

Shape Invariant Potentials in Higher Dimensions

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Abstract

In this paper we investigate the shape invariance property of a potential in one dimension. We show that a simple ansatz allows us to reconstruct all the known shape invariant potentials in one dimension. This ansatz can be easily extended to arrive at a large class of new shape invariant potentials in arbitrary dimensions. A reformulation

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of the shape invariance property and possible generalizations are proposed. These may lead to an important extension of the shape invariance property to Hamiltonians that are related to standard potential problems via space time transformations, which are found useful in path integral formulation of quantum mechanics.

1 Introduction

Supersymmetric quantum mechanics (SUSYQM) [1] played an important role in understanding the problem of solvable potentials. The shape invariance (SI) condition introduced by Gendenshtein [2] gives a sufficient condition for the solvability of a given one dimensional potential. This helped in obtaining a whole class of shape invariant potentials (SIP) which are analytically exactly solvable (ES). It also gave useful insights into the factorization method [3] and has played an important role in the past three decades in constructing new ES potentials in one dimension [4], [5]. More recently supersymmetry (SUSY) has provided another route to the construction of rational potentials [6], [8], which have the new exceptional orthogonal polynomials (EOPs) as a part of their solutions [9], [10].

In higher dimensions the number of known solvable potential models, other than separable ones is severely restricted [11] - [15]. The power of SUSY and the SI condition give a hope that one may construct a new class of ES models in higher dimensions. At present, work on SI in higher dimensional models is very limited. Use of the SI requirement has been shown to lead to known ES potential models in one dimension [16].

In this paper, we present, an alternate route to analyse and obtain the solutions to SI requirement. This is achieved by the use of a simple ansatz which leads to the solution for the superpotential in terms of the free particle Schrödinger equation solutions. This approach and its extension reproduces all the known SI potentials in one dimension, including those recently discovered potentials related to EOPs [6] -[10]. In addition, this route to the solutions of the SI requirement trivially generalizes to higher dimensions and leads to, in the first instance, a large class of SI potentials which get related to the solutions of free particle Schrödinger equation in higher dimension.

Even in one dimension, the SI property alone is not sufficient for obtaining exact solutions for the bound state energy eigenvalues and eigenfunctions. In addition one needs to use the intertwining property of partner Hamiltonians. For separable potential models, the intertwining property works exactly in the same way as in one dimension, as the solution of the problem reduces to solution of several one dimensional problems. The known examples of inter-

twining property in two dimensions indicate that, in practice, this property may have a limited role to play in higher dimensional models and one needs a fresh approach.

A possible route may be the use of space time transformations found useful for obtaining exact solutions of quantum mechanical problems in the path integral approach. In 1984, Duru and Kleinert used space time transformations to provide an exact solution of hydrogen atom problem in three dimensions within the path integral formalism [17], [18]. In the next ten years it was shown to be useful for obtaining exact path integral solutions of many other problems [19] - [25]. Further discussion of space time transformations is given in [21]-[24]. It is important to mention that the use of space time transformations is not limited to the path integral formalism alone and have been found to be useful beyond one dimensional potential problems. Inspired by the success of the space time transformations, a generalization of shape invariance requirements to Hamiltonians of a more general form is presented. As an example, it is applied to the radial equation for free particle equation in three dimensions to show that this leads to known recurrence relations between spherical Bessel functions.

The paper is organized as follows. In the next section we briefly describe SUSYQM and we give an alternate definition of the SI condition and follow a new approach to obtain solutions of SI requirement. It is shown that all the known ES solvable potentials in one dimension are obtained by making use of few simple ansatz. In section 3, we show that, in arbitrary dimensions, this route to the analysis of SI condition does not require anything new and easily leads to SIPs. In section 4, we summarize our results and conclude by giving routes to further generalizations of the SI property.

