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A relation of Banach limit and difference matrix to generate some Orlicz sequence spaces

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Abstract:

In the present paper we study the applications of generalized difference matrices and Orlicz function to generate some spaces of almost convergent sequences. We make an effort to study some algebraic and topological properties of these sequence spaces. Some inclusión relations between these spaces are establish. Furthermore, we study matrix transformations and compute $\beta-$, $\gamma-$ duals of these spaces.

Keywords: Paranorm space; Orlicz function; β-and γ-duals; Matrix transformations.

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1. Introduction and Preliminaries

The notion of almost convergence was introduced by Lorentz [7]. Matrix domains of the generalized difference matrix $B(r,s)$ and triple band matrix $B(r,s,t)$ in sets of almost null and almost convergent sequences have been investigated by Başar and Kirişçi [3] and Sönmez [18], respectively. Let w be the vector space of all real sequences. We shall write c, c_0 and l_∞ for the spaces of all convergent, null and bounded sequences. Moreover, we write bs and cs for the spaces of all bounded and convergent series, respectively. Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex entries, where $n, k \in \mathbf{N}$. Then we say that A defines a matrix mapping from X into Y if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{A_n(x)\}$ is in Y, where

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in \mathbf{N}). \tag{1.1}$$

By (X, Y) we denote the class of all matrices A such that $A : X \to Y$. Thus $A \in (X,Y)$ if and only if the series on the right-hand side of 1.1 converges for each $n \in \mathbf{N}$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

which is a sequence space.

A B-space is a complete normed space. A topological sequence space in which all coordinate functionals $\pi_k, \pi_k(x) = x_k$, are continuous is called a K-space. A BK-space is defined as a K-space which is also a B-space, that is, a BK-space is a Banach space with continuous coordinates. For example, the space $l_p (1 \le p < \infty)$ is a BK-space with $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{\frac{1}{p}}$ and c_0, c and l_∞ are BK-spaces with $\|x\|_\infty = \sup_k |x_k|$. A sequence (b_n) in a normed space X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_m \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$.

The Cesàro matrix $C = (c_{nk})$ of order one is a triangle matrix defined by

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbf{N}$.
One of the best known regular matrix is the Riesz matrix $R = (r_{nk})$, which is a triangle matrix and is defined by

$$r_{nk} = \begin{cases} \frac{r_k}{R_n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbf{N}$, where (r_k) is a real sequence with $r_0 > 0, r_k \geq 0$ and $R_n = \sum_{k=0}^{n} r_k$. The Riesz matrix R is regular if and only if $R_n \to \infty$ as $n \to \infty$ [13]. The matrix domain X_A of a sequence space X has a basis if and only if X has a basis and $A = (a_{nk})$ is a triangle matrix.
Let r, s be non-zero real numbers and define the generalized difference matrix $B(r, s) = (b_{nk}(r, s))$ for all $k, n \in \mathbf{N}$ as follows:

$$b_{nk}(r,s) = \begin{cases} r, & k = n, \\ s, & k = n-1, \\ 0, & 1 \leq k \leq n-1 \text{ or } k > n. \end{cases} \tag{1.3}$$

It is easy to calculate that the inverse $B^{-1}(r, s) = (\hat{b}_{nk}(r, s))$ of the generalized difference matrix is given by

$$\hat{b}_{nk}(r,s) = \begin{cases} \frac{1}{r}(-\frac{s}{r})^{n-k}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

for all $k, n \in \mathbf{N}$.

We now focus on sets of almost convergent sequences. A continuous linear functional ϕ on l_∞ is called a Banach limit if

(i) $\phi(x) \geq 0$ for $x = (x_k)$, $x_k \geq 0$ for every k,

(ii) $\phi(x_{\sigma(k)}) = \phi(x_k)$, where σ is shift operator which is defined on w by $\sigma(k) = k + 1$ and

(iii) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$.
A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent to the generalized limit a if all Banach limits of x are a (see [7]) and denoted by $f - \lim x = a$. In other words, $f - \lim x_k = a$ uniformly in n if and only if $\lim_{p\to\infty} \frac{(x_n + x_{n+1} + ... + x_{n+p-1})}{p} = a$, uniformly in n. We denote the space of all almost convergent and almost null sequences by f and f_0, respectively.

