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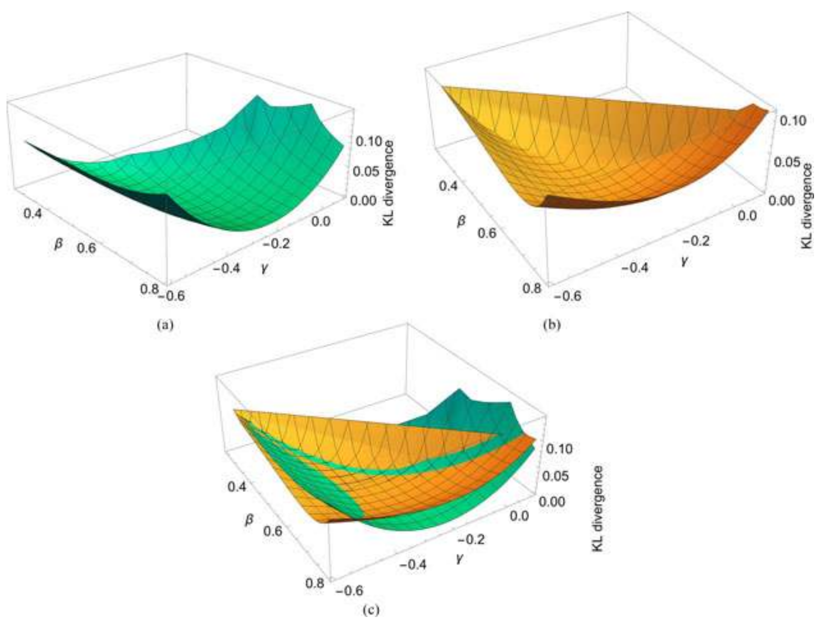

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A Novel Approximation for K Distribution: Closed-Form BER Using DPSK Modulation in Free-Space Optical Communication

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Abstract: A new analytical approximate expression for *K* distribution is proposed by expanding it in terms of orthogonal associated Laguerre polynomial. The expansion is truncated after first three terms, which yields a fairly close approximation to *K* distribution. The advantage of the proposed approximation is that the analytical closed form expression for bit error rate can be easily derived. KL measure is used to show the accuracy of the proposed approximation. The proposed approximate probability density function and bit error rate work well within the desired range of the channel parameter α, which is $1 < \alpha < 2$ and corresponds to the scintillation index value ranging from 2 to 3. We have also demonstrated the utility of our approximation for other quality of service metric such as fade probability.

Index Terms: *K* distribution, orthogonal polynomials, associated Laguerre polynomials, DPSK, BER, fade probability.

1. Introduction

Free space optical (FSO) communication systems have been attracting considerable attention these days for a variety of applications because of high bandwidth [1], efficient solution for last mile access, free licensing, green communication, easy deployment, cost effectiveness and security [2]–[4]. Free space optical communication also known as Optical Wireless Communication (OWC) is a line of sight (LOS) technology which helps in the last mile connectivity and more so in the 5G wireless broadband network [5].

A major challenging factor hampering the wide adoption of FSO is the atmospheric turbulence induced fading which is also known as scintillation [6]. Inhomogenieties in the temperature and pressure of the atmosphere leads to variation of the refractive index on account of atmospheric turbulence [7]. Due to the random fluctuations in the received signal FSO system performance degrades severely particularly over long ranges.

Several statistical models have been proposed to model the random fluctuations. One of the widely accepted models under strong turbulence regime is the K distribution [8]. Jakeman and Pusey [9] showed the significance of K distribution in modelling turbulence. The K distribution has been found to be a suitable model for strong turbulence channel, since it provides an excellent agreement between theoretical and experimental data [9]–[11].

Kiasaleh [12] in his paper notes that when the normalized scintillation index (SI) is confined to the range $2 < SI < 3$, or when moderate propagation distances are encountered, K distribution may be a better model [12], [13]. Although many researchers [12], [14], [15] have obtained the bit error rate (BER) over K-distributed channel model, the expressions obtained are complex and do not lead to easy insight about parameter dependence.

Tekel and Cohen [16] used Laguerre polynomials to approximate the K distribution which works pretty decently. However, a better efficacy of K distribution is desired in the range $1 < \alpha < 2$ which corresponds to $2 < SI < 3$ in view of the relationship $\alpha = 2/(SI - 1)$ [12]. To make the approximation better in this desired range of channel parameter α, we propose an associated Laguerre polynomial based approximation. This paper derives a novel approximation of K probability density function (PDF) using associated Laguerre polynomials and thereby obtains a closed form BER expression based on differential phase shift keying (DPSK) scheme. For proving the efficacy of the proposed model, Kullback–Leibler (KL) measure is analyzed. In addition, it also gives a novel K-PDF approximation which gives excellent result in large SNR range.

The rest of the paper is organized as follows: K distribution and its moment expressions are summarized in Section 2. In Section 3, it gives the approximation based on associated Laguerre polynomial. In Section 4, we fix the parameters of the approximation to optimize its efficacy in the α range 1 to 2. In Section 5 BER expression based on the proposed approximation is derived. Section 6 provides yet another expression for BER which is suited for high Signal to Noise Ratio (SNR) regime. In Section 7 we also examine another Quality of Service (QoS) metric – the fade probability (FP), which follows from the expression of cumulative distribution function (CDF). The paper concludes with Section 8.

