, as value of Lyapunov characteristic exponent is negative always.

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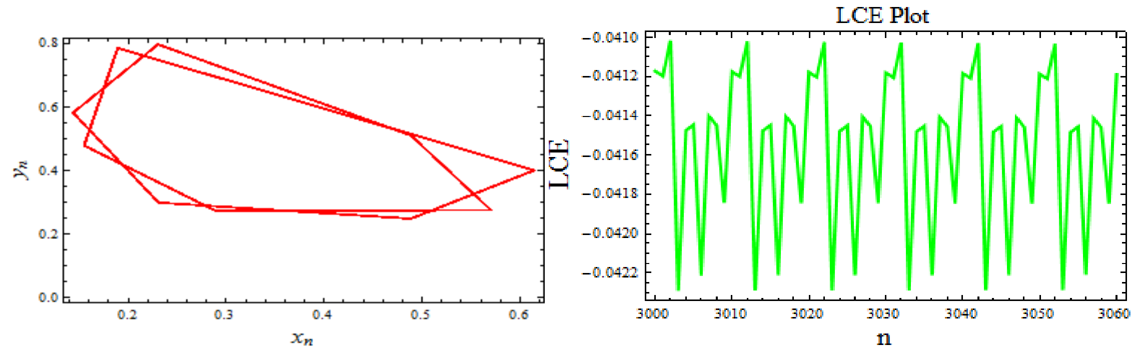

Figure 2. Phase plot and LCE plot of controlled system.

(ii) Food Chain Model: (F – C Model)

Next**,** we have considered three dimensional food chain model,(Elsadany, 2012) written as

$$
\begin{aligned}
x_{n+1} &= a\,x_n\,(1-x_n) - b\,x_n y_n \\
y_{n+1} &= c\,x_n y_n - d\,y_n z_n \\
z_{n+1} &= r\,y_n z_n
\end{aligned}
\tag{9}
$$

For values a = 4.1, b = 3.7, c = 3, d = 3.5, r = 3.8, all five fixed points P_0, P_1, P_2, P_3, P_4 of above system be obtained as: $P_0(0, 0, 0)$, $P_1(0, 0.2632, 0.2857)$, $P_2(0.518614, 0.263158, 0.158812)$, $P_3(0.7561, 0, 0)$ and $P_4(0.3333, 0.4685, 0)$. Then, by using steps of stability analysis it has been checked that the fixed points $P_2(0.518614, 0.263158, 0.158812)$ and $P_4(0.3333, 0.4685, 0)$ are unstable. Thus, the Orbits originating nearby P_2 and P_4 would also be unpredictable & may be chaotic. Nearby P_2, let us assume a desired initial point as P*(0.5, 0.3, 0.2). To proceed further, first we fix two parameters say b and c with same values, i.e., b = 3.7 and c=3.0.

Bifurcation diagrams of the above F – C map (9) along x–axis, for three ranges of parameter c are obtained and given by Figure 3.

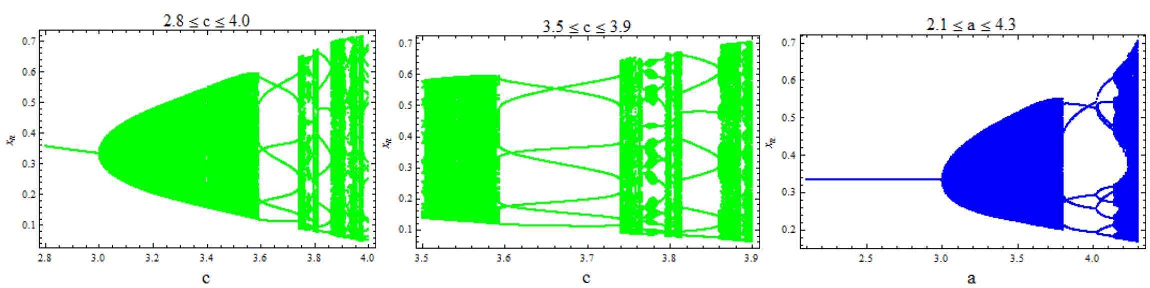

Figure 3. Bifurcation diagrams for map (9).

