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Admissible fundamental operators ☆

Tirthankar Bhattacharyya [a], Sneh Lata [b], Haripada Sau [a,*]

[a] *Department of Mathematics, Indian Institute of Science, Bangalore 560012, India*
[b] *Department of Mathematics, Shiv Nadar University, School of Natural Sciences, Gautam Budh Nagar 203207, Uttar Pradesh, India*

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ABSTRACT

Let F and G be two bounded operators on two Hilbert spaces. Let their numerical radii be no greater than one. This note investigates when there is a Γ-contraction (S,P) such that F is the fundamental operator of (S,P) and G is the fundamental operator of (S^*,P^*). Theorem 1 puts a necessary condition on F and G for them to be the fundamental operators of (S,P) and (S^*,P^*) respectively. Theorem 2 shows that this necessary condition is also sufficient provided we restrict our attention to a certain special case. The general case is investigated in Theorem 3. Some of the results obtained for Γ-contractions are then applied to tetrablock contractions to figure out when two pairs (F_1,F_2) and (G_1,G_2) acting on two Hilbert spaces can be fundamental operators of a tetrablock contraction (A,B,P) and its adjoint (A^*,B^*,P^*) respectively. This is the content of Theorem 3.

1. Introduction

The symmetrized bidisc is

$$\Gamma = \big\{(z_1+z_2, z_1z_2) : |z_1|, |z_2| \le 1\big\}.$$

Its distinguished boundary, i.e., the Shilov boundary with respect to the algebra of functions continuous on Γ and holomorphic in the interior of Γ is $b\Gamma = \{(z_1+z_2, z_1z_2) : |z_1| = 1 = |z_2|\}$. A pair of commuting bounded operators (S,P) on a Hilbert space $\mathcal{H}$ having the symmetrized bidisc as a spectral set is called a Γ-contraction. This means that the joint spectrum $\sigma(S,P) \subset \Gamma$ and

$$\big\|f(S,P)\big\| \le \sup\big\{\big|f(s,p)\big| : (s,p) \in \Gamma\big\}$$

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* Corresponding author.
E-mail addresses: tirtha@math.iisc.ernet.in (T. Bhattacharyya), sneh.lata@snu.edu.in (S. Lata), sau10@math.iisc.ernet.in (H. Sau).

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for all $f \in \mathbb{C}[z_1, z_2]$. The study of Γ-contractions was introduced and carried out very successfully over several papers by Agler and Young, see [3] and the references therein. It follows that the operator P is a contraction and $\|S\| \leq 2$. It can be seen directly from the definition that (S^*, P^*) is a Γ contraction too. Let $D_P = (I - P^*P)^{1/2}$ and $\mathcal{D}_P = \overline{Ran} D_P$. The fundamental operator is the unique bounded operator on $\mathcal{D}_P$ that satisfies the fundamental equation

$$S - S^*P = D_P F D_P.$$

It has numerical radius $w(F)$ no greater than one. The fundamental operator of a Γ-contraction was introduced in [8]. There it is shown that the fundamental equation has a unique solution. The discovery of the fundamental operator of a Γ-contraction put a spurt in the activities around it. In particular, we would like to mention Sarkar's work [11] which made a significant contribution to the understanding of Γ-contractions.

In this paper, $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ will denote the algebra of all bounded operators on $\mathcal{H}$. Since (S^*, P^*) is also a Γ-contraction, it has its own fundamental operator $G \in \mathcal{B}(\mathcal{D}_{P^*})$ with $w(G) \leq 1$. Note how both F and G feature in the following explicit construction of a boundary normal dilation.

A boundary normal dilation of a Γ-contraction (S, P) is a pair of commuting normal operators (R, U) on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that (R, U) is a *dilation* of the given pair (S, P) and $\sigma(R, U)$, the joint spectrum is contained in the distinguished boundary $b\Gamma$. *Dilation* means that

$$P_{\mathcal{H}} R^m U^n \big|_{\mathcal{H}} = S^m P^n.$$

Such a pair (R, U) is also called a Γ-unitary. The following construction, done by two of the authors of the present paper in [9] and independently by Pal in [10], is one of the very few explicit constructions of dilations known, the only other ones being Schaeffer's construction of the minimal unitary dilation of a contraction in [13] and Ando's construction of a commuting unitary dilation of a pair of commuting bounded operators in [4].

Known Theorem. *Let (S, P) be a Γ-contraction. Let F and G be the fundamental operators of (S, P) and (S^*, P^*) respectively. Consider the space $\mathcal{K}$ defined as*

$$\mathcal{K} = \cdots \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \mathcal{H} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \mathcal{D}_{P^*} \oplus \cdots.$$

Let R and U be defined on $\mathcal{K}$ as follows.

$$R = \left[\begin{array}{cccc|c|cccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & F & F^* & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & F & F^* & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & F & F^* D_P & -F^* P^* & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & S & D_{P^*} G & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 0 & 0 & G^* & G & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & G^* & G & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & G^* & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right], \tag{1.1}$$

$$U = \left[\begin{array}{cccc|c|cccc} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & I & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & I & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & D_P & -P^* & 0 & 0 & \cdots \\ \hline \cdots & 0 & 0 & 0 & P & D_{P^*} & 0 & 0 & \cdots \\ \hline \cdots & 0 & 0 & 0 & 0 & 0 & I & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & I & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]. \tag{1.2}$$

Then the pair (R, U) is a Γ-unitary dilation of (S, P).

This shows that it is of interest to know which pair of operators F and G, defined on different Hilbert spaces in general, satisfying $w(F) \leq 1$ and $w(G) \leq 1$, qualify as fundamental operators. In other words, does there always exist a Γ-contraction (S, P) such that F is the fundamental operator of (S, P) and G is the fundamental operator of (S^*, P^*)? In this note, our first result says that if there is such an (S, P), then it forces a relation between F, G and P.

For a contraction P on a Hilbert space $\mathcal{H}$, define

$$\Theta_P(z) = \left[-P + zD_{P^*}\left(I_{\mathcal{H}} - zP^*\right)^{-1} D_P\right]\Big|_{\mathcal{D}_P} \quad \text{for all } z \in \mathbb{D}.$$

The function Θ_P is called the *characteristic function* of the contraction P. By virtue of the relation $PD_P = D_{P^*}P$ (see Chapter 1, Section 3 of [14]), it follows that $\Theta_P(z)$ is an operator from $\mathcal{D}_P$ into $\mathcal{D}_{P^*}$. For a given Hilbert space $\mathcal{E}$, the symbol $H^2_{\mathcal{E}}(\mathbb{D})$ stands for the Hilbert space of $\mathcal{E}$ valued analytic functions on $\mathbb{D}$ with square summable Taylor series coefficients at the origin. The characteristic function induces the operator M_{Θ_P} in $\mathcal{B}(H^2_{\mathcal{D}_P}(\mathbb{D}), H^2_{\mathcal{D}_{P^*}}(\mathbb{D}))$ defined by

$$M_{\Theta_P} f(z) = \Theta_P(z) f(z) \quad \text{for all } z \in \mathbb{D}.$$

Theorem 1. *Let (S, P) on a Hilbert space $\mathcal{H}$ be a Γ-contraction and F, G be the fundamental operators of (S, P) and (S^*, P^*) respectively. Then*

$$\Theta_P(z)\left(F + F^*z\right) = \left(G^* + Gz\right)\Theta_P(z) \tag{1.3}$$

holds, where Θ_P is the characteristic function of P.

Since the theorem above gives a necessary condition, it is natural to ask about sufficiency. A contraction P is called *pure* if P^{*n} strongly converges to 0 as n goes to infinity. This is Arveson's terminology, see [5]. Sz.-Nagy and Foias called it a $C_{\cdot 0}$ contraction.

A Γ-contraction (S, P) is called pure if the contraction P is pure.

Theorem 2. *Let P be a pure contraction on a Hilbert space $\mathcal{H}$. Let $F \in \mathcal{B}(\mathcal{D}_P)$ and $G \in \mathcal{B}(\mathcal{D}_{P^*})$ be two operators with numerical radius not greater than one. If (1.3) holds, then there exists an operator S on $\mathcal{H}$ such that (S, P) is a Γ-contraction and F, G are fundamental operators of (S, P) and (S^*, P^*) respectively.*

A contraction P is called *completely-non-unitary* (c.n.u.) if it has no reducing subspaces on which its restriction is unitary.

A Γ-contraction (S, P) is called completely-non-unitary if the contraction P is completely-non-unitary.

If P is not pure, the sufficiency condition is more complicated. The result for the c.n.u. case will be stated and proved in Section 3.

In the last section, we study when two pairs of operators can be fundamental operators of a tetrablock contraction and its adjoint. The set *tetrablock* is defined by

$$E = \{\underline{x} = (x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \text{ whenever } |z| < 1 \text{ and } |w| < 1\}.$$

See [1] and [2] to learn more about the geometric properties of the domain. A commuting triple of operators (A, B, P) on a Hilbert space $\mathcal{H}$ is called a tetrablock contraction if $\overline{E}$ is a spectral set. Like Γ-contractions, tetrablock contractions also possess fundamental operators and these are introduced in [6]. Fundamental equations for a tetrablock contraction are

$$A - B^*P = D_P F_1 D_P \quad \text{and} \quad B - A^*P = D_P F_2 D_P \tag{1.4}$$

where F_1, F_2 are bounded operators on $\mathcal{D}_P$. Theorem 1.3 in [6] says that the two fundamental equations can be solved and the solutions F_1 and F_2 are unique. The unique solutions F_1 and F_2 of Eqs. (1.4) are called the *fundamental operators* of the tetrablock contraction (A, B, P). Moreover, $w(F_1)$ and $w(F_2)$ are not greater than 1.