2. K Distribution

K distribution is one of the widely used distribution to capture the effect of turbulence in free space optical communication. The K distribution is given by the PDF [6]

$$f_K(x) = \frac{2\alpha^{\frac{\alpha+1}{2}} x^{\frac{\alpha-1}{2}}}{\mu^{\frac{\alpha+1}{2}}\Gamma(\alpha)} K_{\alpha-1}\left(2\sqrt{\frac{\alpha x}{\mu}}\right), x \geq 0 \tag{1}$$

where $K_n(z)$ represents the modified Bessel function of the second kind, and $\Gamma(z)$ is the Gamma function. The parameter μ is the mean irradiance and α, as already introduced, is the channel parameter. The moments of K distribution are known to be [16]

$$E[x^m] = \int_0^\infty x^m f_K(x)\, dx = m! \left(\frac{\mu}{\alpha}\right)^m \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)}. \tag{2}$$

These moments satisfy the recursion relation

$$E[x^{m+1}] = \frac{\mu}{\alpha}(m+1)(m+\alpha)E[x^m], \tag{3}$$

with $E[x^0] = 1$ which signifies that the PDF is normalized to 1. K distribution is less sensitive at large values of α. A limiting case of the K distribution is the negative exponential model, which is considered when the number of discrete scattering regions in the turbulent medium is sufficiently large [6]. It is derived from K-PDF when $\alpha \to \infty$. The PDF of irradiance in this case is given by:

$$f_X(x) = e^{-x}, \; x \geq 0. \tag{4}$$

3. Approximations of K Distribution

Although the K distribution captures the realistic features of turbulence, the analytical expression of QoS gets marred resulting from complicated expression due to the presence of modified Bessel function. In light of this it would be desirable to look for an analytical expression which fairly approximates the shape and form of K distribution [17].

3.1 Approximation Based on Polynomials

The basic idea behind the K distribution approximation is based on expanding a given distribution in terms of orthogonal polynomials [18], [19] ($p_m(x)$) which constitute a complete orthogonal basis in a desired domain ($\mathcal{D}$), i.e.

$$f(x) = w(x)\sum_{m=0}^{\infty} c_m p_m(x), \tag{5}$$

where the orthogonal polynomials satisfy the orthogonality relation

$$\int_{\mathcal{D}} w(x)p_m(x)p_n(x)\,dx = N_m\delta_{m,n}. \tag{6}$$

Here $w(x)$ is the weight function, N_m is the normalization constant, and $\delta_{m,n}$ is the Kronecker delta. The coefficients c_m in (5) can be obtained using orthogonality as

$$c_m = \frac{1}{N_m}\int_{\mathcal{D}} p_m(x)f(x)\,dx. \tag{7}$$

The approximation for $f(x)$ can be obtained by truncating the infinite series in (5) to a finite number of terms. For K distribution we have $\mathcal{D} = [0, \infty)$, therefore we need to choose a suitable set of orthogonal polynomials to find an approximation. A natural choice is to use the associated Laguerre polynomials which are orthogonal with respect to the weight function $w(x) = x^{\gamma}e^{-x}(\gamma > -1)$ in $[0, \infty)$ [18]–[20]. This includes the case of Laguerre polynomials which is obtained for $\gamma = 0$.

3.2 Approximation Based on Associated Laguerre Polynomials

We consider the associated Laguerre weight function [18]–[20]

$$w(x) = x^{\gamma}e^{-\beta x};\ \gamma > -1, \beta > 0, \tag{8}$$

and the corresponding associated Laguerre polynomials $L_m^{(\gamma)}(\beta x)$. These satisfy the orthogonality relation

$$\int_0^{\infty} x^{\gamma}e^{-\beta x}L_m^{(\gamma)}(\beta x)L_n^{(\gamma)}(\beta x)\,dx = N_m\delta_{m,n}, \tag{9}$$

with

$$N_m = \frac{1}{\beta^{\gamma+1}}\frac{\Gamma(\gamma+m+1)}{m!}. \tag{10}$$

The series expansion for associated Laguerre polynomials is given by [19]

$$L_m^{(\gamma)}(z) = \sum_{j=0}^{m}\frac{\Gamma(m+\gamma+1)(-z)^j}{\Gamma(j+\gamma+1)\Gamma(m-j+1)\Gamma(j+1)}. \tag{11}$$

Expanding the K-distribution in terms of associated Laguerre polynomials, we get,

$$f_K(x) = x^{\gamma}e^{-\beta x}\sum_{m=0}^{\infty} c_m L_m^{(\gamma)}(\beta x), \tag{12}$$

where the coefficients are given by

$$c_m = \frac{1}{N_m}\int_0^{\infty} L_m^{(\gamma)}(\beta x)f_K(x)\,dx. \tag{13}$$

Using (11) in this we get,

$$c_m = \frac{1}{N_m}\sum_{j=0}^{m}\frac{\Gamma(m+\gamma+1)(-\beta)^j}{\Gamma(j+\gamma+1)\Gamma(m-j+1)\Gamma(j+1)}\int_0^{\infty} x^j f_K(x)\,dx. \tag{14}$$

Finally, using (2) and (10) we obtain the coefficients in terms of the parameters μ, α, β and γ:

$$c_m = \frac{\beta^{\gamma+1} m!}{\Gamma(\alpha)\Gamma(\gamma+m+1)} \sum_{j=0}^{m} \frac{\Gamma(m+\gamma+1)\Gamma(j+\alpha)}{\Gamma(j+\gamma+1)\Gamma(m-j+1)} \left(-\frac{\mu\beta}{\alpha}\right)^j. \tag{15}$$

For example, let us consider the associated Laguerre polynomials up to second degree:

$$\begin{aligned} L_0^{(\gamma)}(\beta x) &= 1, \\ L_1^{(\gamma)}(\beta x) &= 1+\gamma-\beta x, \\ L_2^{(\gamma)}(\beta x) &= \frac{1}{2}(\gamma+1)(\gamma+2) - \beta(\gamma+2)x + \frac{\beta^2}{2}x^2. \end{aligned} \tag{16}$$

The corresponding coefficients are

$$\begin{aligned} c_0 &= \frac{\beta^{\gamma+1}}{\Gamma(\gamma+1)}, \\ c_1 &= \frac{\beta^{\gamma+1}(1+\gamma-\beta\mu)}{\Gamma(\gamma+2)}, \\ c_2 &= \frac{\beta^{\gamma+1}\left[\alpha(\gamma+1)(\gamma+2) - 2\alpha\beta\mu(\gamma+2) + 2(\alpha+1)\beta^2\mu^2\right]}{\alpha\Gamma(\gamma+3)}. \end{aligned} \tag{17}$$