In [21] Zararsız and Şengönül defined the concepts of the spaces of rf-convergent and rf-null sequences and it is proved that the spaces rf and rf_0 are Banach spaces with the norm

$$\|x\|_{rf} = \|x\|_{rf_0} = \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} \right|, \text{ uniformly in } n.$$

In addition to these spaces, Zararsız [22] introduced two convergent sequences brf and brf_0 as the sets of all sequences such that their $B(r, s)$-transforms are in the spaces rf and rf_0, respectively.
Let us define the sequence $z = (z_k)$ as the $B(r, s)$-transform of a sequence $x = (x_k)$, that is,

$$z_k = sx_{k-1} + rx_k \quad (k \in \mathbf{N}). \tag{1.4}$$

Corollary 1.1. *[22] The space brf does not have a Schauder basis.*

A set $\lambda \subset w$ is said to be convex if $x, y \in \lambda$ implies $C = \{z \in w : z = tx + (1 - t)y, 0 \le t \le 1\} \subset \lambda$.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

is known as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be Musielak-Orlicz function (see [9], [12]). A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy Δ_2-*condition* if there exist constants a, $K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in \mathbf{N}$ and $u \in R^+$, whenever $M_k(u) \leq a$. The reader can refer to the textbook Başar [2] containing the chapters entitled Normed and Paranormed Sequence Spaces and Matrix Domains in Sequence Spaces together with the paper Dutta and Başar [4] devoted to the generalization of Orlicz sequence spaces. For more details about sequence spaces see For more details about sequence spaces see ([10], [11], [14], [15], [16], [17], [20]) and references therein.

Definition 1.2. *Let X be a linear metric space. A function $p : X \to \mathbf{R}$ is called paranorm, if*
$(P1)$ *$p(x) \geq 0$ for all $x \in X$,*
$(P2)$ *$p(-x) = p(x)$ for all $x \in X$,*
$(P3)$ *$p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,*
$(P4)$ *if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.*

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19, Theorem 10.4.2, p. 183]).

Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. By making the use of $B(r, s)$-transform of sequences $x = (x_k)$, we define the following sequence spaces:

$$[brf, \mathcal{M}, u, p] = \Bigg\{ x = (x_k) \in w : \exists a \in \mathbf{C} \ni \lim_m \frac{1}{R_m} \sum_{k=0}^{m} \left[u_k M_k \left(\frac{r_k |s x_{k+n-1} + r x_{k+n}|}{\rho} \right) \right]^{p_k} = a, \text{ uniformily in } n \text{ and some } \rho > 0 \Bigg\}$$

and

$$[brf_0, \mathcal{M}, u, p] = \Bigg\{ x = (x_k) \in w : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} \left[u_k M_k \left(\frac{r_k |sx_{k+n-1} + rx_{k+n}|}{\rho} \right) \right]^{p_k} = 0,$$

$$\text{uniformly in } n \text{ and some } \rho > 0 \Bigg\}.$$

If $M_k(x) = x$ for all $k \in \mathbf{N}$ and $\rho = 1$, then above sequence spaces reduces to $[brf, u, p]$ and $[brf_0, u, p]$, where

$$[brf, u, p] = \Bigg\{ x = (x_k) \in w : \exists a \in \mathbf{C} \ni \lim_m \frac{1}{R_m} \sum_{k=0}^{m} \left[u_k \Big(r_k |sx_{k+n-1} + rx_{k+n}| \Big) \right]^{p_k} = a,$$

$$\text{uniformly in } n \Bigg\}$$

and

$$[brf_0, u, p] = \Bigg\{ x = (x_k) \in w : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} \left[u_k \Big(r_k |sx_{k+n-1} + rx_{k+n}| \Big) \right]^{p_k} = 0,$$