The adjoint triple (A^*, B^*, P^*) is also a tetrablock contraction as can be seen from the definition. By what we stated above, there are unique $G_1, G_2 \in \mathcal{B}(\mathcal{D}_{P^*})$ such that

$$A^* - BP^* = D_{P^*} G_1 D_{P^*} \quad \text{and} \quad B^* - AP^* = D_{P^*} G_2 D_{P^*}. \tag{1.5}$$

Moreover, $w(G_1)$ and $w(G_2)$ are not greater than 1. A tetrablock contraction (A, B, P) on a Hilbert space $\mathcal{H}$ is called pure tetrablock contraction, if the contraction P is pure. Along the lines of [7], a model theory for pure tetrablock contractions was developed in [12], using the fundamental operators. Our result for tetrablock contractions is the following.

Theorem 3. *Let F_1 and F_2 be fundamental operators of a tetrablock contraction (A, B, P) and G_1 and G_2 be fundamental operators of the tetrablock contraction (A^*, B^*, P^*). Then*

$$\big(G_1^* + G_2 z\big)\Theta_P(z) = \Theta_P(z)\big(F_1 + F_2^* z\big) \tag{1.6}$$

$$\big(G_2^* + G_1 z\big)\Theta_P(z) = \Theta_P(z)\big(F_2 + F_1^* z\big) \quad \text{for all } z \in \mathbb{D}. \tag{1.7}$$

Conversely, let P be a pure contraction on a Hilbert space $\mathcal{H}$. Let $G_1, G_2 \in \mathcal{B}(\mathcal{D}_{P^*})$ have numerical radii no greater than one and satisfy

$$[G_1, G_2] = 0 \quad \text{and} \quad [G_1, G_1^*] = [G_2, G_2^*]. \tag{1.8}$$

Suppose G_1 and G_2 also satisfy Eqs. (1.6) and (1.7), for some operators $F_1, F_2 \in \mathcal{B}(\mathcal{D}_P)$ with numerical radii no greater than one. Then there exists a tetrablock contraction (A, B, P) such that F_1, F_2 are fundamental operators of (A, B, P) and G_1, G_2 are fundamental operators of (A^*, B^*, P^*).

2. Results for pure Γ-contractions

Definition 4. Let $\mathcal{F}$ and $\mathcal{G}$ be two Hilbert spaces. Let $F \in \mathcal{B}(\mathcal{F})$ and $G \in \mathcal{B}(\mathcal{G})$. Then (F, G) is called an admissible pair of operators if there is a Γ-contraction (S, P) on a Hilbert space $\mathcal{H}$ such that $\mathcal{D}_P = \mathcal{F}$, $\mathcal{D}_{P^*} = \mathcal{G}$, F is the fundamental operator of (S, P) and G is the fundamental operator of (S^*, P^*).

The Hilbert spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are unitarily equivalent via the map $z^n \mapsto e^{int}$. Further, for a given Hilbert space $\mathcal{L}$, $H^2_{\mathcal{L}}(\mathbb{D})$ (respectively $H^2_{\mathcal{L}}(\mathbb{T})$) is unitarily equivalent to $H^2(\mathbb{D}) \otimes \mathcal{L}$ (respectively $H^2(\mathbb{T}) \otimes \mathcal{L}$). We shall identify these unitarily equivalent spaces and use them, without mention, interchangeably as per notational convenience.

The following useful characterization of the fundamental operator can be found in [6] (Lemma 4.1).

Lemma 5. *Let (S, P) be a Γ-contraction on a Hilbert space $\mathcal{H}$ and $F \in \mathcal{B}(\mathcal{D}_P)$ be its fundamental operator. Then F is the only operator which satisfies*

$$D_P S = F D_P + F^* D_P P. \tag{2.1}$$

The next lemma gives relations between the fundamental operators of Γ-contractions (S, P) and (S^*, P^*). These can be found in [9] (Lemma 7 and Lemma 11).

Lemma 6. *Let (S, P) be a Γ-contraction and F, G be fundamental operators of (S, P) and (S^*, P^*) respectively. Then*

$$PF = G^* P|_{\mathcal{D}_P} \quad \text{and} \quad D_{P^*} D_P F - P F^* = G^* D_{P^*} D_P - G P|_{\mathcal{D}_P}.$$

Proof of Theorem 1. For $z \in \mathbb{D}$, we have

$$\begin{aligned}
&\Theta_P(z)\big(F + F^* z\big) \\
&\quad = \left[-P + \sum_{n=0}^{\infty} z^{n+1} D_{P^*} P^{*n} D_P\right]\big(F + F^* z\big) \\
&\quad = -PF + z\big(D_{P^*} D_P F - P F^*\big) + \sum_{n=1}^{\infty} z^{n+1} D_{P^*} P^{*n} D_P F + \sum_{n=0}^{\infty} z^{n+2} D_{P^*} P^{*n} D_P F^* \\
&\quad = -PF + z\big(D_{P^*} D_P F - P F^*\big) + \sum_{n=2}^{\infty} z^{n} D_{P^*} P^{*n-2}\big(P^* D_P F + D_P F^*\big) \\
&\quad = -PF + z\big(D_{P^*} D_P F - P F^*\big) + \sum_{n=2}^{\infty} z^{n} D_{P^*} P^{*n-2} S^* D_P \quad [\text{by Lemma 5}] \\
&\quad = -PF + z\big(D_{P^*} D_P F - P F^*\big) + \sum_{n=2}^{\infty} z^{n} D_{P^*} S^* P^{*n-2} D_P.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\big(G^* + Gz\big)\Theta_P(z) \\
&\quad = \big(G^* + Gz\big)\left[-P + \sum_{n=0}^{\infty} z^{n+1} D_{P^*} P^{*n} D_P\right]\Bigg|_{\mathcal{D}_P} \\
&\quad = -G^* P|_{\mathcal{D}_P} + z\big(G^* D_{P^*} D_P - G P|_{\mathcal{D}_P}\big) + \sum_{n=1}^{\infty} z^{n+1} G^* D_{P^*} P^{*n} D_P + \sum_{n=0}^{\infty} z^{n+2} G D_{P^*} P^{*n} D_P \\
&\quad = -G^* P|_{\mathcal{D}_P} + z\big(G^* D_{P^*} D_P - G P|_{\mathcal{D}_P}\big) + \sum_{n=2}^{\infty} z^{n}\big(G^* D_{P^*} P^* + G D_{P^*}\big) P^{*n-2} D_P \\
&\quad = -G^* P|_{\mathcal{D}_P} + z\big(G^* D_{P^*} D_P - G P|_{\mathcal{D}_P}\big) + \sum_{n=2}^{\infty} z^{n} D_{P^*} S^* P^{*n-2} D_P.
\end{aligned}$$

Now the equality in Eq. (1.3) follows from Lemma 6. This completes the proof. □

Define $W : \mathcal{H} \to H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ by $W(h) = \sum_{n=0}^{\infty} z^n \otimes D_{P^*}{P^*}^n h$ for all $h \in \mathcal{H}$. Note that

$$\left\|W(h)\right\|^2 = \sum_{n=0}^{\infty}\left\|D_{P^*}{P^*}^n h\right\|^2 = \sum_{n=0}^{\infty}\left(\left\|{P^*}^n h\right\|^2 - \left\|{P^*}^{n+1} h\right\|^2\right) = \|h\|^2 - \lim_{n\to\infty}\left\|{P^*}^n h\right\|^2.$$

Therefore W is an isometry in the case when P is pure. It is easy to verify that

$$W^*\left(z^n \otimes \xi\right) = P^n D_{P^*}\xi \quad \text{for all } \xi \in \mathcal{D}_{P^*} \text{ and } n \geq 0.$$

It is well known that

Lemma 7. *For every contraction P, the identity*

$$WW^* + M_{\Theta_P} M_{\Theta_P}^* = I_{H^2(\mathbb{D})\otimes\mathcal{D}_{P^*}} \tag{2.2}$$

holds.

See [7] for a proof of Lemma 7.

Proof of Theorem 2. Since P is pure, W is an isometry. We first find a relation between P, W and M_z, the operator of multiplication by z on $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$.

$$M_z^* W h = M_z^*\left(\sum_{n=0}^{\infty} z^n D_{P^*}{P^*}^n h\right) = \sum_{n=0}^{\infty} z^n D_{P^*}{P^*}^{n+1} h = W P^* h. \tag{2.3}$$

Therefore $M_z^* W = W P^*$. Define S on $\mathcal{H}$ by $S = W^* M_{G^*+Gz} W$. Since P is pure, from Lemma 7, we have $(Ran W)^\perp = Ran M_{\Theta_P}$. The equation $M_{\Theta_P} M_{F+F^*z} = M_{G^*+Gz} M_{\Theta_P}$ implies that $Ran M_{\Theta_P}$ is invariant under M_{G^*+Gz}, in other words $Ran W$ is co-invariant under M_{G^*+Gz}. We next show that S and P commute.

$$\begin{aligned}
P^* S^* &= W^* M_z^* W W^* M_{G^*+Gz}^* W \\
&= W^* M_z^* M_{G^*+Gz}^* W \quad \left[\text{since } WW^* \text{ is a projection onto } Ran W\right] \\
&= W^* M_{G^*+Gz}^* M_z^* W \quad \left[\text{since } M_z \text{ and } M_{G^*+Gz} \text{ commute}\right] \\
&= W^* M_{G^*+Gz}^* W W^* M_z^* W = S^* P^*.
\end{aligned}$$