A time series graph and chaotic attractors along the coordinate planes for orbits originated at point (0.5, 0.3, 0.2), which is the neighborhood of unstable Fixed Point P_2, shown in Figure 4 are showing chaos.

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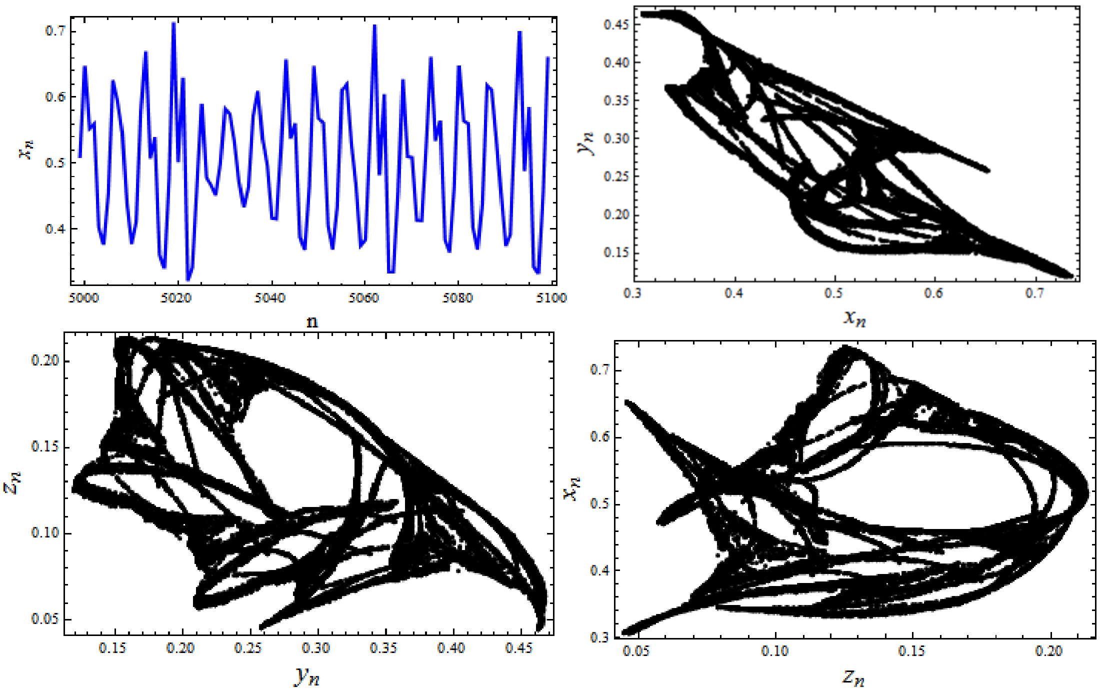

Figure 4. Time series and attractors of unstable system.

In the process of stabilizing the desired point (0.5, 0.3, 0.2), we obtain the following matrices:

$$A_R = \begin{bmatrix} -1.12632 & -1.91887 & 0 \\ 0.789474 & 1.0 & -0.921053 \\ 0 & 0.603486 & 1.0 \end{bmatrix} \qquad A_D = \begin{bmatrix} -1.11 & -1.85 & 0 \\ 0.9 & 0.8 & -1.05 \\ 0 & 0.76 & 1.14 \end{bmatrix}$$

$$B_R = \begin{bmatrix} 0.249654 & 0 & 0 \\ 0 & -0.041793 & 0 \\ 0 & 0 & 0.041793 \end{bmatrix} \qquad B_D = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & -0.06 & 0 \\ 0 & 0 & 0.06 \end{bmatrix}$$

$$C_M = \begin{bmatrix} 0.998614 & 0 & 0 \\ 0 & 0.696544 & 0 \\ 0 & 0 & 0.696544 \end{bmatrix} \qquad C_R = \begin{bmatrix} -8.50528 & -7.67549 & 0 \\ -13.1579 & 0 & 15.3509 \\ 0 & 10.0581 & 6.66667\times10^{-6} \end{bmatrix}$$

$$C_D = \begin{bmatrix} -8.44 & -7.4 & 0 \\ -15.0 & 3.33333 & 17.5 \\ 0 & 12.6667 & 2.33333 \end{bmatrix} \qquad p^* = \begin{bmatrix} 4.1035 \\ 1.05194 \\ 1.02707 \end{bmatrix}$$

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Thus, for fixed value of b = 3.7, c= 3.0, we proceeded above calculations with input

$$p=\begin{pmatrix} a \\ d \\ r \end{pmatrix} = \begin{pmatrix} 4.1 \\ 3.5 \\ 3.8 \end{pmatrix} \quad \text{and obtained} \quad p^*=\begin{pmatrix} a \\ d \\ r \end{pmatrix} = \begin{pmatrix} 4.1035 \\ 1.05194 \\ 1.02707 \end{pmatrix}$$

At these new parameter values of a, d and r, we obtain the phase plot and the plot of Lyapunov exponents as shown in Figure 5 below.