$$\text{uniformly in } n \Bigg\}.$$

By taking $(p_k) = 1$ and $(u_k) = 1$, for all $k \in \mathbf{N}$, then we get the following sequence spaces:

$$[brf, \mathcal{M}] = \Bigg\{ x = (x_k) \in w : \exists a \in \mathbf{C} \ni \lim_m \frac{1}{R_m} \sum_{k=0}^{m} \left[M_k \left(\frac{r_k |sx_{k+n-1} + rx_{k+n}|}{\rho} \right) \right] = a,$$

$$\text{uniformly in } n \text{ and some } \rho > 0 \Bigg\}$$

and

$$[brf_0, \mathcal{M}] = \left\{ x = (x_k) \in w : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} \left[M_k \left(\frac{r_k |s x_{k+n-1} + r x_{k+n}|}{\rho} \right) \right] = 0, \right.$$

$$\left. \text{uniformly in } n \text{ and some } \rho > 0 \right\}.$$

With the notation of 1.1, the sequence spaces $[brf, \mathcal{M}, u, p]$ and $[brf_0, \mathcal{M}, u, p]$ can be redefined as follows:

$$[brf_0, \mathcal{M}, u, p] = \{[rf_0, \mathcal{M}, u, p]\}_{B(r,s)} \text{ and } [brf, \mathcal{M}, u, p] = \{[rf, \mathcal{M}, u, p]\}_{B(r,s)}. \tag{1.5}$$

The following inequality will be use throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.6}$$

for all k and $a_k, b_k \in \mathbf{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbf{C}$.

In this paper, we introduce the sequence spaces $[brf, \mathcal{M}, u, p]$ and $[brf_0, \mathcal{M}, u, p]$. We investigate some topological properties of these new sequence spaces and establish some inclusion relations between these spaces. Also we determine the $\alpha-, \beta-$ and $\gamma-$ duals of these spaces and construct the matrix transformation of these spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $[brf, \mathcal{M}, u, p]$ and $[brf_0, \mathcal{M}, u, p]$ are linear spaces over the complex field $\mathbf{C}$.*

Proof. It is a routine verification so we omit the proof.

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$*

be a sequence of strictly positive real numbers. Then $[brf, \mathcal{M}, u, p]$ *and* $[brf_0, \mathcal{M}, u, p]$ *are paranormed spaces with the paranorm defined by*

$$g(x) = \inf\left\{ (\rho)^{\frac{p_k}{M}} : \left(\frac{1}{R_m} \sum_{k=0}^{m} \left[u_k M_k \left(\frac{r_k |s x_{k+n-1} + r x_{k+n}|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{M}} \leq 1, \right.$$

$$\left. \text{uniformly in } n > 0, \ \ \rho > 0 \right\},$$

where $0 \leq p_k \leq \sup p_k = H, \ \ M = \max(1, H)$.

Proof. For the proof verification see [15].

Theorem 2.3. *Let* $\mathcal{M} = (M_k)$ *be a sequence of Orlicz functions,* $u = (u_k)$ *be a sequence of strictly positive real numbers. If* $p = (p_k)$ *and* $q = (q_k)$ *are bounded sequences of positive real numbers with* $0 \leq p_k \leq q_k < \infty$ *for each* k*, then* $[brf_0, \mathcal{M}, u, p] \subseteq [brf, \mathcal{M}, u, q]$.

Proof. It is easy to prove.

Theorem 2.4. *Let* $\mathcal{M} = (M_k)$ *be a sequence of Orlicz functions which satisfies the* Δ_2*-condition and* $\beta = \lim_{t \to \infty} \frac{M_k(t)}{t} > 0$ *for all* $k \in \mathbf{N}$*. Then* $[brf_0, \mathcal{M}, u, p] \subseteq [brf_0, u, p]$.

Proof. It is easy to prove.
The following theorems can be proved in a similar way as in [15].