Furthermore

$$\begin{aligned}
S^* - S P^* &= W^* M_{G^*+Gz}^* W - W^* M_{G^*+Gz} W W^* M_z^* W \\
&= W^*\left(I \otimes G + M_z^* \otimes G^*\right)W - W^*\left(I \otimes G^* + M_z \otimes G\right)\left(M_z^* \otimes I\right)W \\
&= W^*\left(I \otimes G + M_z^* \otimes G^*\right)W - W^*\left(M_z^* \otimes G^* + M_z M_z^* \otimes G\right)W \\
&= W^*(P_{\mathbb{C}} \otimes G)W \quad \left[P_{\mathbb{C}} \text{ is the projection of } H^2(\mathbb{D}) \text{ onto constants}\right] \\
&= D_{P^*} G D_{P^*}.
\end{aligned}$$

For all $\theta \in (0, 2\pi]$, we have $G^* + e^{i\theta} G = e^{i\frac{\theta}{2}}(e^{-i\frac{\theta}{2}} G^* + e^{i\frac{\theta}{2}} G)$. Hence $\|G^* + e^{i\theta} G\| = \|(e^{-i\frac{\theta}{2}} G^* + e^{i\frac{\theta}{2}} G)\|$. Note that for all $\theta \in (0, 2\pi]$ and $\xi \in \mathcal{D}_{P^*}$ we have

$$\begin{aligned}\left|\left\langle\left(e^{-i\frac{\theta}{2}}G^* + e^{i\frac{\theta}{2}}G\right)\xi,\xi\right\rangle\right| &= \left|e^{-i\frac{\theta}{2}}\left\langle G^*\xi,\xi\right\rangle + e^{i\frac{\theta}{2}}\langle G\xi,\xi\rangle\right| \\ &\le \left|\left\langle G^*\xi,\xi\right\rangle\right| + \left|\langle G\xi,\xi\rangle\right| \le 2 \quad \left[\text{since } w(G)\le 1\right].\end{aligned}$$

Since $(e^{-i\frac{\theta}{2}}G^*+e^{i\frac{\theta}{2}}G)$ is a self-adjoint operator, we have $\|(e^{-i\frac{\theta}{2}}G^*+e^{i\frac{\theta}{2}}G)\| \le 2$. Therefore $\|(G^*+Gz)\| \le 2$ for all $z \in \mathbb{D}$, which implies that $\|M_{G^*+Gz}\| \le 2$. Hence $\|S\| \le 2$.

Hence (S^*, P^*) is a commuting pair of operators on $\mathcal{H}$ such that the spectral radius of S is not greater than two and the operator equation $S^* - SP^* = D_{P^*}XD_{P^*}$ has a solution for X (viz. G) with numerical radius of X not greater than one. Therefore by Theorem 4.4 in [8], (S^*, P^*) is a Γ-contraction and hence so is (S, P) as observed in the introduction.

Now we will show that F is the fundamental operator of (S, P). Note that if X is the fundamental operator of (S, P), then by Theorem 1 we have $M_{\Theta_P}M_{X+X^*z} = M_{G^*+Gz}M_{\Theta_P}$. Also by hypothesis we have $M_{\Theta_P}M_{F+F^*z} = M_{G^*+Gz}M_{\Theta_P}$. Since P is pure contraction, M_{Θ_P} is an isometry and hence we have $M_{X+X^*z} = M_{F+F^*z}$ on $H^2_{\mathcal{D}_P}(\mathbb{D})$, which implies that $X = F$. Therefore F is the fundamental operator of (S, P). This completes the proof of the theorem. □

Remark 8. Theorem 2 shows that given a contraction P and two bounded operators F and G in $\mathcal{B}(\mathcal{D}_P)$ and $\mathcal{B}(\mathcal{D}_{P^*})$ respectively, there need not always exist an S such that (S, P) is a Γ-contraction, F is its fundamental operator and G is the fundamental operator of (S^*, P^*).

We would like to remark that given a pure contraction P and $G \in \mathcal{B}(\mathcal{D}_{P^*})$ with $w(G) \le 1$, there is only one S such that (S^*, P^*) is a Γ-contraction with fundamental operator G. The proof is as follows.

Let S and S' be two different operators such that (S^*, P^*) and $({S'}^*, P^*)$ are Γ-contractions with the same fundamental operator G. Since P is a pure contraction, by Theorem 2.1 in [7], both S and S' are unitarily equivalent to $P_{\mathbb{H}_P}M_{G^*+Gz}|_{\mathbb{H}_P}$, where $\mathbb{H}_P = Ran W$ and the same unitary $W : \mathcal{H} \to Ran W$ works for both operators S and S'. Hence $S = S'$.

3. Results for completely-non-unitary Γ-contractions

In this section we shall prove a version of Theorem 2 that holds for the c.n.u. case. We first recall two minimal isometric dilations of a c.n.u. contraction. Let $P \in \mathcal{B}(\mathcal{H})$ be a c.n.u. contraction.

(i) Note that

$$I \ge PP^* \ge P^2{P^*}^2 \ge \cdots \ge P^n{P^*}^n \ge \cdots \ge 0.$$

Therefore there exists a positive bounded operator, say P^2_∞, such that $P^2_\infty h = \lim_{n\to\infty} P^n P^{*n} h$ for all $h \in \mathcal{H}$. Then $PP^2_\infty P^* = P^2_\infty$, which implies that $\|P_\infty h\| = \|P_\infty P^* h\|$ for all h. This defines an isometry $T \in \mathcal{B}(\overline{Ran(P_\infty)})$ such that $TP_\infty = P_\infty P^*$. Let $U \in \mathcal{B}(\mathcal{K})$ be the minimal unitary extension of T. Then $\Pi_0 : \mathcal{H} \to H^2_{\mathcal{D}_{P^*}}(\mathbb{D}) \oplus \mathcal{K}$, defined as

$$\Pi_0(h) = \begin{pmatrix} W(h) \\ P_\infty \end{pmatrix},$$

is an isometry, where $W : \mathcal{H} \to H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$, $W(h) = \sum_{n=0}^\infty z^n D_{P^*} P^{*n} h$. We can check that $\left(\begin{smallmatrix} M_z\otimes I & 0 \\ 0 & U^* \end{smallmatrix}\right)$ is a minimal isometric dilation of $\Pi_0 P \Pi_0^*$ and

$$\Pi_0 P^* = \begin{pmatrix} M_z \otimes I & 0 \\ 0 & U^* \end{pmatrix}^* \Pi_0. \tag{3.1}$$

(ii) For all $t \in [0, 2\pi)$ define the operator

$$\Delta_P(t) = \left[I - \Theta_P\left(e^{it}\right)^* \Theta_P\left(e^{it}\right)\right]^{\frac{1}{2}}$$

where Θ_P is the characteristic function of P introduced in Section 1. Consider the subspace

$$\mathcal{S}_P = \left\{M_{\Theta_P} f \oplus \Delta_P f : f \in H^2_{\mathcal{D}_P}(\mathbb{D})\right\}.$$

Then $\mathcal{S}_P$ is a closed subspace of $H^2_{\mathcal{D}_{P^*}}(\mathbb{D}) \oplus \overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}$. Let $\mathcal{Q}_P$ be the orthogonal complement of $\mathcal{S}_P$ in $H^2_{\mathcal{D}_{P^*}(\mathbb{D})} \oplus \overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}$.

There exists an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_{P^*}}(\mathbb{D}) \oplus \overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}$ with $\Pi(\mathcal{H}) = \mathcal{Q}_P$ such that $\left(\begin{smallmatrix} M_z & 0 \\ 0 & M_{e^{it}} \end{smallmatrix}\right)$ is a minimal isometric dilation of $\Pi P \Pi^*$ and

$$\Pi P^* = \begin{pmatrix} M_z & 0 \\ 0 & M_{e^{it}} \end{pmatrix}^* \Pi. \tag{3.2}$$

Thus Π and Π_0 give two minimal isometric dilations of P. But the minimal dilation is unique up to unitary equivalence. Thus we get a unitary $\Phi : H^2_{\mathcal{D}_{P^*}}(\mathbb{D}) \oplus \overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})} \longrightarrow H^2_{\mathcal{D}_{P^*}}(\mathbb{D}) \oplus \mathcal{K}$, such that $\Phi\Pi = \Pi_0$ and

$$\Phi \begin{pmatrix} M_z & 0 \\ 0 & M_{e^{it}} \end{pmatrix}^* = \begin{pmatrix} M_z \otimes I & 0 \\ 0 & U^* \end{pmatrix}^* \Phi. \tag{3.3}$$

Since Φ is unitary and satisfies (3.3), by an easy matrix calculation and the fact that any operator intertwining a pure isometry and a unitary is zero (Lemma 2.5 in [3]), we get Φ to be of the form

$$\Phi = \begin{pmatrix} I \otimes V_1 & 0 \\ 0 & V_2 \end{pmatrix} \tag{3.4}$$

where $V_1 \in \mathcal{B}(\mathcal{D}_{P^*})$ and $V_2 \in \mathcal{B}(\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}, \mathcal{K})$ are unitary operators.