Using these results in (5) along with (7) and (10), we obtain the approximation to the K-PDF:

$$\begin{aligned} f_{AL}(x) = \frac{\beta^{\gamma+1}}{\Gamma(\gamma+1)} x^{\gamma} e^{-\beta x} \Bigg[1 + \left(\frac{1+\gamma-\beta\mu}{\gamma+1}\right)(1+\gamma-\beta x) \\ + \left(1 - \frac{2\beta\mu}{\gamma+1} + \frac{2\beta^2\mu^2(\alpha+1)}{\alpha(\gamma+1)(\gamma+2)}\right)\left(\frac{(\gamma+1)(\gamma+2)}{2} - \beta(\gamma+2)x + \frac{\beta^2}{2}x^2\right)\Bigg]. \end{aligned} \tag{18}$$

This may also be re-written in the following form:

$$f_{AL}(x) = \frac{\beta^{\gamma+1}}{\Gamma(\gamma+1)} x^{\gamma} e^{-\beta x} (A\beta^2 x^2 + B\beta x + C), \tag{19}$$

where

$$\begin{aligned} A &= \frac{\beta^2\mu^2(\alpha+1)}{\alpha(\gamma+1)(\gamma+2)} - \frac{\beta\mu}{\gamma+1} + \frac{1}{2}, \\ B &= -\frac{2\beta^2\mu^2(\alpha+1)}{\alpha(\gamma+1)} + \frac{\beta\mu(2\gamma+5)}{\gamma+1} - (\gamma+3), \\ C &= \frac{\beta^2\mu^2(\alpha+1)}{\alpha} - \beta\mu(\gamma+3) + \frac{(\gamma+2)(\gamma+3)}{2}. \end{aligned} \tag{20}$$

We should emphasize here that the dependence of the approximate PDF on the parameters here is algebraic and is comparatively easier to comprehend. In contrast, the exact PDF (1) involves the parameter dependence via Bessel function, along with the prefactor, which makes it difficult to understand the effect of parameters on the distribution.

It is to be noted that the approximate expression in (18) and (19) can be improved further if we include higher moments. Based on numerical computation we find that, the relative error is minimal in the given expression. Here, the overall behavior of the approximation is decided by the parameters $\alpha, \beta, \gamma, \mu$. In what follows, we fix the μ value at 1 and then try to optimize β and γ values to get a close approximation to the K-PDF for the α-value range (1,2). To this end, we use KL divergence [21] to quantify the closeness of approximate PDF with the actual PDF.

TABLE 1
Variation of KL Divergence With β and γ Parameters for Associated Laguerre Approximation With Up To Second Degree Polynomial

β	γ	KL divergence
0.5	−0.5	0.043675648
0.5	−0.4	0.016632878
0.5	−0.3	0.011060965
0.5	−0.2	0.031620095
0.6	−0.5	0.047153415
0.6	−0.4	0.018289510
0.6	**−0.3**	**0.0068528312**
0.6	−0.2	0.013276104
0.6	−0.1	0.041121777
0.6	0.0	0.14293554
0.7	−0.5	0.054937996
0.7	−0.4	0.023668934
0.7	−0.3	0.0081835361
0.7	−0.2	0.0080036108
0.7	−0.1	0.023818450
0.7	0.0	0.059024558

The KL divergence is a non symmetric measure of distance between two probability distributions $p(x)$ and $q(x)$. Specifically, the KL divergence of $q(x)$ from $p(x)$ denoted by $D_{KL}(p(x), q(x))$ is a measure of distance between true and approximated distributions. The KL divergence is defined as [21]:

$$D_{KL}(p(x), q(x)) = \int_X p(x) \ln \frac{p(x)}{q(x)} dx. \tag{21}$$

We take $p(x)$ as the exact K-PDF and $q(x)$ as the approximation for which we want to investigate the closeness to the exact PDF.

In Tables I and II we compile the KL divergence values for several values of the parameters β and γ. The μ has been fixed at 1 while α value has been kept fixed at 1.5, which lies midway in the range of our interest, viz. $1 < \alpha < 2$. Fig. 1 shows the corresponding surface plots. Fig. 1(a) and (b) respectively show the plots using Associated Laguerre polynomial approximation involving up to second degree polynomial (green surface) and up to third degree polynomial (orange surface). In Fig. 1(c) we plot the two together. We see that a lower value of KL divergence can be achieved for a certain choice of β and γ parameters for the second degree polynomial based approximation. Considering higher order terms may lead to improved accuracy, but the price one has to pay is the increased complexity of the expressions. We therefore find that it better to truncate the proposed approximation in (12) at $m = 2$ and use (18).

4. Parameter Optimization in the Proposed Approximation

We analyze more closely the approximation based on associated Laguerre polynomials, (18) and examine its accuracy for different α values. In principle, for each value of α one can look for optimal values of β and γ. However, it would not be an efficient way of using the approximation. It would be

TABLE 2

Variation of KL Divergence With β and γ Parameters for Associated Laguerre Approximation With Up To Third Degree Polynomial

β	γ	KL divergence
0.4	−0.6	0.087305002
0.4	−0.5	0.042027896
0.5	−0.6	0.077632201
0.5	−0.5	0.040491666
0.5	−0.4	0.017216078
0.6	−0.6	0.067192943
0.6	−0.5	0.038501532
0.6	−0.4	0.018878332
0.6	**−0.3**	**0.0098852024**
0.6	−0.2	0.013276104
0.7	−0.6	0.063174223
0.7	−0.5	0.042584054
0.7	−0.4	0.026952702
0.7	−0.3	0.018526651
0.7	−0.2	0.019162160
0.7	−0.1	0.030873671
0.7	0.0	0.056837471

helpful if we find some relationship between parameters β and α, and γ and α. Therefore, considering a range of α values and making β, γ dependant on α in such a way that the approximation works well. We find that the following simple choice of β and γ work reasonably well for $1 < \alpha < 2$, which is the region of our interest.

$$\beta = 0.175\alpha + 0.38, \;\; \gamma = -\frac{0.38}{\alpha}. \tag{22}$$

To arrive at (22), we have taken into consideration the optimal β and γ values at various α values. The above choice works at $\alpha = 1, 1.5, 2$, which is well within our region of interest as can be seen in the KL divergence plot in Fig. 2 where we consider α values ranging from 1 to 3. The μ value is fixed at 1 while the values of β and γ are obtained using (22). We can see that the KL divergence value is quite low which indicates the closeness of our approximation to the exact K-PDF.