Theorem 2.5. *If* $\mathcal{M}' = (M_k')$ *and* $\mathcal{M}'' = (M_k'')$ *are sequences of Orlicz functions satisfying the* Δ_2*-condition, then*

$$[brf_0, \mathcal{M}, u, p] \cap [brf_0, \mathcal{M}', u, p] \subseteq [brf_0, (\mathcal{M}' + \mathcal{M}''), u, p].$$

Theorem 2.6. *Let* $\mathcal{M} = (M_k)$ *and* $\mathcal{M}' = (M_k')$ *be two sequences of Orlicz functions, then*

$$[brf_0, \mathcal{M}', u, p] \subseteq [brf_0, \mathcal{M} \circ \mathcal{M}', u, p],$$

where $\mathcal{M} \circ \mathcal{M}'$ *is the composition of* $\mathcal{M}$ *and* $\mathcal{M}'$.

Theorem 2.7. *The spaces* $[brf, \mathcal{M}, u, p]$ *and* $[brf_0, \mathcal{M}, u, p]$ *are BK-spaces with the norm defined by*

$$(2.1) \qquad \|x\|_{brf_0,\mathcal{M},u,p} = \|x\|_{brf,\mathcal{M},u,p}$$

$$= \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} \left[u_k M_k \left(\frac{r_k |s x_{k+n-1} + r x_{k+n}|}{\rho} \right) \right]^{p_k} \right|, \text{ uniformly in } n.$$

Proof. Since 1.4 holds, brf and brf_0 are the BK-spaces with respect to their norms (see Theorem 3.3 in [22]) and the matrix $B(r, s)$ is normal, Theorem 4.3.12 of Wilansky [19] gives the fact that $[brf, \mathcal{M}, u, p]$ and $[brf_0, \mathcal{M}, u, p]$ are BK-spaces with the given norms. This completes the proof.

Theorem 2.8. *The spaces* $[brf, \mathcal{M}, u, p]$ *and* $[brf_0, \mathcal{M}, u, p]$ *are linearly isomorphic to the spaces* brf *and* brf_0*, respectively.*

Proof. We only consider the sequence spaces $[brf, \mathcal{M}, u, p]$ and brf and other will follow similarly. To show the theorem, we must show the existence of linear bijection between the spaces $[brf, \mathcal{M}, u, p]$ and brf. For this, we consider the transformation T defined with the notation 1.4, from $[brf, \mathcal{M}, u, p]$ to brf by $x \to y = Tx$. The linearity of T is obvious. Further, it is trivial that $x = \theta = (0, 0, 0...)$ whenever $Tx = \theta$ and hence T is injective. Next, let $y = (y_k) \in brf$ and defined the sequence $x = (x_k)$ by $(\{B^{-1}(r, s)y\})_k$ for all $k \in \mathbf{N}$. Then, it is clear that

$$\{B^{-1}(r,s)y\}_k = s x_{k-1} + r x_k = \sum_{j=0}^{k-1} \frac{s}{r} \left(-\frac{s}{r} \right)^j y_{k-j-1} + \sum_{j=0}^{k-1} \left(-\frac{s}{r} \right)^j y_{k-j} = y_k$$

for all $k \in \mathbf{N}$ which shows that

$$\lim_m \frac{1}{R_m} \sum_{k=0}^{m} \left[u_k M_k \left(\frac{r_k |s x_{k+n-1} + r x_{k+n}|}{\rho} \right) \right]^{p_k}$$

$$= \lim_m \frac{1}{R_m} \sum_{k=0}^{m} \left[u_k M_k \left(\frac{r_k |y_{k+n}|}{\rho} \right) \right]^{p_k}$$

$$= [brf, \mathcal{M}, u, p] - \lim y_k, \text{ uniformly in n.}$$

Thus, $x = (x_k) \in [brf, \mathcal{M}, u, p]$. Consequently, it is clear that T is surjective. Because of the fact that is linear bijection, $[brf, \mathcal{M}, u, p]$ and brf are linearly isomorphic. This completes the proof.