Lemma 9. *Let P be a c.n.u. Γ-contraction on $\mathcal{H}$. Let $X \in \mathcal{B}(\mathcal{D}_{P^*})$, $w(X) \le 1$ and $R \in \mathcal{B}(\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})})$ such that $(R, M_{e^{it}})$ is a Γ-unitary on $\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}$. If*

$$\begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix} \mathcal{S}_P \subseteq \mathcal{S}_P, \tag{3.5}$$

then there exists $Y \in \mathcal{B}(\mathcal{D}_P)$ with $w(Y) \le 1$ such that

$$\begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} = \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} M_{Y+zY^*}.$$

Proof. Eq. (3.5) allows us to define an operator $T \in \mathcal{B}(H^2_{\mathcal{D}_P}(\mathbb{D}))$ so that

$$\begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} = \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} T. \tag{3.6}$$

In other words,

$$T = \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix}^* \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix}. \tag{3.7}$$

To prove the result, it is enough to show that (T, M_z) is a Γ-isometry. Since $w(X) \le 1$, as shown in the previous section, we have $\|M_{X^*+zX}\| \le 2$. Also, $(R, M_{e^{it}})$ is a Γ-unitary, therefore $\|R\| \le 2$. Thus, from Eq. (3.6), we can easily deduce that $\|T\| \le 2$, since the operator $\left(\begin{smallmatrix} M_{\Theta_P} \\ \Delta_P \end{smallmatrix}\right)$ is an isometry. We shall now show that T commutes with M_z.

From Eq. (3.6) we have

$$M_{X^*+zX} M_{\Theta_P} = M_{\Theta_P} T \tag{3.8}$$

$$R\Delta_P = \Delta_P T. \tag{3.9}$$

Note that M_z commute with M_{X^*+zX} and M_{Θ_P}. Therefore applying M_z on both sides of Eq. (3.8) we get

$$M_{\Theta_P} T M_z = M_{\Theta_P} M_z T. \tag{3.10}$$

Also, $M_{e^{it}}|_{\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}}$ commutes with R and Δ_P, therefore applying $M_{e^{it}}$ on both sides of Eq. (3.9) we get

$$\Delta_P T M_z = \Delta_P M_z T. \tag{3.11}$$

Eqs. (3.10) and (3.11) together with the fact that $\left(\begin{smallmatrix} M_{\Theta_P} \\ \Delta_P \end{smallmatrix}\right)$ is an isometry yield $TM_z = M_z T$.

Lastly, we shall show that $T = T^* M_z$. To accomplish this, consider

$$\begin{aligned} M_z^* T &= M_z^* \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix}^* \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} \\ &= \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix}^* \begin{pmatrix} M_z^* & 0 \\ 0 & M_{e^{it}}^* \end{pmatrix} \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} \\ &= T^*. \end{aligned}$$

Consequently, $M_z^* T = T^*$, that is, $T = T^* M_z$. Therefore we can conclude that (T, M_z) is a Γ-isometry. Agler and Young showed in [3] that the only way this can happen is that T is of the form M_{Y+zY^*} for some $Y \in \mathcal{B}(\mathcal{D}_P)$, $w(Y) \le 1$. This completes the proof. □

The next result, apart from its usefulness in proving the main theorem of this section, is interesting in its own right and depends on the beautiful model theory for a Γ-contraction developed by Agler and Young in [3]. They proved, by a Stinespring like method, that if (S, P) is a Γ-contraction on a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ can be isometrically embedded in a Hilbert space $\mathcal{K}$ (by an isometry Π_{AY}, say) on which a Γ-isometry $(\tilde{S}, \tilde{P})$ acts such that the isometric image of $\mathcal{H}$ is a common invariant subspace of $\tilde{S}^*$ and $\tilde{P}^*$ and

$$\Pi_{AY} S^* = \tilde{S}^*|_{\Pi_{AY}\mathcal{H}}, \qquad \Pi_{AY} P^* = \tilde{P}^*|_{\Pi_{AY}\mathcal{H}}.$$

Moreover, the Γ-isometry $(\tilde{S}, \tilde{P})$ has a Wold decomposition, viz., $\mathcal{K}$ has an orthogonal decomposition $\mathcal{K}_1 \oplus \mathcal{K}_2$ such that $\mathcal{K}_1$ and $\mathcal{K}_2$ reduce both $\tilde{S}$ and $\tilde{P}$, the pair $(\tilde{S}|_{\mathcal{K}_1}, \tilde{P}|_{\mathcal{K}_1})$ is a pure Γ-isometry and

$$(\tilde{S}_u, \tilde{P}_u) \overset{\text{def}}{=} (\tilde{S}|_{\mathcal{K}_2}, \tilde{P}|_{\mathcal{K}_2})$$

is a Γ-unitary. In addition to this, the structure of a pure Γ-isometry was completely deciphered by them. It is as follows. There exists a Hilbert space $\mathcal{E}$ and a bounded operator Y on $\mathcal{E}$ such that $w(Y) \leq 1$ and $(\tilde{S}|_{\mathcal{K}_1}, \tilde{P}|_{\mathcal{K}_1})$ is unitarily equivalent to (T_ψ, T_z) acting on $H^2_{\mathcal{E}}(\mathbb{D})$, where $\psi \in L^\infty(\mathcal{B}(\mathcal{E}))$ is given by $\psi(z) = Y^* + Yz$ for all $z \in \mathbb{T}$. In short,

$$\Pi_{AY} S^* = \begin{pmatrix} M_{Y^*+zY} & 0 \\ 0 & \tilde{S}_u \end{pmatrix}^* \Pi_{AY} \quad \text{and} \quad \Pi_{AY} P^* = \begin{pmatrix} M_z & 0 \\ 0 & \tilde{P}_u \end{pmatrix}^* \Pi_{AY}. \tag{3.12}$$

Let P be a c.n.u. contraction and Π be as above. Then in Theorem 4.1 of [11], Sarkar showed that there is a unique isometry $\Psi : H^2_{\mathcal{D}_{P^*}}(\mathbb{D}) \oplus \overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})} \to \mathcal{K}_1 \oplus \mathcal{K}_2$ such that $\Pi_{AY} = \Psi \Pi$. Indeed, Ψ is defined by sending Πh to $\Pi_{AY} h$. What Sarkar showed next in Theorem 4.1 of [11] is significant for our purpose, viz., Ψ is of the form $(I_{H^2(\mathbb{D})} \otimes \hat{V}_1) \oplus \hat{V}_2$, for some isometries $\hat{V}_1 \in \mathcal{B}(\mathcal{D}_{P^*}, \mathcal{E})$ and $\hat{V}_2 \in \mathcal{B}(\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}, K_2)$. Taking all this into account, we have from (3.12),

$$\begin{aligned} \Pi S^* &= \big((I_{H^2(\mathbb{D})} \otimes \hat{V}_1^*) \oplus \hat{V}_2^*\big)\big((I_{H^2(\mathbb{D})} \otimes Y^* + M_z \otimes Y) \oplus \tilde{S}_u\big)^*\big((I_{H^2(\mathbb{D})} \otimes \hat{V}_1) \oplus \hat{V}_2\big)\Pi \\ &= \big((I_{H^2(\mathbb{D})} \otimes \hat{V}_1^* Y^* \hat{V}_1 + M_z \otimes \hat{V}_1^* Y \hat{V}_1) \oplus \hat{V}_2^* \tilde{S}_u \hat{V}_2\big)^* \Pi. \end{aligned}$$

Therefore writing $X = \hat{V}_1^* Y \hat{V}_1$ and $R = \hat{V}_2^* \tilde{S}_u \hat{V}_2$, we get the following neat relation

$$\Pi S^* = \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix}^* \Pi \tag{3.13}$$

for some operator $X \in \mathcal{B}(\mathcal{D}_{P^*})$ with $w(X) \leq 1$ and $R \in \mathcal{B}(\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})})$ such that $(R, M_{e^{it}}|_{\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}})$ is a Γ-unitary on $\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}$. We are going to show that X is unitarily equivalent to the fundamental operator of (S^*, P^*). Using (3.13) and (3.2) we get

$$\begin{aligned} S^* - SP^* &= \Pi^* \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix}^* \Pi \\ &\quad - \Pi^* \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix} \Pi \Pi^* \begin{pmatrix} M_z & 0 \\ 0 & M_{e^{it}} \end{pmatrix}^* \Pi \\ &= \Pi^* \begin{pmatrix} P_{\mathbb{C}} \otimes X & 0 \\ 0 & 0 \end{pmatrix} \Pi \quad \big[\text{since } (R, M_{e^{it}}|_{\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}}) \text{ is a } \Gamma\text{-unitary}\big] \\ &= \Pi_0^* \begin{pmatrix} P_{\mathbb{C}} \otimes (V_1 X V_1^*) & 0 \\ 0 & 0 \end{pmatrix} \Pi_0 \\ &= D_{P^*}(V_1 X V_1^*) D_{P^*}. \end{aligned}$$

Therefore $G = V_1 X V_1^*$ is the fundamental operator of (S^*, P^*). By Eq. (3.13) we have that $\Pi\mathcal{H} = \mathcal{Q}_P$ is an invariant subspace for $\begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix}^*$. In other words, $\mathcal{S}_P = \mathcal{Q}_P{}^\perp$ is invariant under $\begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix}$. Hence, using Lemma 9, we have proved the following.

Lemma 10. *Let (S, P) be a c.n.u. Γ-contraction. Then there exists $Y \in \mathcal{B}(\mathcal{D}_P)$ with $w(Y) \leq 1$ such that*

$$\begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} = \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} M_{Y+zY^*},$$

where X in the representation of S, i.e., Eq. (3.13), is unitarily equivalent to the fundamental operator for (S^*, P^*).

The following result reveals a beautiful and useful relation between the operators S, P and P_∞, when (S,P) is a special Γ-contraction.