In Fig. 3, the plots of the K-PDF and its approximation (18) are shown below for different values of α, with μ fixed at 1. We can see that approximation works quite well in the desired range of α values, viz. $1 < \alpha < 2$.

5. Bit Error Rate (BER)

The SNR is a common measure of performance in analog communication. However, in digital communication, BER is considered to be a more effective measure for performance analysis [8]. Under DPSK scheme, assuming a shot noise limited heterodyne detection, the signal dependent

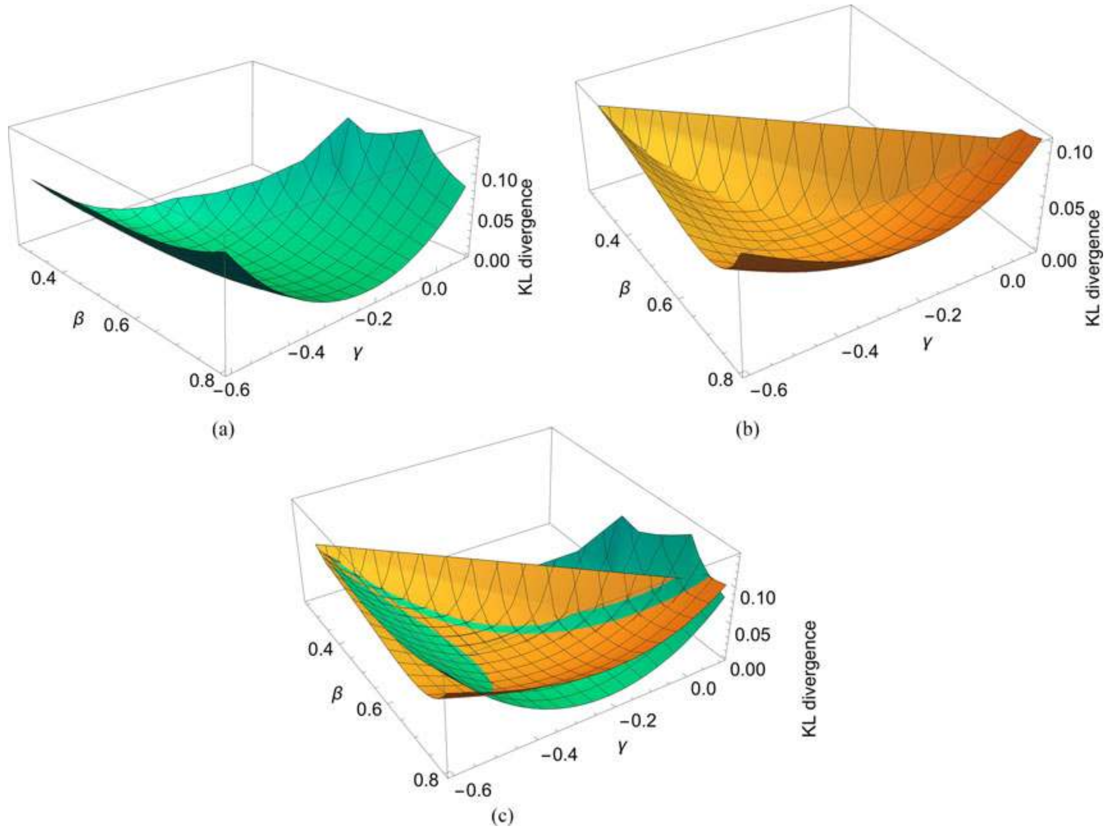

Fig. 1. Variation of KL divergence for the parameters β and γ for associated Laguerre polynomial approximation. (a) $\alpha = 1$. (b) $\alpha = 1.5$. (c) $\alpha = 2$.

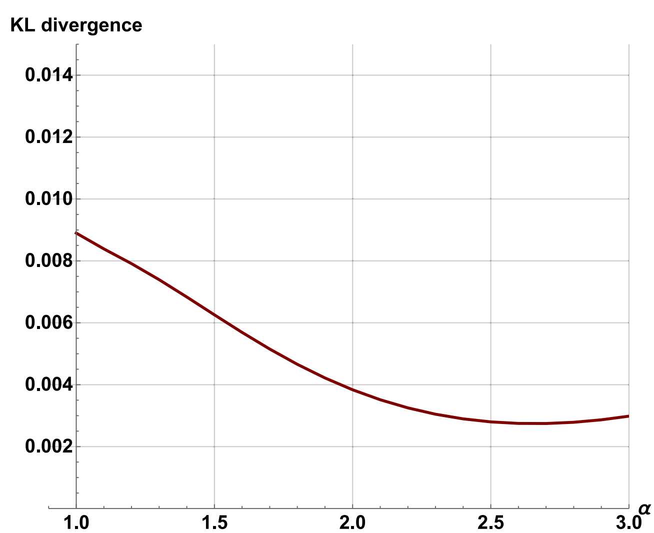

Fig. 2. Variation of KL divergence of the proposed approximation (18) with α.