2 Supersymmetric quantum mechanics

In SUSYQM [1], [5], we have a pair of partner potentials $V^\pm(x)$ defined in terms of the superpotential $W(x)$ as

$$V^{(\pm)}(x) = W^2(x) \pm W'(x), \quad (1)$$

where the prime denotes differentiation with respect to x and $W(x)$ is defined as

$$W(x) = -\frac{d}{dx} \log \psi_0^{(-)}(x). \quad (2)$$

Here $\psi_0^{(-)}(x)$ is the ground state wave function of $V^-(x)$ and $E_0^- = 0$. The wave functions $\psi_n^{(\pm)}(x)$ of the partners are related by

$$\psi_{n+1}^{(-)}(x) = A^\dagger \psi_n^{(+)}(x) ; \quad \psi_n^{(+)}(x) = A \psi_{n+1}^{(-)}(x), \quad (3)$$

where A and A^\dagger are the intertwining operators

$$A = \frac{d}{dx} + W(x) ; \quad A^\dagger = -\frac{d}{dx} + W(x). \quad (4)$$

SUSY is known to be unbroken (exact) between the partners $V^{(\pm)}(x)$, if $E_0^- = 0$, $\psi_0^{(-)}$ is normalizable and $A\psi_0^{(-)} = 0$. In this case, $\psi_0^{(+)}(x)$ is non-normalisable and the partners are isospectral except for the ground states. SUSY is said to be broken, when the ground states of both the partners are non-normalisable and the partners are isospectral including the ground states. For more details we refer the reader to [5], [4].

It should be noted that for a given potential $V^{(-)}(x)$, superpotential $W(x)$ is not uniquely determined. *Moreover, each $W(x)$ associated with $V^{(-)}(x)$, in general, gives a different partner associated with it. Therefore, it would appear that the shape invariance property depends on the choice of the superpotential; this statement is not correct.* In the next section a sufficient condition for shape invariance is formulated in terms of the solutions of quantum Hamilton-Jacobi (QHJ) equation [26], [27]. This condition makes no reference to any particular choice of the superpotential.

Shape invariance

The SUSY partners are said to be shape invariant if, for some $f(\lambda)$, one has

$$V^{(+)}(x, \lambda) = V^{(-)}(x, f(\lambda)) + R(\lambda), \quad (5)$$

where λ is the potential parameter and $R(\lambda)$ is a function of λ . It is clear that given a potential $V(x, \lambda)$, depending on a set of parameters λ , one introduces a superpotential $W(x, \lambda)$ such that

$$V(x) = W^2(x, \lambda) - W'(x, \lambda) + E_0, \quad (6)$$

which has the form of Riccati equation and is also known as the QHJ equation. Substituting (2) in the above equation gives the Schrödinger equation. It may be remarked that several solutions of the above equation for $W(x)$ exist. For a mapping of parameters $\rho : \lambda \rightarrow \rho(\lambda)$, we introduce functions w_1, w_2 through equations

$$w_1(x, \lambda) = -W(x, \lambda), \quad \text{and} \quad w_2(x, \lambda) = W(x, \tau(\lambda)), \quad (7)$$

and define two potentials V_1, V_2 by

$$V_k = w_k^2(x, \lambda) - w_k'(x, \lambda), \quad k = 1, 2. \quad (8)$$

A potential $V(x)$ is called shape invariant, if one can find a superpotential $W(x)$ and a mapping $\lambda \rightarrow \tau(\lambda)$ such that $V_1(x, \lambda)$ and $V_2(x, \lambda)$ differ by a constant. Though the use of partner potential has been bypassed here, it is apparent that this definition of shape invariance is equivalent to that used in literature.

It is to be noted that an obvious sufficient condition for shape invariance of a potential $V(x, \lambda)$ is the existence of superpotential $W(x)$ and a map τ such that

$$W(x, \tau(\lambda)) = -W(x, \lambda). \quad (9)$$

The above sufficient condition for shape invariance can be restated as follows. A potential $V(x, \lambda)$ is shape invariant if there exists a superpotential $W(x, \lambda)$ satisfying the QHJ equation

$$W^2(\lambda, x) - W'(\lambda, x) = V(x, \lambda) - E_1 \quad (10)$$

and a map τ such that the function \underline{W} defined by

$$\underline{W}(\lambda, x) \equiv -W(\tau(\lambda), x), \quad (11)$$

satisfies the QHJ equation

$$\underline{W}^2(x, \lambda) - \underline{W}'(x, \lambda) = V(x) - E_2, \quad (12)$$

for some energy E_2 .