Theorem 2.9. *The spaces $[brf_0, \mathcal{M}, u, p]$ and $[brf, \mathcal{M}, u, p]$ are convex spaces.*

Proof. The proof is clear from the definition of convexity.

Corollary 2.10. *The space $[brf, \mathcal{M}, u, p]$ does not have a Schauder basis.*

3. β and γ-Duals

In this section, we determine the β and γ-duals of the spaces $[brf, \mathcal{M}, u, p]$ and $[brf_0, \mathcal{M}, u, p]$. For the sequence spaces X and Y, define the set $S(X, Y)$ by

$$S(X,Y) = \{z = (z_k) \in w : xz = (x_k z_k) \in Y \text{ for all } x = (x_k) \in X\}. \tag{3.1}$$

With the notation of 3.1 the α-, β- and γ-duals of a sequence space X, which are, respectively, denoted by X^α, X^β and X^γ are defined by $S(X, l_1)$, $S(X, cs)$ and $S(X, bs)$.
The following theorems are proved by using some lemmas of [21].

Theorem 3.1. *The γ-dual of the space $[brf, \mathcal{M}, u, p]$ is the set $d_1(r, s)$, where*

$$d_1(r,s) = \left\{ a = (a_k) \in w : \sup_n \sum_{k=0}^{n} \left[u_k M_k \left(\frac{|\sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_j|}{\rho} \right) \right]^{p_k} < \infty \right\}.$$

Proof. The proof of the theorem is clear, so we omit it.

Theorem 3.2. *Let us write the sets $d_2(r, s), d_3(r, s)$ and $d_4(r, s)$ by*

$$d_2(r,s) = \left\{ a = (a_k) \in w : \lim_n \sum_{k=0}^{n} \left[u_k M_k \left(\frac{|\sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_j|}{\rho} \right) \right]^{p_k} \text{ exists} \right\},$$

$$d_3(r,s) = \left\{ a = (a_k) \in w : \lim_n \sum_{k=0}^{n} \left[u_k M_k \left(\frac{\left|\Delta\left(\sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_j - a_k\right)\right|}{\rho} \right) \right]^{p_k} = 0 \right\},$$

where

$$\Delta\Big(\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}a_j - a_k\Big) = \sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}a_j - \sum_{j+1=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j+1-k}a_{j+1} - a_k + a_{k+1}.$$

$$d_4(r,s) = \left\{ a=(a_k)\in w : \lim_n \sum_{k=0}^{n}\left[u_k M_k\left(\frac{\left|\left[\frac{1-(-\frac{s}{r})^{k+1}}{1+\frac{s}{r}}a_k\right]\right|}{\rho}\right)\right]^{p_k} \text{ exists}\right\}.$$

for all $j,k\in\mathbf{N}$*. Then,* $D=[brf,\mathcal{M},u,p]^{\beta}=\bigcap_{i=1}^{4}d_i(r,s)$.

Proof. Let us define the matrix $V=(v_{nk})$ via the sequence $z=(z_k)\in w$ by

$$v_{nk} = \begin{cases} \sum_{k=0}^{n}\left[u_k M_k\left(\frac{|\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}z_j|}{\rho}\right)\right]^{p_k}, & 0\le k\le n,\\ 0, & k>n,\end{cases}$$

for all $n,k\in\mathbf{N}$. By considering the relation

$$x_k = \sum_{k=0}^{n}\left[u_k M_k\left(\frac{|\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}y_j|}{\rho}\right)\right]^{p_k}, \text{ we have}$$

$$(3.2)\quad \sum_{k=0}^{n} z_k x_k = \sum_{k=0}^{n}\left[u_k M_k\left(\frac{|\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}z_j y_k|}{\rho}\right)\right]^{p_k} = (Vy)_n \quad (n\in\mathbf{N}).$$

From 3.2, we see that $zx=(z_kx_k)\in cs$ whenever $x=(x_k)\in[brf,\mathcal{M},u,p]$ if and only if $Vy\in c$ whenever $y=(y_k)\in brf$. Then, we have $[brf,\mathcal{M},u,p]^{\beta}=\bigcap_{i=1}^{4}d_i(r,s)$.