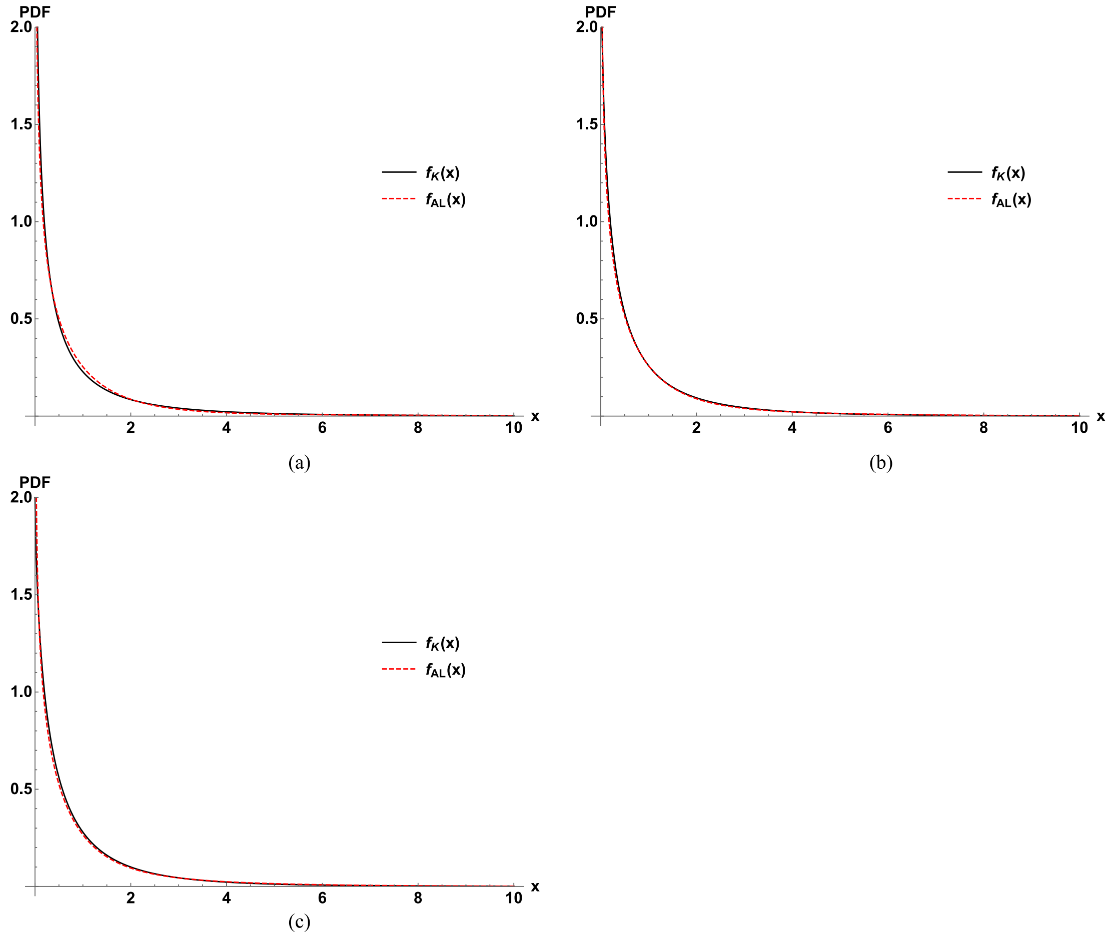

Fig. 3. Direct comparison of the exact K-PDF and the proposed approximation (18) for three α values.

BER is given by [12]

$$\mathcal{P}_e(x) = \frac{1}{2} e^{-\mathrm{SNR}(x)}, \tag{23}$$

where SNR(x) represents the signal intensity-dependent SNR and is given by

$$\mathrm{SNR}(x) = sx; \; s = \frac{\eta A T}{h\nu}. \tag{24}$$

Here η is the quantum efficiency of the detector, A is the detector area in m^2, T is the DPSK symbol duration in seconds, h is Planck's constant in Joules/Hz, and ν denotes the frequency of the received optical signal in Hz.

In presence of optical turbulence, the probability of error in (23) must be averaged over the period of the random signal to derive the unconditional BER. Given a slow-fading environment, the unconditional BER is obtained using

$$P_e = \int_0^{\infty} \mathcal{P}_e(x) f(x)\, dx, \tag{25}$$

where $f(x)$ represents the K-PDF or one of its approximations, as described above. The exact and approximation results for P_e are given below.

5.1 Simple Exact Expression

Kiasaleh [12] has derived the exact expression for BER under DPSK using complex Whittaker function and hypergeometric function, which gives a complex expression and does not easily lend to easy interpretation of parameter dependance. We derive BER expression for DPSK using easy mathematical expressions given below. It is convenient to introduce a variable [12]

$$\zeta = \frac{\alpha}{\mu s}. \tag{26}$$

Also, since μ is the mean intensity level, $\mathbb{E}[\mathrm{SNR}(x)] = \mu s$ may be viewed as the averaged SNR [12]. In the plots we, therefore, use averaged SNR for analysis. We also note that, we may use $\mu = 1$ without loss of any generality, and then averaged SNR is $\mathbb{E}[\mathrm{SNR}(x)] = s$. This is because for $\mu \neq 1$, we can still work with $\mu = 1$ by scaling s accordingly. Using $f(x) = f_K(x)$ given by (1), we obtain a more simplified exact expression for BER under the DPSK scheme as

$$P_e^{(\mathrm{DPSK})} = \frac{1}{2}\zeta e^{\zeta} E_{\alpha}(\zeta) = \frac{1}{2}\zeta^{\alpha} e^{\zeta}\Gamma(1-\alpha, \zeta). \tag{27}$$

Here $E_a(z) = \int_1^\infty dt\, e^{-zt}/t^a$ is the exponential integral function, and $\Gamma(a, z) = \int_z^\infty dt\, t^{a-1} e^{-t}$ is the upper incomplete Gamma function. The proof of the above derivation is given in the appendix. Other equivalent results for P_e have been provided in [12].[1]

5.2 Approximate Explicit Expression

We use the approximation (18) of K-PDF in (25) to obtain the BER approximation for the DPSK scheme. Due to the simple functional form of (18), we need no more than the integral $\int_0^\infty x^a e^{-bx}\, dx = \Gamma(a+1)/b^{a+1}$ for this derivation. We have

$$P_e^{(\mathrm{DPSK})} = \frac{\beta^{\gamma+1}}{4\alpha(\beta+s)^{\gamma+3}}\Big[\big\{2(\alpha+1)\beta^2\mu^2 - 2\alpha\beta(\gamma+3)\mu + \alpha(\gamma+2)(\gamma+3)\big\}s^2 + 2\alpha\beta(-\beta\mu+\gamma+3)s + 2\alpha\beta^2\Big]. \tag{28}$$

We use the β and γ values in this equation based on (22). The comparison between the exact result and the approximation is shown in Fig. 4. We can see very good agreement in all cases. In the next section we derive approximations for the BER, for large SNR, that work remarkably well.

6. Large SNR Approximation for BER

We have derived closed form approximation of the K distribution using associated Laguerre polynomial suitable for Free Space Optical Communication. For the sake of the completeness of the approximation of K distribution, we are also providing, a much simpler approximation which is valid under high SNR regime.