In the next section we re-derive the one dimensional SIPs listed in [4], in a very simple and straightforward fashion, by making a simple ansatz for the superpotential. Our approach has the advantage of being very simple and also leads one to a large class of shape invariant potentials in higher dimensions.

A simple derivation of SI potentials in one dimension

In the table below, we list the known SIPs in one dimension along with the corresponding superpotentials [4], [5].

Name of Potential	Superpotential	Name of Potential	Superpotential
Shifted Oscillator	$\frac{1}{2}\omega x - b$	Radial Oscillator	$\frac{1}{2}\omega r - \frac{\ell + 1}{r}$
Coulomb	$\frac{e^2}{2(\ell + 1)} - \frac{\ell + 1}{r}$	Morse	$A - B \exp(-ax)$
Scarf II (hyperbolic)	$A \tanh ax + B \operatorname{sech} ax$	Rosen-Morse II (hyperbolic)	$A \tanh ax + B/A$
Eckart	$-A \coth ar + B/A$	Scarf I (trigonometric)	$-A \tan ax + B \sec ax$
Gen. Pöschl-Teller	$A \coth ar - B \operatorname{cosech} ar$	Rosen-Morse trigonometric	$-A \cot ax - B/A$

We will now present a simple derivation of all these SIPs. We will show in the next section how the simplicity of this derivation makes it possible to generalize the results to higher dimensions.

Let the SUSY partner potentials, $V^{(\pm)}(x, \lambda)$, be specified by a superpotential $W(x)$:

$$V^{(\pm)}(x, \lambda) = W^2(x, \lambda) \pm W'(x, \lambda), \quad (13)$$

where λ denotes a parameter appearing in the potential.

On using a simple ansatz

$$W(x, \lambda) = \lambda F(x), \quad (14)$$

the shape invariance requirement (13) becomes

$$(\lambda^2 - \mu^2)F(x)^2 + (\lambda + \mu)F'(x) + R = 0. \quad (15)$$

Defining a scaled independent variable $(\lambda - \mu)x \equiv \xi$, we get

$$\tilde{F}^2(\xi) + \tilde{F}'(\xi) + K = 0, \quad (16)$$

where $K = R/(\mu + \lambda)$ is a constant and $\tilde{F}(\xi) = F(x(\xi))$. This equation has the form of the Riccati equation and can be linearised by introducing a function, $u(\xi)$, by $\tilde{F}(\xi) \equiv u'(\xi)/u(\xi)$. The equation for $u(\xi)$ turns out to be the familiar free particle Schrödinger equation:

$$u''(\xi) + Ku(\xi) = 0, \quad (17)$$

whose solutions can be written down immediately for different cases $K > 0$, $K = 0$ and $K < 0$. Thus for each case of K , we use the corresponding solution of (17) and calculate $W(x)$ as follows.

Case I $K = 0$:

$$u(\xi) = A\xi + B, \quad W(x) = \lambda \frac{A}{A(\alpha x) + B} \quad (18)$$

$$\text{where } \alpha = (\lambda - \mu) \quad (19)$$

and A, B are constants.

Case II $K > 0$: Using the notation $k = \sqrt{K}$ the two independent solutions are

$$u(\xi) = \sin \xi, \cos \xi \quad (20)$$

$$W(x) = \lambda k \tan(k\alpha x), -\lambda k \cot(k\alpha x). \quad (21)$$

Case III $K < 0$: In this case, the two independent solutions are given in terms of the hyperbolic functions and

$$u(\xi) = \sinh c\xi, \cosh c\xi \quad (22)$$

$$W(x) = \lambda c \tan(c\alpha x), \lambda c \cot(c\alpha x). \quad (23)$$

where $c^2 = -K$.