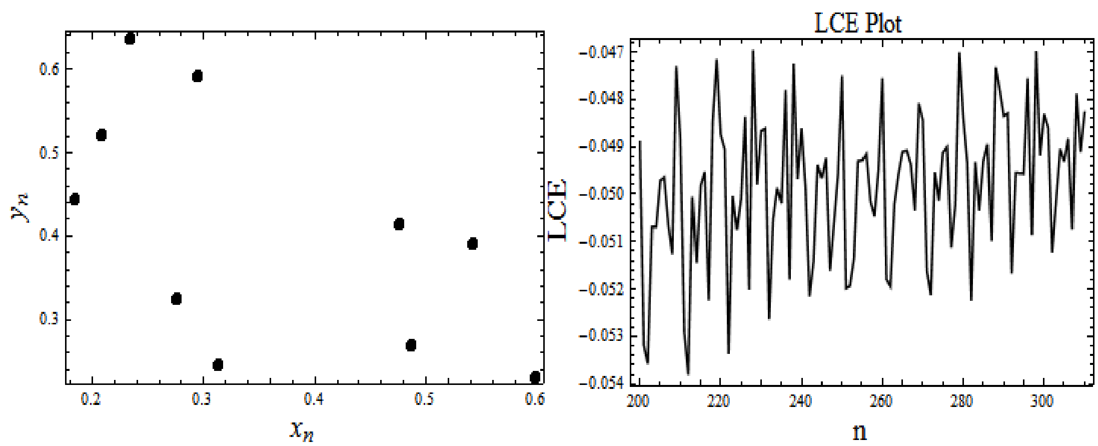

Figure 5. Phase plot and LCE plot of controlled system.

Clearly phase plot on XY plane shows finite number of points. Similar is the case with also other two planes. Also, LCE plot shows the Lyapunov Exponents all are negative. Hence, the system is no more chaotic and chaos is controlled.

(iii) 3–D Arneodo-Coullet-Tresser (ACT) Map

Further, let us consider an another 3-dimensional map, known as the Arneodo-Coullet-Tresser (or ACT) Map, (Arneodo et al., 1981, 1982),written as

$$\begin{aligned} x_{n+1} &= a\,x_n - b(y_n - z_n) \\ y_{n+1} &= bx_n + a(y_n - z_n) \\ z_{n+1} &= cx_n - d\,x_n k + e\,z_n \end{aligned} \tag{10}$$

For parameter values a = 0.6, b = 0.5, c = 0.41, d = 1, e = 1, k =3, ACT map (10) has an unstable fixed point given by (0.640312,0.0128062, 0.525056) and thus, the neighboring points this point (0.55, 0.01, 0.5) is also unstable. Time series graphs and chaotic attractor obtained for orbit originating from this neighboring point are shown in Figure 6.