Consider the transformation $x = y^2$ in (25). This gives

$$P_e^{(\mathrm{DPSK})} = 2\int_0^\infty y\mathcal{P}_e(y^2) f(y^2)\, dy. \tag{29}$$

In Fig. 5 we examine the behavior of the integrand $y\mathcal{P}_e(y^2)$ and find that it decays rapidly with y for large SNR. Therefore, in (25), the contribution to the integral comes mainly from the region where y is close to zero. Consequently, we may use a small-y approximation for the K-PDF. One such approximation is

$$K_\nu(z) \approx \frac{1}{2}\left[\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\left(1+\frac{z^2}{4(1-\nu)}\right) + \Gamma(-\nu)\left(\frac{z}{2}\right)^{\nu}\left(1+\frac{z^2}{4(1+\nu)}\right)\right], \tag{30}$$

[1] A factor of 2 has been missed in [12, (7)] which has led to the same factor missing in subsequent results for BER.

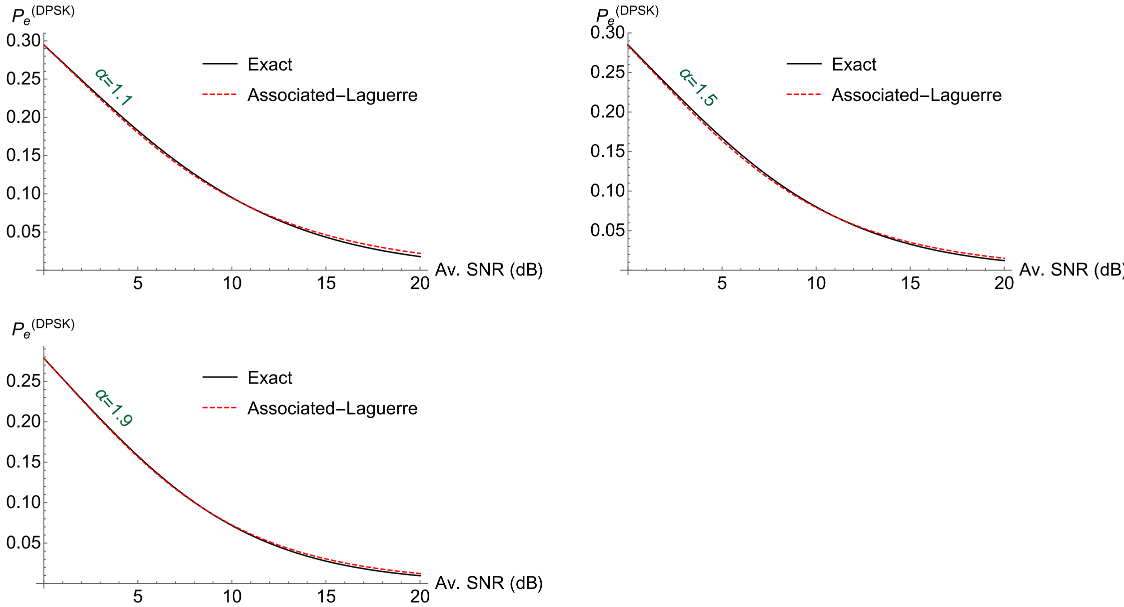

Fig. 4. Comparison between exact BER (black-solid) and approximate BER (red-dashed) and associated-Laguerre approximations for 0 $\leq$ averaged SNR $\leq$ 20 dB for DPSK scheme.

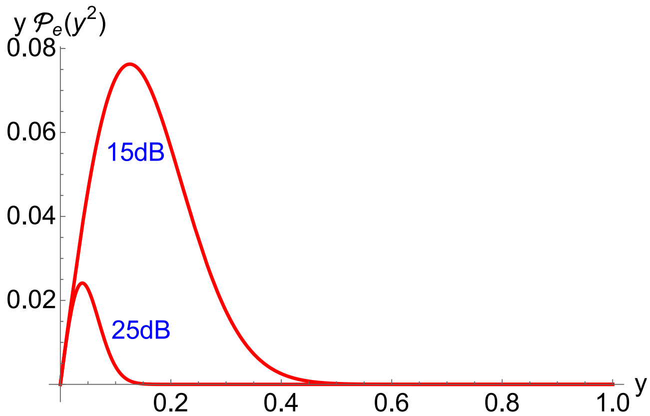

Fig. 5. Behavior of $y\mathcal{P}_e(y^2)$ for large averaged-SNR; see (29).

which holds for non-integer values of ν [22]. It should be noted that Gamma function is well defined in the entire complex plane except at 0 and negative integers. With this approximation we obtain the BER result for DPSK scheme as

$$P_e^{(\mathrm{DPSK})} \approx \frac{1}{2}\left(\frac{\zeta}{\alpha-1} + \frac{\zeta^2}{(\alpha-1)(2-\alpha)} + \zeta^{\alpha}(\zeta+1)\,\Gamma(1-\alpha)\right), \tag{31}$$

In Fig. 6 we compare this approximation with the exact result (27). We can see that the approximation work remarkably well and complements the approximation (28) which is very effective in the low SNR region. We should note that by considering a few more terms in (30) the approximations can be made to work better in low SNR region as well.

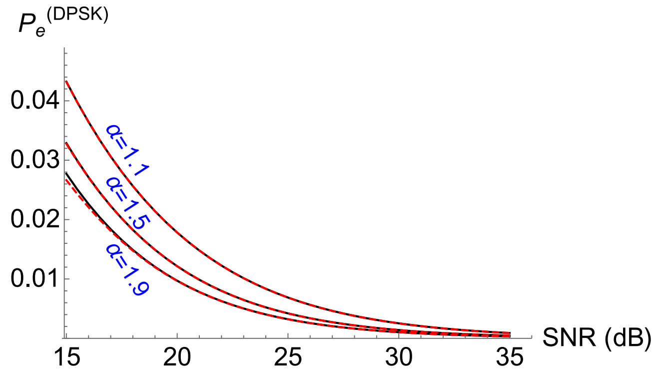

Fig. 6. Comparison between exact BER (black) and approximate BER (red-dashed) for averaged-SNR $\geq$ 15 dB for DPSK scheme.