The superpotentials constructed above correspond to the one parameter potentials like the harmonic oscillator, trigonometric Scarf, Rosen-Morse, hyperbolic Pösch-Teller etc., with suitable redefinition of parameters, listed in the table.

Extensions

As a next step we wish to extend the above results to cover the other cases listed in the table. We will construct superpotentials corresponding to the two parameter families of potentials. We will use the fact that a solution of Riccati equation can be used to construct a second solution [28]. Denoting a solution of (16) by $f(\xi)$ and writing $W(\xi) = \lambda f(\xi) + \phi(\xi)$, assuming $\phi(\xi)$ to be independent of λ , an analysis of the shape invariance requirement leads to

$$f(\xi)\phi(\xi) + \phi'(\xi) = C, \quad (24)$$

where C is a constant. The solution for $\phi(\xi)$ in terms of $f(\xi)$ is given by

$$\phi(\xi) = e^{-\int f(\xi) d\xi} \left(C e^{\int f(\xi) d\xi} + D \right), \quad (25)$$

where D is the constant of integration. Thus each of the above solutions, given in (18), (21) and (23), can be used to construct a new solution of (16).

At this stage, we get the radial oscillator, Scarf I and II, generalized Pöschl-Teller and $B = 0$ cases of the Rosen Morse-I, Rosen-Morse-II and the Eckart potentials.

For the remaining potentials, *i.e.*, the shifted oscillator, Coulomb, Morse, Eckart Rosen-Morse-I and II potentials with $B \neq 0$, we proceed as follows. Let $F(x)$ be one of the solutions as obtained above. We now look for solutions of the shape invariance condition of the form

$$W(x) = \lambda F(x) + g(\lambda, x). \quad (26)$$

The general case, when g depends on both x and λ is complicated to analyse. For the present purpose, it turns out to be sufficient to assume that g is independent of x . In this case we get

$$\left(\lambda F(x) + g(\lambda)\right)^2 + \lambda F'(x) - \left(\mu F(x) + g(\mu)\right)^2 + \mu F'(x) + R = 0. \quad (27)$$

Use of (15) and the fact that $F(x)$ is one of the solutions already giving a shape invariant potential, simplifies the above equation to give

$$g(\lambda) = \text{const}/\lambda. \quad (28)$$

This completes the construction of all the superpotentials listed in the table and we can reproduce all the known SIPs in one dimension with suitable choice of the constants.

Let us assume that we have found a set of solutions of the SI requirement (27). We now ask if we can extend these solutions to further generate a new set of SIPs? Let $W(x, \lambda)$ be a superpotential corresponding to a shape invariant potential. Let us introduce $\tilde{W}(x, \lambda) = W(x, \lambda) + \chi(x, \lambda)$ and the requirement that the corresponding potential $\tilde{V}(x, \lambda)$ being shape invariant gives

$$\tilde{W}(x, \lambda)^2 - \tilde{W}'(x, \lambda) = \tilde{W}(x, \mu)^2 + \tilde{W}'(x, \mu) + R(\mu, \lambda), \quad (29)$$

where $R(\mu, \lambda)$ is a constant. Substituting for $\tilde{W}(x, \lambda)$ and making use of the shape invariance property of $W(x, \lambda)$ and simplifying we get

$$\chi(x, \lambda)^2 + 2W(x, \lambda)\chi(x, \lambda) + \chi'(x, \lambda) \quad (30)$$

$$= \chi(x, \mu)^2 + 2W(x, \mu)\chi(x, \mu) - \chi'(x, \mu) + R(\mu, \lambda). \quad (31)$$

Solutions of the above equation, for the case when $\chi(x, \lambda)$ is a constant have already been found. The general case where $\chi(x, \lambda)$ is a function of x , does not appear to be easy to solve. Recall for shape invariance to be satisfied,

there exists a mapping τ such that $W(x, \tau(\lambda)) = -W(x, \lambda)$. The above equation can then be cast in the form

$$\chi(x, \lambda)^2 + 2W(x, \lambda)\chi(x, \lambda) + \chi'(x, \lambda) \quad (32)$$

$$= \chi(x, \tau(\mu))^2 + 2W(x, \tau(\mu))\chi(x, \tau(\mu)) + \chi'(x, \tau(\mu)) + R(\mu, \lambda). \quad (33)$$