Lemma 11. *Let (S,P) be a c.n.u. Γ-contraction such that $R = M_{e^{it}} + I = M_{e^{it}+I}$ in the representation (3.13) of S, then*

$$P_\infty^2 + PP_\infty^2 - PP_\infty^2 S^* = 0.$$

Proof. Let $R = M_{e^{it}+I}$. Using relations (3.2), (3.3), (3.13) and $\Phi\Pi = \Pi_0$ we can write

$$S = \Pi_0^* \begin{pmatrix} M_{G^*+zG} & 0 \\ 0 & U^*+I \end{pmatrix} \Pi_0 \quad \text{and} \quad P = \Pi_0^* \begin{pmatrix} M_z & 0 \\ 0 & U^* \end{pmatrix} \Pi_0,$$

where $G = V_1 X V_1^*$.

Consider

$$P^* + PP^* - PP^*S^* = \Pi_0^* \begin{pmatrix} M_z^* & 0 \\ 0 & U \end{pmatrix} \Pi_0 + \Pi_0^* \begin{pmatrix} M_z M_z^* & 0 \\ 0 & I \end{pmatrix} \Pi_0 - \Pi_0^* \begin{pmatrix} M_z M_z^* M_{G^*+zG}^* & 0 \\ 0 & U+I \end{pmatrix} \Pi_0.$$

Applying the property (3.1) of Π_0, we get

$$P^* + PP^* - PP^*S^* = P^* + PP^* - PP^*S^* - P_\infty^2 P^* - P_\infty^2 + P_\infty^2 S^*.$$

Hence, $P_\infty^2 P^* + P_\infty^2 - P_\infty^2 S^* = 0$, or equivalently, $P_\infty^2 + PP_\infty^2 - PP_\infty^2 S^* = 0$. $\square$

We are now in a position to state and prove the main result of this section.

Theorem 12. *Let (S,P) be a c.n.u. Γ-contraction on a Hilbert space $\mathcal{H}$ such that $R = M_{e^{it}+I}$ in the representation (3.13) of S. Then*

$$\begin{pmatrix} M_{G^*+zG} & 0 \\ 0 & M_{e^{it}+I} \end{pmatrix} \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} = \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} M_{F+zF^*}, \tag{3.14}$$

where $F \in \mathcal{B}(\mathcal{D}_P)$, $G \in \mathcal{B}(\mathcal{D}_{P^*})$ are the fundamental operators for (S,P) and (S^*,P^*) respectively. Moreover, if V_1 is as in (3.4), then

$$\begin{pmatrix} M_{G^*+zG} & 0 \\ 0 & M_{e^{it}+I} \end{pmatrix} \begin{pmatrix} M_{V_1} M_{\Theta_P} \\ \Delta_P \end{pmatrix} = \begin{pmatrix} M_{V_1} M_{\Theta_P} \\ \Delta_P \end{pmatrix} M_{Y+zY^*} \tag{3.15}$$

holds for some $Y \in \mathcal{B}(\mathcal{D}_P)$ with $w(Y) \le 1$.

Conversely, if P is a c.n.u. contraction on $\mathcal{H}$ and $F, Y \in \mathcal{B}(\mathcal{D}_P)$ with $w(F) \le 1$, $w(Y) \le 1$ and $G \in \mathcal{B}(\mathcal{D}_{P^*})$ with $w(G) \le 1$ satisfy Eqs. (3.14) and (3.15), then there exists $S \in \mathcal{B}(\mathcal{H})$ so that (S,P) is a c.n.u. Γ- contraction, F is the fundamental operator for (S,P) and G is the fundamental operator for (S^*,P^*).

Proof. We have seen that if (S,P) is a c.n.u. Γ-contraction and S has the form (3.13), then $S^*-SP^* = D_{P^*}V_1XV_1^*D_{P^*}$ where X is as above. Thus, $V_1XV_1^*$ is the fundamental operator of (S^*,P^*). Let $G=V_1XV_1^*$ and F denote the fundamental operator for (S,P). Then by Theorem 1, we have

$$M_{\Theta_P}M_{F+zF^*} = M_{G^*+zG}M_{\Theta_P}. \tag{3.16}$$

We claim that

$$M_{e^{it}+I}\Delta_P = \Delta_P M_{F+zF^*}. \tag{3.17}$$

As Δ_P commutes with $M_{e^{it}+I}$ and Δ_P is non-negative, therefore Eq. (3.17) is equivalent to

$$\Delta_P^2 M_{e^{it}+I} = \Delta_P^2 M_{F+zF^*}. \tag{3.18}$$

Using the fact that

$$\Delta_P(t) = \left[1-\Theta_P\left(e^{it}\right)^*\Theta_P\left(e^{it}\right)\right]^{\frac{1}{2}}$$

and the representation

$$\Theta_P\left(e^{it}\right) = \left[-P+\sum_{n=0}^{\infty}e^{i(n+1)t}D_{P^*}P^{*n}D_P\right]\bigg|_{\mathcal{D}_P}$$

we get

$$\begin{aligned}
\Delta_P^2 M_{e^{it}+I} = {} & D_PPP_\infty^2D_P + D_PP_\infty^2D_P \\
& + e^{it}\left[D_PP_\infty^2D_P + D_PP_\infty^2P^*D_P\right] \\
& + \sum_{n=2}^{\infty}e^{int}\left[D_PP_\infty^2P^{*(n-1)}D_P + D_PP_\infty^2P^{*n}D_P\right] \\
& + \sum_{n=-\infty}^{-1}e^{int}\left[D_PP^{1-n}P_\infty^2D_P + D_PP^{1-n}P_\infty^2P^*D_P\right]
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
\Delta_P^2 M_{F+zF^*} = {} & D_P^2F + D_PD_{P^*}GP - D_PSD_P + D_PPP_\infty^2S^*D_P \\
& + e^{it}\left[F^*D_P^2 + P^*G^*D_{P^*}D_P - D_PS^*D_P + D_PP_\infty^2S^*D_P\right] \\
& + \sum_{n=2}^{\infty}e^{int}\left[D_PP_\infty^2P^{*(n-1)}S^*D_P\right] \\
& + \sum_{n=-\infty}^{-1}e^{int}\left[D_PP^{1-n}P_\infty^2S^*D_P\right],
\end{aligned} \tag{3.20}$$

where to simplify the expressions that appear in the expansion of $\Delta_P^2M_{F+zF^*}$ we have used that G being the fundamental operator for (S^*,P^*) satisfies the equations $D_{P^*}GD_{P^*} = S^*-SP^*$ and $D_{P^*}S^* = GD_{P^*} + G^*D_{P^*}P^*$. We defer the proofs of these two equations till Appendix A. Using these equations, we shall now show that the coefficients of e^{int} are the same in Eqs. (3.19) and (3.20). For this, let L_n and R_n denote the coefficients of e^{int} in the right hand side of Eqs. (3.19) and (3.20), respectively.

We first look at

$$L_0 = D_P P P_\infty^2 D_P + D_P P_\infty^2 D_P = D_P P P_\infty^2 S^* D_P,$$

since $PP_\infty^2 + P_\infty^2 - PP_\infty^2 S^* = 0$.

Now, consider

$$\begin{aligned}
R_0 &= D_P^2 F + D_P D_{P^*} G P - D_P S D_P + D_P P P_\infty^2 S^* D_P \\
R_0 D_P &= D_P\big[D_P F D_P + D_{P^*} G P D_P - S D_P^2 + P P_\infty^2 S^* D_P^2\big] \\
&= D_P\big[S - S^* P + \big(S^* - S P^*\big)P - S\big(1 - P^* P\big)\big] + D_P P P_\infty^2 S^* D_P^2 \\
&= 0 + D_P P P_\infty^2 S^* D_P^2 \\
&= L_0 D_P.
\end{aligned}$$

Thus $L_0 = R_0$, since $L_0, R_0 \in \mathcal{B}(\mathcal{D}_P)$.

From Eq. (3.19),

$$L_1 = D_P P_\infty^2 D_P + D_P P_\infty^2 P^* D_P = D_P P_\infty^2 S^* D_P,$$

since $P_\infty^2 + PP_\infty^2 P^* = P_\infty^2 S^*$.

Further, from Eq. (3.20),

$$\begin{aligned}
R_1 &= F^* D_P^2 + P^* G^* D_{P^*} D_P - D_P S^* D_P + D_P P_\infty^2 S^* D_P \\
D_P R_1 &= D_P\big[F^* D_P^2 + P^* G^* D_{P^*} D_P - D_P S^* D_P + D_P P_\infty^2 S^* D_P\big] \\
&= \big[D_P F^* D_P + D_P P^* G^* D_{P^*} - D_P^2 S^*\big] D_P + D_P^2 P_\infty^2 S^* D_P \\
&= \big[S^* - P^* S + P^*\big(S^* - S P^*\big)^* - \big(1 - P^* P\big) S^*\big] D_P + D_P^2 P_\infty^2 S^* D_P \\
&= D_P^2 P_\infty^2 S^* D_P \\
&= D_P L_1.
\end{aligned}$$

Therefore, $D_P R_1 = D_P L_1$ which implies that $R_1 = L_1$, as $R_1, L_1 \in \mathcal{B}(\mathcal{D}_P)$.

We shall now show the equality of L_n and R_n for $n \geq 2$.

$$\begin{aligned}
L_n &= D_P P_\infty^2 P^{*(n-1)} D_P + D_P P_\infty^2 P^{*n} D_P \\
&= D_P P_\infty^2 S^* P^{*(n-1)} D_P = R_n.
\end{aligned}$$

Lastly, we shall show that $L_n = R_n$ for all $n \leq -1$. For $n \leq -1$,

$$\begin{aligned}
L_n &= D_P P^{1-n} P_\infty^2 D_P + D_P P^{1-n} P_\infty^2 P^* D_P \\
&= D_P P^{1-n} P_\infty^2 S^* D_P = R_n.
\end{aligned}$$

All these above computations show that $L_n = R_n$ for all n. Therefore, $\Delta_P^2 M_{e^{it}+I} = \Delta_P^2 M_{F+zF^*}$ which implies that $M_{e^{it}+I}\Delta_P = \Delta_P M_{F+zF^*}$. Hence, Eq. (3.14) holds true.