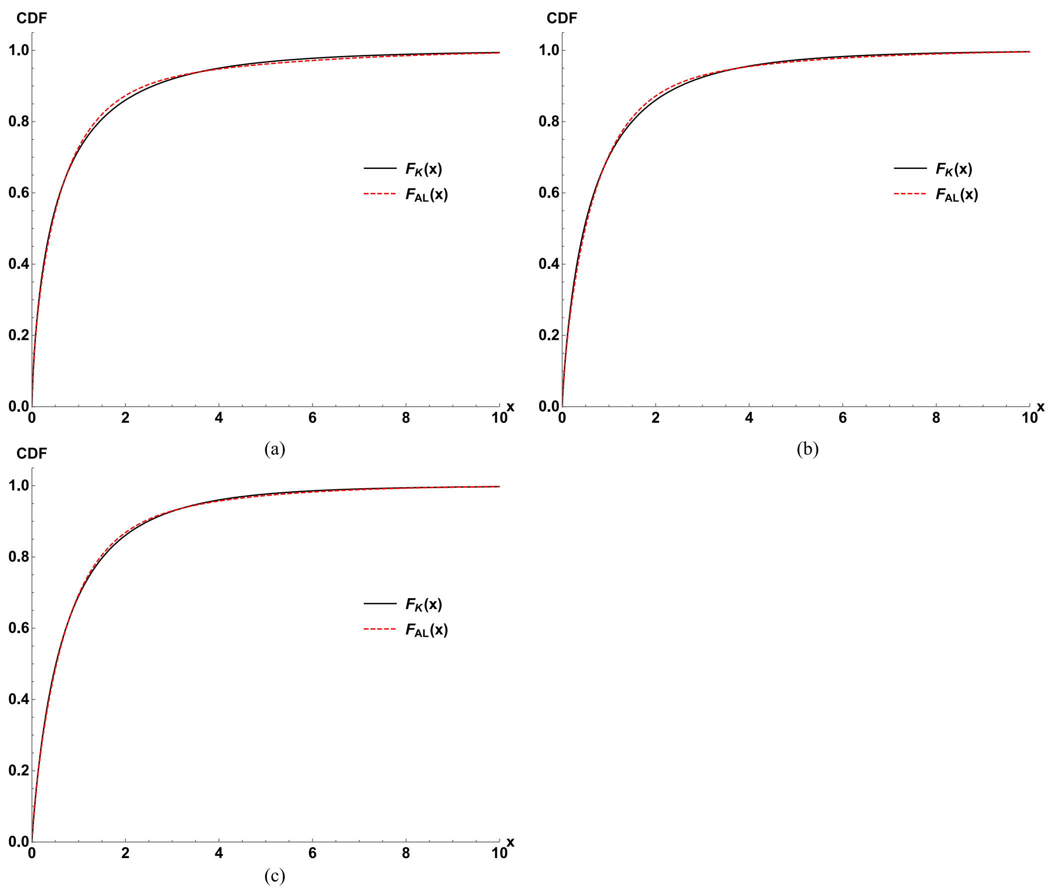

Fig. 7. Comparison of the exact CDF (34) and that based on proposed approximation (36) for three α values. (a) $\alpha = 1$. (b) $\alpha = 1.5$. (c) $\alpha = 2$.

7. Cumulative Distribution Function and Fade Probability

In addition to obtaining BER, we can obtain QoS measures like Fade Probability by using the proposed approximate PDF. The advantage accrued by this PDF is that it is mathematically tractable. The fade probability or miss probability is an estimation of the likelihood of the output of the intensity

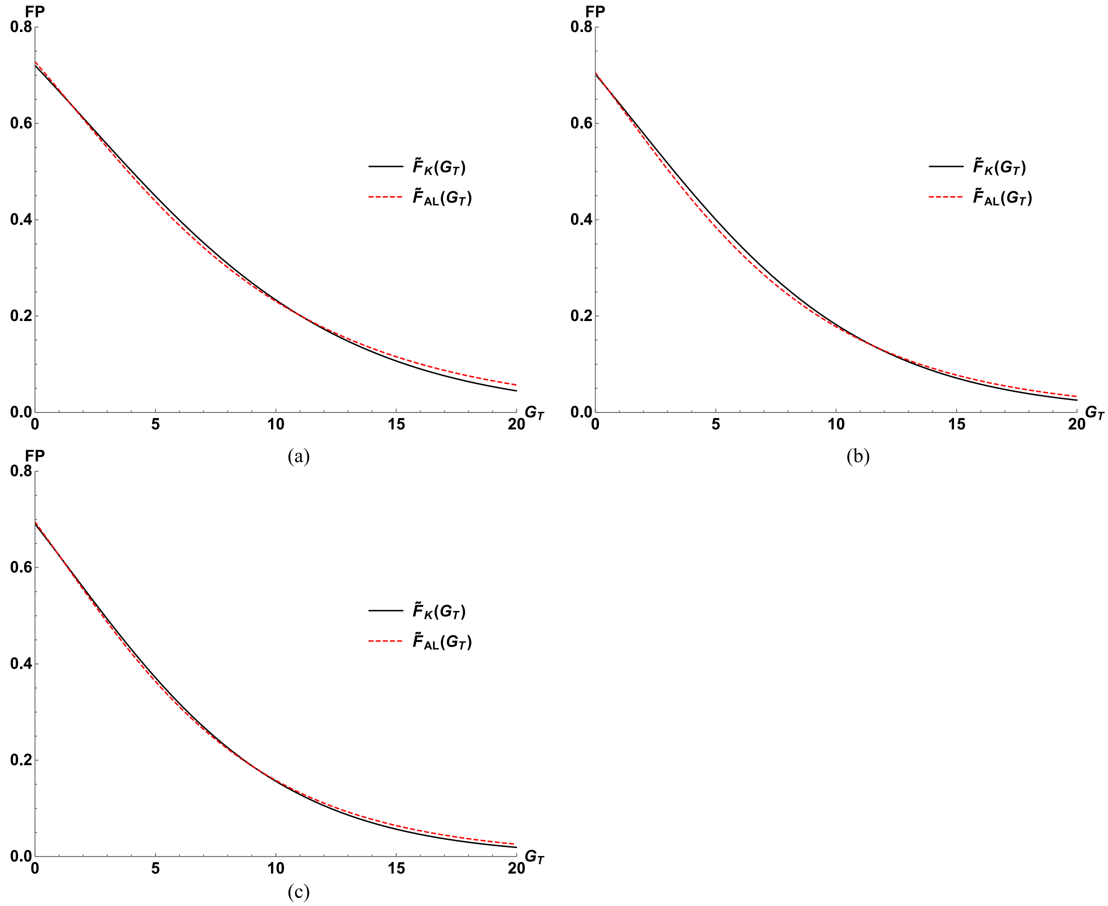

Fig. 8. Comparison of the exact fade probability (35) and that based on proposed approximation (37) for three α values. (a) $\alpha = 1$. (b) $\alpha = 1.5$. (c) $\alpha = 2$.