4. Matrix Transformations

Başar [1], Kuttner [5] and Lorentz and Zeller [8] have been used the methods of dual summability. Now, let us review these methods.
Let us suppose that the sequences $x=(x_k)$ and $y=(y_k)$ are connected

with 1.4 and let A-transform of the sequence $x = (x_k)$ be $z = (z_k)$ and B-transform of the sequence $y = (y_k)$ be $t = (t_k)$, that is,

$$z_k = (Ax)_k = \sum_k a_{nk} x_k, \quad (k \in \mathbf{N}) \tag{4.1}$$

$$t_k = (By)_k = \sum_k b_{nk} y_k, \quad (k \in \mathbf{N}). \tag{4.2}$$

Method B is applied to the $B(r,s)$-transform of the sequence $x = (x_k)$ while the method A is directly applied to the terms of the sequence $x = (x_k)$. So it is clear that A and B are essentially different [1]. Let us suppose that the matrix product $BB(r,s)$ exists. If z_k turns into t_k (or vice versa), under the application of the formal summation by parts, then the methods A and B as in 4.1 and 4.2 are called generalized difference dual type matrices. It means that $BB(r,s)$ exists and is equal to A. This condition is equivalent to the following equality:

$$b_{nk} = \sum_{k=0}^{n} \left[u_k M_k \left(\frac{|\frac{1}{r} \sum_{j=k}^{n} \left(-\frac{s}{r} \right)^{j-k} a_{nj}|}{\rho} \right) \right]^{p_k} \tag{4.3}$$

$$\text{or } a_{nk} = \sum_{k=0}^{n} \left[u_k M_k \left(\frac{|s b_{n,k-1} + r b_{nk}|}{\rho} \right) \right]^{p_k},$$

for all $n, k \in \mathbf{N}$.

Theorem 4.1. *Let μ be any given sequence space and the matrices $A = (a_{nk})$ and $B = (b_{nk})$ be generalized difference dual type matrices. Then, $A \in ([brf, \mathcal{M}, u, p] : \mu)$ if and only if $B \in (brf : \mu)$ and $(a_{nk})_{k \in \mathbf{N}} \in [brf, \mathcal{M}, u, p]^{\beta}$ for all $n \in \mathbf{N}$.*

Proof. Let μ be any sequence space and $A = (a_{nk})$ and $B = (b_{nk})$ be generalized difference dual type matrices, that is, 4.3 holds. Furthermore, the spaces $[brf, \mathcal{M}, u, p]$ and brf are isomorphic. Let $A \in ([brf, \mathcal{M}, u, p] : \mu)$ and $y = (y_k) \in brf$. Then $BB(r,s)$ exists and $(a_{nk})_{k \in \mathbf{N}} \in D$, it means that $(b_{nk})_{k \in \mathbf{N}} \in l_1$ for each $n \in \mathbf{N}$. Hence, we have

$$\sum_k b_{nk} y_k = \sum_k a_{nk} x_k, \tag{4.4}$$

for all $n \in \mathbf{N}$ which concluded that $B \in (brf : \mu)$. On the contrary, let $(a_{nk})_{k\in\mathbf{N}} \in [brf, \mathcal{M}, u, p]^{\beta}$ for each $n \in \mathbf{N}$ and $B \in (brf : \mu)$ and $x = (x_k) \in [brf, \mathcal{M}, u, p]$. Then it is clear that Ax exists. Thus, we attain from the following equality for all $n \in \mathbf{N}$

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m} \left[u_k M_k \left(\frac{|\frac{1}{r}\sum_{j=k}^{n} \left(-\frac{s}{r} \right)^{j-k} a_j|}{\rho} \right) \right]^{p_k} y_k = \sum_{k=0}^{m} b_{nk}y_k$$

as $m \to \infty$ that $Ax = By$ and it is easy to show that $A \in ([brf, \mathcal{M}, u, p] : \mu)$. This completes the proof.