One can make further progress by looking for solutions that satisfy the ansatz

$$\chi(x, \lambda)^2 + 2W(x, \lambda)\chi(x, \lambda) + \chi'(x, \lambda) = K(\lambda), \quad (34)$$

as suggested by the above equation (33). Here $K(\lambda)$ is a constant. Among other results, we are then led to the recently discovered SIPs with solutions related to EOPs [6], [7], [9], [10].

This ansatz turns out to be equivalent to an isospectral shift deformation of the original potential, first used in [8] for the construction of new rational potentials with EOPs as solutions. In this work it was not clear if the extended potentials were shape invariant, in fact it was erroneously concluded that the deformed potentials are not shape invariant. Our approach with SI as guiding principle guarantees that the deformed potentials indeed turn out to be shape invariant. The simple process outlined in this section allows us to construct these new potentials from the requirements of shape invariance. The details of this study are reported elsewhere [29].

In the next section, we show how our approach to SI not only allows us to extend the concept of SI to higher dimensions but also provide us with a method to construct SI potentials in higher dimensions.

3 SI in higher dimensions

In this section, we reformulate the shape invariance property in one dimension in a manner which can be generalized to higher dimensions. To construct SIPs in higher dimensions is straight forward. For this it turns out to be useful and simpler to discard the superpotential and to write the Hamiltonian in a different form. To elucidate, we consider the one dimensional Hamiltonian

$$H = p^2 + W^2(x, \lambda) - W'(x, \lambda), \quad (35)$$

which can be rewritten as

$$H = -e^{\Omega(x)} \frac{d}{dx} e^{-2\Omega(x)} \frac{d}{dx} e^{\Omega(x)}, \quad (36)$$

where $\Omega(x, \lambda)$, called prepotential, is related to the superpotential by

$$\Omega(x, \lambda) = \int W(x, \lambda) dx. \quad (37)$$

It is easy to see that

$$-e^{-\Omega(x)} \frac{d}{dx} e^{\Omega(x)} = \frac{d}{dx} + W(x, \lambda) = A; \quad e^{\Omega(x)} \frac{d}{dx} e^{-\Omega(x)} = -\frac{d}{dx} + W(x, \lambda) = A^\dagger \quad (38)$$

and therefore

$$H = A^\dagger A. \quad (39)$$

The SUSY partner of H is obtained by simply changing the sign of the prepotential Ω . The Hamiltonian written in this form is suitable for generalization to higher dimensions.

Working in three dimensions, a Hamiltonian for a potential problem can be written as

$$H_- = -e^{\Omega(\mathbf{r})} \nabla e^{-2\Omega(\mathbf{r})} \nabla e^{\Omega(\mathbf{r})}, \quad (40)$$

where ∇ is the gradient operator in three dimensions. The partner Hamiltonian is defined by

$$H_+ = -e^{-\Omega(\mathbf{r})} \nabla e^{2\Omega(\mathbf{r})} \nabla e^{-\Omega(\mathbf{r})}. \quad (41)$$

Again, we analyze the simplest case of the prepotential depending on a single parameter λ by means of the ansatz

$$\Omega(\mathbf{r}, \lambda) = \lambda F(\mathbf{r}, \lambda). \quad (42)$$

Proceeding as in the case of one dimension, the SUSY SI condition leads to the Riccati equation for $F(\mathbf{r}, \lambda)$

$$|\nabla F(\mathbf{r}, \lambda)|^2 - \nabla^2 F(\mathbf{r}, \lambda) = K, \quad (43)$$

where K is a constant. This equation can be transformed into a free particle equation by writing $F(\mathbf{r}, \lambda) = \log \chi(\mathbf{r})$:

$$\nabla^2 \chi(\mathbf{r}) + K \chi(\mathbf{r}) = 0. \quad (44)$$

Here again we will have three cases of $K > 0$, $K = 0$ and $K < 0$, separately. It is to be noted that, for a fixed constant K , in higher dimensions there are an infinite number of independent solutions, as against two solutions in one dimension. Hence the class of SIPs is already very large in higher dimensions as compared to that in one dimension.