dropping below a given threshold [6], i.e.

$$P(X < x_T) = \int_0^{x_T} f_X(x)dx = F_X(x_T) \tag{32}$$

Here x_T is the threshold, $f_X(x)$ is the PDF of intensity and $F_X(x_T)$ is the cumulative distribution function (CDF). Following Andrews *et al.* [8], the fade probability can also be written in terms of the fade threshold parameter G_T. It is defined as

$$G_T = 10\log_{10}\left(\frac{\mu}{x_T}\right) \text{dB}, \tag{33}$$

and represents the number of decibels the threshold x_T is setup below the mean intensity.

The exact cumulative distribution function (CDF) for KPDF is known to be

$$F_K(x) = \int_0^x f_K(x')dx' = 1 - \frac{2}{\Gamma(\alpha)}\left(\frac{\alpha x}{\mu}\right)^{\alpha/2} K_{-\alpha}\left(2\sqrt{\frac{\alpha x}{\mu}}\right), \tag{34}$$

Correspondingly, the FP is given by

$$\widetilde{F}_K(G_T) = 1 - \frac{2}{\Gamma(\alpha)}\left(\alpha e^{-\lambda G_T}\right)^{\alpha/2} K_{-\alpha}\left(2\sqrt{\alpha e^{-\lambda G_T}}\right), \tag{35}$$

where $\lambda = \ln(10)/10 \approx 0.230259$. We see that the above expressions still involve the Bessel function. On the other hand using our approximate PDF (19), we obtain the CDF as

$$F_{AL}(x) = \int_0^x f_{AL}(x')dx' = \frac{1}{\Gamma(\gamma+1)}\left[A\,\Upsilon(\gamma+3,\beta x) + B\,\Upsilon(\gamma+2,\beta x) + C\,\Upsilon(\gamma+1,\beta x)\right], \quad (36)$$

which gives the FP as

$$\widetilde{F}_{AL}(x) = \frac{1}{\Gamma(\gamma+1)}\left[A\,\Upsilon(\gamma+3,\beta\mu e^{-\lambda G_T}) + B\,\Upsilon(\gamma+2,\beta\mu e^{-\lambda G_T}) + C\,\Upsilon(\gamma+1,\beta\mu e^{-\lambda G_T})\right]. \quad (37)$$

Here A, B, C are given by (20), and $\Upsilon(a,z) = \int_0^z t^{a-1}e^{-t}dt$ is the lower incomplete gamma function [23]. With μ fixed at 1 as before, we compare the above exact results and approximations for the CDF and FP in Figs. 7 and 8, and find excellent agreement.

8. Conclusion

We have proposed a novel approximation for K Distribution using associated Laguerre Polynomial to facilitate computation of analytical closed form expression for performance measures such as BER and Fade Probability. The advantage of this approximation is that it not only works fairly well in the region of interest but also forms expressions which are not obscured by complicated mathematical functions. This approximation can be viewed as a weighted sum of three Gamma PDF's which makes the analytical computation of all performance measures in FSO tractable. The low value of KL measure further confirms the accuracy of the proposed model. Based on Jaynes' Concentration Theorem, it is known that the distance between the approximate and exact PDF follows asymptotically χ^2 distribution [21]. This will enable to test the null hypothesis for a given level of statistical significance.

Appendix

We use the K-PDF expression (1) in (23) to obtain

$$\begin{aligned}\mathcal{P}_e^{(\mathrm{DPSK})} &= \int_0^\infty \frac{1}{2}e^{-sx}\frac{2\alpha^{\frac{\alpha+1}{2}}x^{\frac{\alpha-1}{2}}}{\mu^{\frac{\alpha+1}{2}}\Gamma(\alpha)}K_{\alpha-1}\left(2\sqrt{\frac{\alpha x}{\mu}}\right)dx \\ &= \frac{\alpha^{\frac{\alpha+1}{2}}}{\mu^{\frac{\alpha+1}{2}}\Gamma(\alpha)}\int_0^\infty e^{-sx}\,x^{\frac{\alpha-1}{2}}\,K_{\alpha-1}\left(2\sqrt{\frac{\alpha x}{\mu}}\right)dx. \quad (38)\end{aligned}$$

Now, substituting $x = y^2$, so that $dx = 2ydy$, we get

$$\mathcal{P}_e^{(\mathrm{DPSK})} = \frac{2\alpha^{\frac{\alpha+1}{2}}}{\mu^{\frac{\alpha+1}{2}}\Gamma(\alpha)}\int_0^\infty e^{-sy^2}\,y^\alpha\,K_{\alpha-1}\left(2y\sqrt{\frac{\alpha}{\mu}}\right)dy \quad (39)$$

As noted in [12], this leads to

$$\mathcal{P}_e^{(\mathrm{DPSK})} = \frac{1}{2}\zeta^\alpha e^\zeta\int_\zeta^\infty t^{-\alpha}e^{-t}\,dt, \quad (40)$$

where $\zeta = \alpha/(\mu s)$, as defined in (26). The integral in the above equation is easily expressible in terms of upper incomplete gamma function $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}\,dt$ [18], thereby giving the second equality in (27). Next we introduce the variable $u = t/\zeta$, this gives

$$\mathcal{P}_e^{(\mathrm{DPSK})} = \frac{1}{2}\zeta e^\zeta\int_1^\infty u^{-\alpha}e^{-\zeta u}\,dt, \quad (41)$$

and enables us to write an expression in terms of the exponential integral $E_a(z) = \int_1^\infty u^{-a}e^{-zu}\,du$ [18]. This gives the first equality in (27)

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