Theorem 4.2. *Let us assume that the components of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the following relation*

$$e_{nk} = \sum_{k=0}^{n} \left[u_k M_k \left(\frac{|sa_{n-1,k} + ra_{nk}|}{\rho} \right) \right]^{p_k}, \tag{4.5}$$

for all $n \in \mathbf{N}$ and μ be any given sequence space. Then, $A \in (\mu : [brf, \mathcal{M}, u, p])$ if and only if $E \in (\mu : brf)$.

Proof. It is easy to prove.
Now, we list the following conditions;

$$\sup_n \sum_{k=0}^{n} \left[u_k M_k \left(\frac{|\sum_{j=k}^{n} \frac{1}{r} \left(-\frac{s}{r} \right)^{j-k} a_{nj}|}{\rho} \right) \right]^{p_k} < \infty, \tag{4.6}$$

$$\lim_n \left[u_k M_k \left(\frac{|\sum_{j=k}^{n} \frac{1}{r} \left(-\frac{s}{r} \right)^{j-k} a_{nj}|}{\rho} \right) \right]^{p_k} = a_k \ \forall k \in \mathbf{N}, \tag{4.7}$$

$$\lim_n \sum_{k=0}^{n} \left[u_k M_k \left(\frac{\left|\Delta \left(\sum_{j=k}^{n} \frac{1}{r} \left(-\frac{s}{r} \right)^{j-k} a_{nj} - a_k \right)\right|}{\rho} \right) \right]^{p_k} = 0, \tag{4.8}$$

for each fixed $k \in \mathbf{N}$,

where $\Delta\Big(\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}a_{nj}-a_k\Big)$

$$= \sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}a_{nj}-\sum_{j+1=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j+1-k}a_{n,j+1}-a_k+a_{k+1}.$$

$$brf-\lim_n\left[u_kM_k\left(\frac{|\sum_{j=k}^{n}\frac{1}{r}\Big(-\frac{s}{r}\Big)^{j-k}a_{nj}|}{\rho}\right)\right]^{p_k} = a_k \tag{4.9}$$

exists for each fixed $k \in \mathbf{N}$,

$$\sup_n\sum_k\left[u_kM_k\left(\frac{|sa_{n-1,k}+ra_{nk}|}{\rho}\right)\right]^{p_k} < \infty, \tag{4.10}$$

$$brf-\lim_n\left[u_kM_k\left(\frac{|sa_{n-1,k}+ra_{nk}|}{\rho}\right)\right]^{p_k} = a_k, \tag{4.11}$$

exists for each $k \in \mathbf{N}$,

$$brf-\lim_n\left[u_kM_k\left(\frac{|sa_{n-1,k}+ra_{nk}|}{\rho}\right)\right]^{p_k} = a, \tag{4.12}$$

By using the lemmas of [21] and Theorems 4.1 and 4.2, we derive the following results:

Corollary 4.3. *The following statements hold:*
(i) $A=(a_{nk})\in([brf,\mathcal{M},u,p]:l_\infty)$ *if and only if* $(a_{nk})_{k\in\mathbf{N}}\in[brf,\mathcal{M},u,p]^\beta$ *for all* $n\in\mathbf{N}$ *and 4.6 holds.*
(ii) $A=(a_{nk})\in([brf,\mathcal{M},u,p]:c)$ *if and only if* $(a_{nk})_{k\in\mathbf{N}}\in[brf,\mathcal{M},u,p]^\beta$ *for all* $n\in\mathbf{N}$ *and 4.6, 4.7, 4.8 and 4.9 hold.*
(iii) $A=(a_{nk})\in(l_\infty:[brf,\mathcal{M},u,p])$ *if and only if 4.10, 4.11 and 4.12 hold.*
(iv) $A=(a_{nk})\in(c:[brf,\mathcal{M},u,p])$ *if and only if 4.11 and 4.12 hold.*

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