Now it is very easy to give examples of non-separable SIPs in three dimensions, keeping in mind that any solution of free particle Schrödinger equation leads to a SIP. As an example, consider a particular solution of (44) for $K = 0$, in the polar coordinates

$$\chi(r, \theta) = \sum_n \{r^n P_n(\cos(\theta)) + r^{-(n+1)} P_n(\cos(\theta))\}, \quad (45)$$

with $\Omega(r, \theta, \lambda) = \lambda \log \chi(r, \theta)$. The corresponding Hamiltonians H_{\pm} become

$$H_{\pm} = -\nabla^2 + V^{(\pm)}(\vec{r}), \quad (46)$$

with the potentials $V^{(\pm)}(r)$ given by

$$V^{(\pm)}(\vec{r}) = \lambda^2(\nabla\chi)^2 \pm \lambda\nabla^2\chi. \quad (47)$$

Although we have found solutions to the SI requirement for higher dimensional potentials, SI alone is insufficient to obtain solutions for the energy eigenvalue problems. We also require intertwining operators connecting the partner Hamiltonians. A straightforward extension of conventional intertwining relation rapidly becomes more and more complex as one takes up problems in higher dimensions and is likely to become intractable. This suggests a need to look for a replacement for the intertwining property of the partner Hamiltonians. The work on exact solutions of potential problems within the path integral, mentioned in the introduction, suggests use of reparametrisation of time, to be called space time transformation in this paper. A formulation of classical mechanics with an alternate parametrization of trajectories is well known and has been used extensively for obtaining solutions of potential problems in one and more than one dimensions within the path integral approach [17] - [25], [30], [31].

For a classical problem defined by a Hamiltonian H , the use of alternate parametrization of classical trajectories of energy E leads to formulating the classical problem in terms of a pseudo Hamiltonian $\mathcal{H} = f(\mathbf{q})(H - E)$ where $f(\mathbf{q})$ is a function of the generalized coordinates \mathbf{q} and specifies the transformation to a 'new time parameter'. This motivates an extension of SI requirement to more general Hamiltonians of the form

$$\mathcal{H} = f(\mathbf{q})\left(\frac{\mathbf{p}^2}{2m} + (V(\mathbf{q}) - E)\right). \quad (48)$$

The extension of SI to quantum Hamiltonians corresponding to (48) can be done in several possible ways. Here we present a particular scheme and an example of application to the radial equation coming from the separation of variables in a spherically symmetric problem in three dimensions.

Factorization and generalisations of shape invariance: The solution to the factorization problem [3] of writing the Schrödinger Hamiltonian

$$H = p^2 + V(x) - E \quad (49)$$

in the factorized form $A^\dagger A$, where

$$A = \frac{d}{dx} + Q(x), \quad A^\dagger = -\frac{d}{dx} + Q(x) \quad (50)$$

is seen to be provided by the QHJ equation

$$Q(x)^2 - \frac{dQ(x)}{dx} = V(x) - E. \quad (51)$$

The solution $Q(x)$ of the Riccati equation (50) is equivalent to solving the Schrödinger equation

$$-\frac{d^2\psi(x)}{dx^2} + (V(x) - E)\psi(x) = 0, \quad (52)$$

since

$$Q(x) = -\frac{\psi'(x)}{\psi(x)}. \quad (53)$$

Thus a solution of (51) can be used to factorize the Hamiltonian H as $A^\dagger A$.