To show the validity of Eq. (3.15), note that

$$\begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & R \end{pmatrix}^* \Pi(\mathcal{H}) \subseteq \Pi(\mathcal{H}).$$

Therefore, by Lemma 9, we have Eq. (3.15).

Conversely, let P be a c.n.u. contraction on $\mathcal{H}$, and $F, Y \in \mathcal{B}(\mathcal{D}_P)$ with $w(F) \leq 1$, $w(Y) \leq 1$ and $G \in G(\mathcal{D}_{P^*})$ with $w(G) \leq 1$, satisfy Eqs. (3.14) and (3.15).

Let

$$S = \Pi^* \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & M_{e^{it}+I} \end{pmatrix} \Pi,$$

where $X = V_1^* G V_1$.

From Eq. (3.15) we can easily deduce that $\Pi(\mathcal{H})$ is invariant under

$$\begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & M_{e^{it}+I} \end{pmatrix}^*.$$

Also,

$$P = \Pi^* \begin{pmatrix} M_z & 0 \\ 0 & M_{e^{it}} \end{pmatrix} \Pi \quad \text{and} \quad \begin{pmatrix} M_z & 0 \\ 0 & M_{e^{it}} \end{pmatrix}^* \Pi(\mathcal{H}) \subseteq \Pi(\mathcal{H}).$$

Therefore,

$$S^* P^* = P^* S^*.$$

Thus, (S, P) is a commuting pair of bounded operators on $\mathcal{H}$ with $\|S\| \leq 2$.

Now to show that G is the fundamental operator for (S^*, P^*), consider

$$\begin{aligned} S^* - SP^* &= \Pi^* \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & M_{e^{it}+I} \end{pmatrix}^* \Pi \\ &\quad - \Pi^* \begin{pmatrix} M_{X^*+zX} & 0 \\ 0 & M_{e^{it}+I} \end{pmatrix} \Pi \Pi^* \begin{pmatrix} M_z & 0 \\ 0 & M_{e^{it}} \end{pmatrix}^* \Pi \\ &= \Pi^* \begin{pmatrix} P_{\mathbb{C}} \otimes X & 0 \\ 0 & 0 \end{pmatrix} \Pi \\ &= \Pi_0^* \begin{pmatrix} P_{\mathbb{C}} \otimes G & 0 \\ 0 & 0 \end{pmatrix} \Pi_0 \\ &= D_{P^*} G D_{P^*}. \end{aligned}$$

Thus, $S^* - SP^* = D_{P^*} G D_{P^*}$. Therefore, G is the fundamental operator for (S^*, P^*).

Applying the first part of this result to the c.n.u Γ-contraction (S, P), we obtain

$$\begin{pmatrix} M_{G^*+zG} & 0 \\ 0 & M_{e^{it}+I} \end{pmatrix} \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} = \begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix} M_{C+zC^*}, \tag{3.21}$$

where $C \in \mathcal{B}(\mathcal{D}_P)$ is the fundamental operator for (S, P). Then from the given equation, that is, Eq. (3.14) and Eq. (3.21) and the fact that

$$\begin{pmatrix} M_{\Theta_P} \\ \Delta_P \end{pmatrix}$$

is an isometry we get $M_{F+zF^*} = M_{C+zC^*}$. Thus $F = C$. This completes the proof. □

Remark 13. Every pure contraction is a c.n.u. contraction. So, for a pure contraction $P \in \mathcal{B}(\mathcal{H})$, we have two results, Theorem 2 and the converse of Theorem 12. Theorem 12 demands two conditions, namely Eqs. (3.14) and (3.15), for the existence of $S \in \mathcal{B}(\mathcal{H})$ so that the operators F and G are the fundamental operators for (S, P) and (S^*, P^*), respectively, whereas in Theorem 2 the same conclusion holds just by assuming Eq. (3.14). Does this make Theorem 12 a weaker result? The answer is no as we shall see from the following discussion that if P is a pure contraction Eq. (3.14) holds if and only if Eq. (3.15) holds.

Let $P \in \mathcal{B}(\mathcal{H})$ be a pure contraction. Then $\mathbf{P}_\infty$ and Δ_P are both zero. Therefore, for the pure contraction P, Eqs. (3.14) and (3.15) become

$$M_{G^*+zG} M_{\Theta_P} = M_{\Theta_P} M_{F+zF^*} \tag{3.22}$$

and

$$M_{G^*+zG} M_{V_1} M_{\Theta_P} = M_{V_1} M_{\Theta_P} M_{Y+zY^*}, \tag{3.23}$$

respectively. Further, now since P is pure, $\varPhi = I \otimes V_1$, $\varPi_0 \varPi_0^* + M_{\Theta_P} M_{\Theta_P}^* = I$ and $\varPi_0 = W$. This implies that M_{Θ_P} and $(I \otimes V_1) M_{\Theta_P}$ are both isometries in $\mathcal{B}(H^2_{\mathcal{D}_P}(\mathbb{D}), H^2_{\mathcal{D}_{P^*}}(\mathbb{D}))$ and they satisfy the following equation

$$M_{\Theta_P} M_{\Theta_P}^* = (I \otimes V_1) M_{\Theta_P} M_{\Theta_P}^* \big(I \otimes V_1^*\big).$$

Consequently, $Ran M_{\Theta_P} = Ran M_{V_1} M_{\Theta_P}$. Hence, by using Lemma 9, we can easily conclude that if Eq. (3.23) holds, then Eq. (3.22) will also hold. Lastly, if Eq. (3.22) holds, then by using arguments similar to the ones used in the proof of Lemma 9, Eq. (3.23) will also hold.

4. Results for pure tetrablock contractions

In this section, we prove a result for pure tetrablock contractions similar to the result stated in Theorem 1 and Theorem 2 for pure $\varGamma$-contractions.

Before we state and prove the main results of this section, we need to recall a result from [6] which will come very handy in proving the main results.

Lemma 14. *The fundamental operators F_1 and F_2 of a tetrablock contraction (A, B, P) are the unique bounded linear operators on $\mathcal{D}_P$ that satisfy the pair of operator equations*

$$D_P A = X_1 D_P + X_2^* D_P P \quad \textit{and} \quad D_P B = X_2 D_P + X_1^* D_P P.$$

The next two lemmas give analogous results for a tetrablock contraction to Lemma 6. These two lemmas can be found in [12]. We just state the results here without giving the proofs.

Lemma 15. *Let (A, B, P) be a tetrablock contraction on a Hilbert space $\mathcal{H}$ and F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then*

$$PF_i = G_i^* P|_{\mathcal{D}_P}, \quad \textit{for } i = 1 \textit{ and } 2.$$

Lemma 16. *Let (A, B, P) be a tetrablock contraction on a Hilbert space $\mathcal{H}$ and F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then*

$$\big(F_1^* D_P D_{P^*} - F_2 P^*\big)\big|_{\mathcal{D}_{P^*}} = D_P D_{P^*} G_1 - P^* G_2^* \quad \text{and}$$
$$\big(F_2^* D_P D_{P^*} - F_1 P^*\big)\big|_{\mathcal{D}_{P^*}} = D_P D_{P^*} G_2 - P^* G_1^*.$$

The fundamental operators of a tetrablock contraction always abide by two relations (like in the case of Γ-contractions, Theorem 1). The next theorem, which was proved in [12] (Corollary 12), gives the relations between them.

Lemma 17. *Let F_1 and F_2 be fundamental operators of a tetrablock contraction (A, B, P) and G_1 and G_2 be fundamental operators of the tetrablock contraction (A^*, B^*, P^*). Then*

$$\big(F_1^* + F_2 z\big)\Theta_{P^*}(z) = \Theta_{P^*}(z)\big(G_1 + G_2^* z\big) \quad \text{and} \tag{4.1}$$
$$\big(F_2^* + F_1 z\big)\Theta_{P^*}(z) = \Theta_{P^*}(z)\big(G_2 + G_1^* z\big) \quad \text{hold for all } z \in \mathbb{D}. \tag{4.2}$$

Proof.