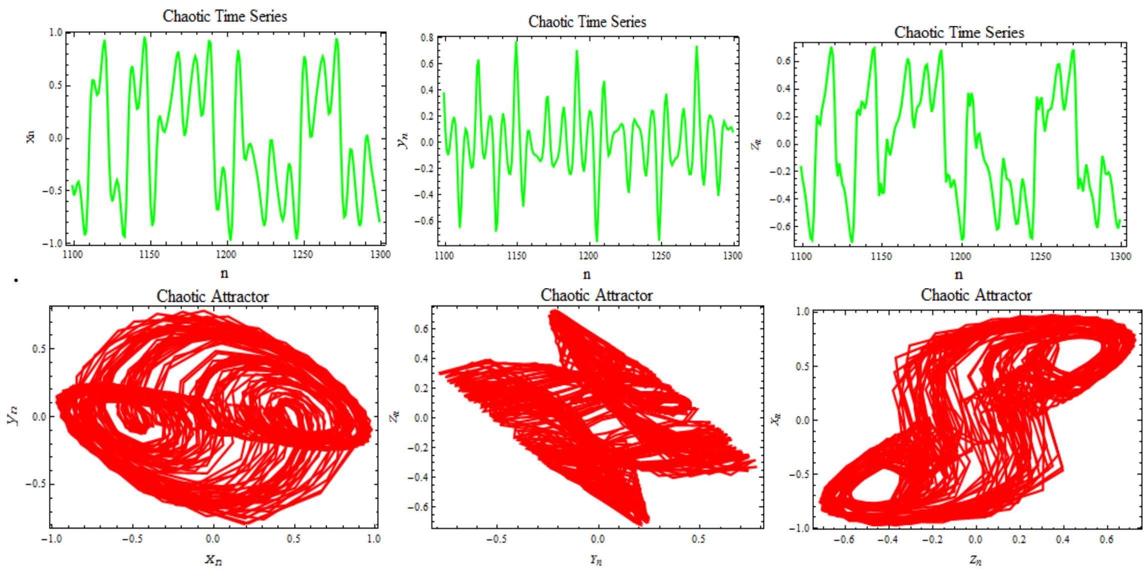

Figure 6. Time series graphs and Chaotic Attractor Showing chaos in ACT map (10).

Plot of Lyapunov exponents (LCE) with a = 0.6, b = 0.5, c = 0.41, d = 1, e = 1, k = 3 and x=0.55, y=0.01, z=0.5 is given by Figure 7.

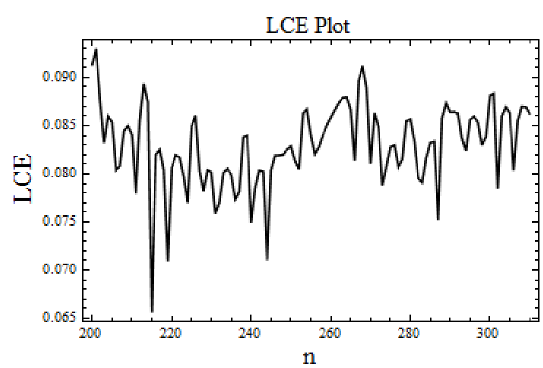

Figure 7. LCE plot for ACT map in chaotic case.

With such a desired initial point $(x_0, y_0, z_0) = (0.55, 0.01, 0.5)$, we proceed for steps of asymptotic stability analysis and obtained

$$A_R = \begin{bmatrix} 0.6 & -0.5 & 0.5 \\ 0.5 & 0.6 & -0.6 \\ -0.819998 & 0 & 1 \end{bmatrix} \qquad A_D = \begin{bmatrix} 0.6 & -0.5 & 0.5 \\ 0.5 & 0.6 & -0.6 \\ -0.4975 & 0 & 1 \end{bmatrix}$$

$$B_R = \begin{bmatrix} 0.640312 & -0.51225 & 0 \\ -0.51225 & 0.640312 & 0 \\ 0 & 0 & 0.640312 \end{bmatrix} \qquad B_D = \begin{bmatrix} 0.55 & 0.49 & 0 \\ -0.49 & 0.55 & 0 \\ 0 & 0 & 0.55 \end{bmatrix}$$

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$$C_M = \begin{bmatrix} 1.11164 & -0.590039 & 0 \\ 0.590039 & 1.11164 & 0 \\ 0 & 0 & 1.11164 \end{bmatrix} \quad C_R = \begin{bmatrix} -0.856985 & -0.145595 & 1.04865 \\ 0.145595 & -0.856985 & -0.156653 \\ -1.49091 & 0 & 0 \end{bmatrix}$$

$$C_D = \begin{bmatrix} -0.856985 & -0.145595 & 1.04865 \\ 0.145595 & -0.856985 & -0.156653 \\ -0.904545 & 0 & 0 \end{bmatrix} \quad p^* = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.585951 \\ 0.59804 \\ 0.0201784 \end{bmatrix}$$

with these new parameter values, a = 0.585951, b = 0.59804, c = 0.0201784 together with d = e = 1, k = 3, x=0.55, y = 0.01, z = 0.5 the chaotic system shown earlier is controlled and evolve regularly. A phase diagram and the plot for Lyapunov exponents for this case are given in Figure 8.