$$\begin{aligned}
&\big(F_1^* + F_2 z\big)\Theta_{P^*}(z) \\
&\quad = \big(F_1^* + F_2 z\big)\left(-P^* + \sum_{n=0}^{\infty} z^{n+1} D_P P^n D_{P^*}\right) \\
&\quad = \left(-F_1^* P^* + \sum_{n=1}^{\infty} z^n F_1^* D_P P^{n-1} D_{P^*}\right) + \left(-z F_2 P^* + \sum_{n=2}^{\infty} z^n F_2 D_P P^{n-2} D_{P^*}\right) \\
&\quad = -F_1^* P^* + z\big(-F_2 P^* + F_1^* D_P D_{P^*}\big) + \sum_{n=2}^{\infty} z^n \big(F_1^* D_P P^{n-1} D_{P^*} + F_2 D_P P^{n-2} D_{P^*}\big) \\
&\quad = -F_1^* P^* + z\big(-F_2 P^* + F_1^* D_P D_{P^*}\big) + \sum_{n=2}^{\infty} z^n \big(F_1^* D_P P + F_2 D_P\big) P^{n-2} D_{P^*} \\
&\quad = -P^* G_1 + z\big(D_P D_{P^*} G_1 - P^* G_2^*\big) + \sum_{n=2}^{\infty} z^n D_P B P^{n-2} D_{P^*} \quad \text{[using Lemmas 14, 15 and 16]}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\Theta_{P^*}(z)\big(G_1 + G_2^* z\big) \\
&\quad = \left(-P^* + \sum_{n=0}^{\infty} z^{n+1} D_P P^n D_{P^*}\right)\big(G_1 + G_2^* z\big) \\
&\quad = \left(-P^* G_1 + \sum_{n=1}^{\infty} z^n D_P P^{n-1} D_{P^*} G_1\right) + \left(-z P^* G_2^* + \sum_{n=2}^{\infty} z^n D_P P^{n-2} D_{P^*} G_2^*\right) \\
&\quad = -P^* G_1 + z\big(D_P D_{P^*} G_1 - P^* G_2^*\big) + \sum_{n=2}^{\infty} z^n \big(D_P P^{n-1} D_{P^*} G_1 + D_P P^{n-2} D_{P^*} G_2^*\big) \\
&\quad = -P^* G_1 + z\big(D_P D_{P^*} G_1 - P^* G_2^*\big) + \sum_{n=2}^{\infty} z^n D_P P^{n-2} \big(P D_{P^*} G_1 + D_P G_2^*\big)
\end{aligned}$$

$$= -P^*G_1 + z\big(D_PD_{P^*}G_1 - P^*G_2^*\big) + \sum_{n=2}^{\infty} z^n D_P P^{n-2} B D_{P^*}$$
$$= -P^*G_1 + z\big(D_PD_{P^*}G_1 - P^*G_2^*\big) + \sum_{n=2}^{\infty} z^n D_P B P^{n-2} D_{P^*}.$$

Hence $(F_1^* + F_2 z)\Theta_{P^*}(z) = \Theta_{P^*}(z)(G_1 + G_2^* z)$ for all $z \in \mathbb{D}$. Similarly one can prove that $(F_2^* + F_1 z)\Theta_{P^*}(z) = \Theta_{P^*}(z)(G_2 + G_1^* z)$ holds for all $z \in \mathbb{D}$. □

We end with the proof of Theorem 3.

Proof of Theorem 3. The first part is obtained by applying Lemma 17 to the tetrablock contraction (A^*, B^*, P^*).

For the converse, let W be the isometry defined above. Since P is pure contraction, we have $WP^* = M_z^* W$ as seen in Eq. (2.3). Eqs. (1.8) imply that $(M_{G_1^*+G_2 z}, M_{G_2^*+G_1 z}, M_z)$ is a commuting triple of bounded operators on $H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$. Using Theorem 5.7 (part (3)) of [6] one can easily check that $(M_{G_1^*+G_2 z}, M_{G_2^*+G_1 z}, M_z)$ is actually a tetrablock isometry. Define $A = W^* M_{G_1^*+G_2 z} W$ and $B = W^* M_{G_2^*+G_1 z} W$. Eqs. (1.6) and (1.7) tell that $Ran M_{\Theta_P}$ is invariant under $M_{G_1^*+G_2 z}$ and $M_{G_2^*+G_1 z}$. In other words $Ran W = (Ran M_{\Theta_P})^{\perp}$ is invariant under $M^*_{G_1^*+G_2 z}$ and $M^*_{G_2^*+G_1 z}$.

Commutativity of A and B with P can be checked easily. To show that A and B commute, we proceed as follows.

$$\begin{aligned} A^*B^* &= W^* M^*_{G_1^*+G_2 z} W W^* M^*_{G_2^*+G_1 z} W \\ &= W^* M^*_{G_1^*+G_2 z} M^*_{G_2^*+G_1 z} W \quad \big[\text{since } Ran W \text{ is invariant under } M^*_{G_2^*+G_1 z}\big] \\ &= W^* M^*_{G_2^*+G_1 z} M^*_{G_1^*+G_2 z} W \\ &= W^* M^*_{G_2^*+G_1 z} W W^* M^*_{G_1^*+G_2 z} W \quad \big[\text{since } Ran W \text{ is invariant under } M^*_{G_1^*+G_2 z}\big] \\ &= B^*A^*. \end{aligned}$$

Therefore (A, B, P) is a commuting triple of bounded operators. Now we shall show that (A, B, P) is a tetrablock contraction. Note that for every polynomial f in three variables we have $f(A^*, B^*, P^*) = W^* f(T_1^*, T_2^*, T_3^*) W$, where $(T_1, T_2, T_3) = (M_{G_1^*+G_2 z}, M_{G_2^*+G_1 z}, M_z)$. Let f be any polynomial in three variables. Then we have

$$\big\| f\big(A^*, B^*, P^*\big)\big\| = \big\| W^* f\big(T_1^*, T_2^*, T_3^*\big) W\big\| \le \big\| f\big(T_1^*, T_2^*, T_3^*\big)\big\| \le \|f\|_{\overline{E},\infty},$$

where the last inequality follows from the fact that (T_1, T_2, T_3) is a tetrablock contraction.

$$\begin{aligned} A^* - BP^* &= W^* M^*_{G_1^*+G_2 z} W - W^* M_{G_2^*+G_1 z} W W^* M_z^* W \\ &= W^* M^*_{G_1^*+G_2 z} W - W^* M_{G_2^*+G_1 z} M_z^* W \quad \big[\text{since } Ran W \text{ is invariant under } M_z^*\big] \\ &= W^* \big((I \otimes G_1) + \big(M_z \otimes G_2^*\big) - \big(M_z^* \otimes G_2^*\big) - \big(M_z M_z^* \otimes G_1\big)\big) W \\ &= W^* (P_{\mathbb{C}} \otimes G_1) W = D_{P^*} G_1 D_{P^*}. \end{aligned}$$

Similarly one can show that $B^* - AP^* = D_{P^*} G_2 D_{P^*}$. This shows that G_1, G_2 are the fundamental operators of (A^*, B^*, P^*). Let X_1, X_2 be the fundamental operators of (A, B, P). Then we have, by first part of Theorem 3,

$$\big(G_1^* + G_2 z\big)\Theta_P(z) = \Theta_P(z)\big(X_1 + X_2^* z\big) \quad \text{and}$$
$$\big(G_2^* + G_1 z\big)\Theta_P(z) = \Theta_P(z)\big(X_2 + X_1^* z\big) \quad \text{hold for all } z \in \mathbb{D}.$$

By this and the fact that G_1 and G_2 satisfy Eqs. (1.6) and (1.7), for some operators $F_1, F_2 \in \mathcal{B}(\mathcal{D}_P)$ with numerical radii no greater than one, we have $F_1 + F_2^* z = X_1 + X_2^* z$ and $F_2 + F_1^* z = X_2 + X_1^* z$, for all $z \in \mathbb{D}$. Which shows that $X_1 = F_1$ and $X_2 = F_2$. Hence F_1, F_2 are the fundamental operators of (A, B, P). This completes the proof of the theorem. □

Appendix A

A.1. Proof of Eq. (3.19)

$$\begin{aligned}
\Delta_P(t)^2\big(e^{it} + I\big) &= \big[I - \Theta_P\big(e^{it}\big)^* \Theta_P\big(e^{it}\big)\big]\big[e^{it} + I\big] \\
&= \left[I - \left(-P^* + \sum_{n=0}^{\infty} e^{-i(n+1)t} D_P P^n D_{P^*}\right)\left(-P + \sum_{n=0}^{\infty} e^{i(n+1)t} D_{P^*} P^{*n} D_P\right)\right]\big[e^{it} + I\big] \\
&= \big[e^{it} + I\big] - \left[P^* + \sum_{n=-\infty}^{-1} e^{int} D_P P^{-n-1} D_{P^*}\right] \\
&\quad \times \left[-P + e^{it}(D_{P^*} D_P - P) + \sum_{n=2}^{\infty} e^{int}\big(D_{P^*} P^{*(n-2)}\big(I + P^*\big) D_P\big)\right] \\
&= \big[e^{it} + I\big] - P^*P - e^{it}\big(P^*P - P^* D_{P^*} D_P\big) \\
&\quad + \sum_{n=2}^{\infty} e^{int} P^* D_{P^*} P^{*(n-2)}\big(I + P^*\big) D_P + \sum_{n=-\infty}^{-1} e^{int} D_P P^{-n-1} D_{P^*} P \\
&\quad - \sum_{n=-\infty}^{0} e^{int} D_P P^{-n} D_{P^*}(D_{P^*} D_P - P) \\
&\quad - \sum_{n=-\infty}^{0} e^{int}\left[\sum_{k=-\infty}^{n-2} D_P P^{-k-1} D_{P^*}^2 P^{*(n-k-2)}\big(I + P^*\big) D_P\right] \\
&\quad - \sum_{n=1}^{\infty} e^{int}\left[\sum_{k=-\infty}^{-1} D_P P^{-k-1} D_{P^*}^2 P^{*(n-k-2)}\big(I + P^*\big) D_P\right].
\end{aligned}$$

We shall now simplify the coefficients of e^{int}, $n \in \mathbb{Z}$. Let C_n denote the coefficient of e^{int}. In the following simplifications we shall be repeatedly using $D_{P^*}^2 = I - PP^*$, $D_P P^* = P^* D_{P^*}$, $P_\infty^2 h = \lim_n P^n P^{*n} h$ for all h and $P P_\infty^2 P^* = P_\infty^2$.