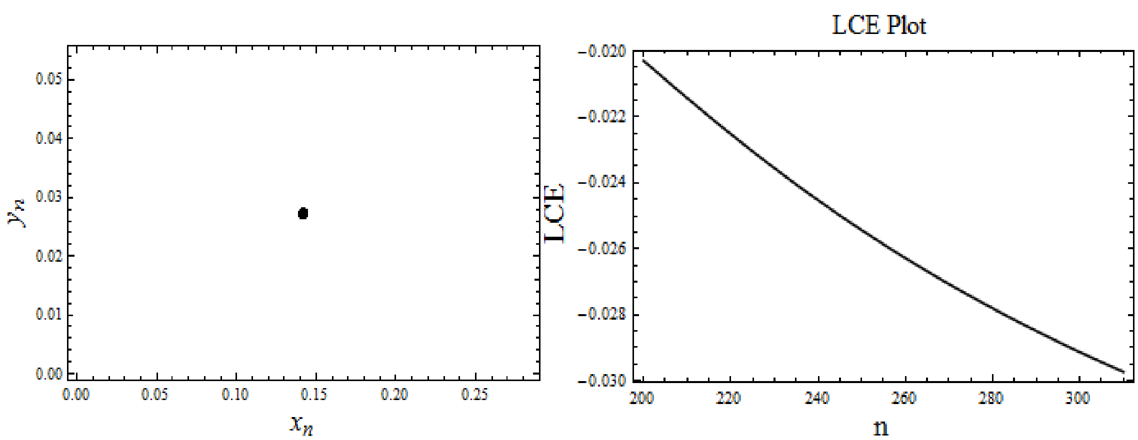

Figure 8. Phase Plot in XY plane and the plot for Lyapunov exponents.

Clearly phase plot on XY plane shows a single point and same is the case with other two planes. Also, the values of Lyapunov Exponents all are negative. This shows that the system is no more chaotic and showing one periodic regular case. Hence, chaos is controlled.

4. Limitation

The method of asymptotic analysis to control chaos has some limitations & it does not work for all dynamical systems. Suppose we desire to have a solution of system of equations

$$AX = B,$$

where A and B are known matrices of order, m × n and m × r respectively and X is unknown matrix of order n × r which is to be determined. X can be written as X≡ ($X_1, X_2 ... X_r$) and B as B≡ (B_1, B_2, Br) where X_i (i = 1,2,3, r) are n × 1 columns and B_i (i = 1,2,3…, r) are m × 1 columns.
The system A X = B will be consistent if and only if

Rank of A = Rank of augmented matrix (A, Bi), ∀ i =1, 2, 3, r (11)

Considering equation (5), (6), (7) again

$$B_D\ (C_D - C_R) = A_D - A_R \tag{12}$$

$$B_D\ C_M = B_R \tag{13}$$

$$A_R - B_D\ C_R = -I \tag{14}$$

Equations (12), (13) and (14) are consistent if and only if each one of them satisfies condition(s) giving by equation (11).For example (Saha et al., 2004), consider the model:

$$\begin{aligned} "f\ (x, y) &= x + y + k\ \sin\ x, \\ g\ (x, y) &= y + k\ \sin\ x" \end{aligned} \tag{15}$$

Fixed points for this map are given by ($\pm$ n π, 0). The fixed point (0, 0) for k=1 is unstable & proceeding above calculations one finds matrix $B_D = 0$, hence Rank (B_D) = 0.

But, Rank (B_D: B_R) is not equal to zero. So, the equation $B_R = B_D C_M$ is inconsistent in this case. Similarly, there are many more systems where this technique fails.

5. Conclusion

We observed that the asymptotic stability method can be used appropriately to have chaos control for maps with n dimensions and m parameters such that m ≥ n. One can fix the values of some m-n parameters in case m > n, with appropriate considerations. This method could be applied in systems of 2, 3 or higher dimensions. However, it does have its drawbacks and it cannot be used for every chaotic system. Therefore, there is no evergreen strategy for treating chaos management. In order to have stability, different methods are needed in different non-linear systems.

Declaration of Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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