$$\begin{aligned}
C_0 &= I - P^*P - D_P D_{P^*}(D_{P^*} D_P - P) - \sum_{k=-\infty}^{-2} D_P P^{-k-1} D_{P^*}^2 P^{*(-k-2)}\big(I + P^*\big) D_P \\
&= D_P P D_P + D_P P P^* D_P - \sum_{k=2}^{\infty} D_P P\big(P^{k-2} P^{*(k-2)} - P^{k-1} P^{*(k-1)}\big)\big(I + P^*\big) D_P \\
&= D_P P P_\infty^2 D_P + D_P P_\infty^2 D_P.
\end{aligned}$$

$$\begin{aligned}
C_1 &= I - P^*P + P^*D_{P^*}D_P - \sum_{k=-\infty}^{-1} D_P P^{-k-1} D_{P^*}^2 P^{*(-k-1)} \big(I + P^*\big) D_P \\
&= D_P^2 + D_P P^* D_P - \sum_{k=1}^{\infty} D_P \big(P^{k-1}P^{*(k-1)} - P^k P^{*k}\big)\big(I + P^*\big) D_P \\
&= D_P P_\infty^2 D_P + D_P P_\infty^2 P^* D_P.
\end{aligned}$$

Next we look at C_n, $n \geq 2$. For $n \geq 2$,

$$\begin{aligned}
C_n &= P^* D_{P^*} P^{*(n-2)} \big(I + P^*\big) D_P - \sum_{k=-\infty}^{-1} D_P P^{-k-1} D_{P^*}^2 P^{*(n-k-2)} \big(I + P^*\big) D_P \\
&= D_P P^{*(n-1)} D_P + D_P P^{*n} D_P - \sum_{k=1}^{\infty} D_P \big(P^{k-1} P^{*(k-1)} - P^k P^{*k}\big) P^{*(n-1)} \big(I + P^*\big) D_P \\
&= D_P P_\infty^2 P^{*(n-1)} D_P + D_P P_\infty^2 P^{*n} D_P.
\end{aligned}$$

Lastly, we simplify C_n, $n \leq -1$. For $n \leq -1$,

$$\begin{aligned}
C_n &= D_P P^{-n-1} D_{P^*} P - D_P P^{-n} D_{P^*} (D_{P^*} D_P - P) \\
&\quad - \sum_{k=-\infty}^{n-2} D_P P^{-k-1} D_{P^*}^2 P^{*(n-k-2)} \big(I + P^*\big) D_P \\
&= D_P P^{-n+1} P^* D_P + D_P P^{-n+1} D_P - \sum_{k=0}^{\infty} D_P P^{1-n} \big(P^k P^{*k} - P^{k+1} P^{*(k+1)}\big)\big(I + P^*\big) D_P \\
&= D_P P^{1-n} P_\infty^2 D_P + D_P P^{1-n} P_\infty^2 P^* D_P.
\end{aligned}$$

Thus, Eq. (3.19) holds.

A.2. Proof of Eq. (3.20)

$$\begin{aligned}
\Delta_P(t)^2 \big(F + e^{it} F^*\big) &= \big[I - \Theta_P\big(e^{it}\big)^* \Theta_P\big(e^{it}\big)\big]\big[F + e^{it} F^*\big] \\
&= F + e^{it} F^* - \Theta_P\big(e^{it}\big)^* \big[G^* + e^{it} G\big] \Theta_P\big(e^{it}\big) \\
&\quad \big(\text{since } \Theta_P\big(e^{it}\big)\big[F + e^{it} F^*\big] = \big[G^* + e^{it} G\big]\Theta_P\big(e^{it}\big)\big) \\
&= F + e^{it} F^* - \left[-P^* + \sum_{n=0}^{\infty} e^{-i(n+1)t} D_P P^n D_{P^*}\right] \big[G^* + e^{it} G\big] \\
&\quad \times \left[-P + \sum_{n=0}^{\infty} e^{i(n+1)t} D_{P^*} P^{*n} D_P\right] \\
&= F + e^{it} F^* - \left[-P^* + \sum_{n=-\infty}^{-1} e^{int} D_P P^{-n-1} D_{P^*}\right] \\
&\quad \times \left[-G^* P + e^{it}\big(G^* D_{P^*} D_P - GP\big) + \sum_{n=2}^{\infty} e^{int} \big(G^* D_{P^*} P^* + G D_{P^*}\big) P^{*(n-2)} D_P\right] \\
&= F + e^{it} F^* - \left[-P^* + \sum_{n=-\infty}^{-1} e^{int} D_P P^{-n-1} D_{P^*}\right] \\
&\quad \times \left[-G^* P + e^{it}\big(G^* D_{P^*} D_P - GP\big) + \sum_{n=2}^{\infty} e^{int} D_{P^*} S^* P^{*(n-2)} D_P\right].
\end{aligned}$$

To get the last equality we used that G being the fundamental operator for (S^*, P^*) satisfies $D_{P^*}S^* = GD_{P^*} + G^*D_{P^*}P^*$. Next we multiply the last two terms, as we did to obtain (3.19), and collect coefficients of e^{int}.

$$\begin{aligned}\Delta_P(t)^2\big(F + e^{it}F^*\big) = &\Bigg[F - P^*G^*P - D_PD_{P^*}\big(G^*D_{P^*}D_P - GP\big)\\ &- \sum_{k=-\infty}^{-2} D_PP^{-k-1}D^2_{P^*}P^{*(-k-2)}S^*D_P\Bigg]\\ &+ e^{it}\Bigg[F^* - P^*GP + P^*G^*D_{P^*}D_P - \sum_{k=1}^{\infty} D_PP^{k-1}D^2_{P^*}P^{*(k-1)}S^*D_P\Bigg]\\ &+ \sum_{n=2}^{\infty} e^{int}\Bigg[P^*D_{P^*}S^*P^{*(n-2)}D_P - \sum_{k=1}^{\infty} D_PP^{k-1}D^2_{P^*}P^{*(n+k-2)}S^*D_P\Bigg]\\ &+ \sum_{n=-\infty}^{-1} e^{int}\Bigg[D_PP^{-n-1}D_{P^*}G^*P - D_PP^{-n}D_{P^*}\big(G^*D_{P^*}D_P - GP\big)\\ &- \sum_{k=2-n}^{\infty} D_PP^{k-1}D^2_{P^*}P^{*(n+k-2)}S^*D_P\Bigg].\end{aligned}$$

Next we simplify the coefficients of e^{int}, $n \in \mathbb{Z}$. Let D_n denote the coefficient of e^{int}. To simplify $D_n's$ we shall be repeatedly using $D_P^2 = I - P^*P$, $D^2_{P^*} = I - PP^*$, $PD_P = D_{P^*}P$, $P^*F = G^*P$ and $D_{P^*}GD_{P^*} = S^* - SP^*$.

$$\begin{aligned}D_0 &= \Bigg[F - P^*G^*P - D_PD_{P^*}\big(G^*D_{P^*}D_P - GP\big)\\ &\quad - \sum_{k=-\infty}^{-2} D_PP^{-k-1}D^2_{P^*}P^{*(-k-2)}S^*D_P\Bigg]\\ &= F - PP^*F + D_PD_{P^*}GP - D_PSD_P + D_PPS^*D_P\\ &\quad - \sum_{k=2}^{\infty} D_PP\big(P^{k-2}P^{*(k-2)} - P^{k-1}P^{*(k-1)}\big)S^*D_P\\ &= D_P^2F + D_PD_{P^*}GP - D_PSD_P + D_PPP^2_\infty S^*D_P.\end{aligned}$$

$$\begin{aligned}D_1 &= F^* - P^*GP + P^*G^*D_{P^*}D_P - \sum_{k=1}^{\infty} D_PP^{k-1}D^2_{P^*}P^{*(k-1)}S^*D_P\\ &= F^* - F^*P^*P + P^*G^*D_{P^*}D_P - \sum_{k=1}^{\infty} D_P\big(P^{k-1}P^{*(k-1)} - P^kP^{*k}\big)S^*D_P\\ &= F^*D_P^2 + P^*G^*D_{P^*}D_P - D_PS^*D_P + D_PP^2_\infty S^*D_P.\end{aligned}$$

For $n \geq 2$,

$$\begin{aligned}D_n &= P^*D_{P^*}S^*P^{*(n-2)}D_P - \sum_{k=1}^{\infty} D_PP^{k-1}D^2_{P^*}P^{*(n+k-2)}S^*D_P\\ &= P^*D_{P^*}S^*P^{*(n-2)}D_P - \sum_{k=1}^{\infty} D_P\big(P^{k-1}P^{*(k-1)} - P^kP^{*k}\big)P^{*(n-1)}S^*D_P\\ &= D_PP^2_\infty P^{*(n-1)}S^*D_P.\end{aligned}$$

Lastly, for $n \leq -1$,

$$\begin{aligned}
D_n &= D_P P^{-n-1} D_{P^*} G^* P - D_P P^{-n} D_{P^*} \big(G^* D_{P^*} D_P - GP \big) \\
&\quad - \sum_{k=2-n}^{\infty} D_P P^{k-1} D_{P^*}^2 P^{*(n+k-2)} S^* D_P \\
&= D_P P^{-n-1} D_{P^*} G^* P - D_P P^{-n} \big(S^* - SP^* \big)^* D_P + D_P P^{-n} D_{P^*} GP \\
&\quad - \sum_{k=0}^{\infty} D_P P^{1-n} \big(P^k P^{*k} - P^{k+1} P^{*(k+1)} \big) S^* D_P \\
&= D_P P^{1-n} P_\infty^2 S^* D_P.
\end{aligned}$$

For each $n \in \mathbb{Z}$, the expression for D_n is the same as required in Eq. (3.20). This proves Eq. (3.20).

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