We now make use of this observation to extend shape invariance to more general situations which arise when one is trying to solve a Schrödinger equation by means of a point transformation. As a concrete example we look for a function $Q(x)$ such that the operator appearing in a generalised form of the Schrödinger equation given by

$$-\frac{1}{f(x)}\frac{d}{dx}f(x)\frac{d\psi(x)}{dx} + (V(x) - E)\psi(x) = 0. \quad (54)$$

can be factorized and the above equation can be cast in the form

$$\frac{1}{f(x)} \left[-\left(\frac{d}{dx} - Q(x)\right)f(x)\left(\frac{d}{dx} + Q(x)\right) \right] \psi(x) = 0. \quad (55)$$

The requirement that (54) and (55) coincide, implies that $Q(x)$ be a solution of the QHJ equation

$$Q^2(x) - \frac{1}{f(x)}\frac{d}{dx}\left(f(x)Q(x)\right) = V(x) - E. \quad (56)$$

Using factorization we can now define a SUSY partner equation

$$\frac{1}{f(x)} \left[\left(\frac{d}{dx} + Q(x)\right)f(x)\left(\frac{d}{dx} - Q(x)\right) \right] \psi(x) = 0. \quad (57)$$

This then leads us to a generalization of SI property if we demand that the potentials V_\pm obtained from

$$\frac{1}{f(x)}A^\dagger f(x)A = p^2 + V_-(x) - E, \quad (58)$$

be equal to the potential $V^{(+)}(x)$ obtained from

$$\frac{1}{f(x)}Af(x)A^\dagger = p^2 + V_+(x) - E \quad (59)$$

up to a constant and a redefinition of parameters in the usual fashion. Using this extension of shape invariance property, we can apply methods of SUSY QM to a larger class of equations. The partner potentials in the above equations can be written as

$$V^{(\pm)}(x) = Q^2(x) \pm \frac{1}{f(x)}\frac{d}{dx}(f(x)Q(x)). \quad (60)$$

Note that, for $f(x) = 1$, the above expressions reduce to the forms familiar from the standard SUSY quantum mechanics.

In an alternative approach to define partner potentials and shape invariance, we write the Hamiltonian given in (54) as a product CB where

$$B = \frac{d}{dx} + Q(x), \quad C = -\frac{d}{dx} - \frac{f'(x)}{f(x)} + Q(x) \quad (61)$$

and define the partner Hamiltonian as $H_+ = BC$. Expanding the products BC and CB , the two Hamiltonians H_\pm are seen to be

$$H_\pm = -\frac{d^2}{dx^2} - \frac{f'(x)}{f(x)}\frac{d}{dx} + V_\pm(x) \quad (62)$$

where the 'partner potentials' $V_\pm(x)$ are given by

$$V^{(-)}(x) = Q^2(x) - Q'(x) - Q(x)\left(\frac{d \log f(x)}{dx}\right), \quad (63)$$

$$V^{(+)}(x) = Q^2(x) + Q'(x) - Q(x)\left(\frac{d \log f(x)}{dx}\right) - \left(\frac{d^2 \log f(x)}{dx^2}\right). \quad (64)$$

An application to radial equation

In general there are several possible ways of defining the partner potentials and hence different ways of bringing in SI. Without loss of generality, we illustrate this by means of an example

Considering the radial equation in three dimensions,

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR(r)}{dr} + V(r)R(r) = ER(r) \quad (65)$$

which is of the form (54), if we make replacements $x \rightarrow r$ and take $f(r) = r^2$, $Q = (\ell + 1)/r$. The method described above can now be used to find SIPs.

These potentials will get related to the solutions of the radial equation instead of the free particle equation in one dimension. Thus we will get a whole new class of shape invariant spherically symmetric potentials.

In this case the partner potentials for the radial equation are

$$V^{(-)}(r) = Q^2(r) - \frac{1}{f(r)} \frac{d}{dr}(f(r)Q(r)), \quad (66)$$

$$= \frac{\ell(\ell+1)}{r^2}. \quad (67)$$

$$V^{(+)}(r) = Q^2(r) + \frac{dQ(r)}{dr} - Q(r) \frac{d \log f(r)}{dr} - \frac{d^2 \log f(r)}{dr^2}, \quad (68)$$

$$= \frac{\ell(\ell-1)}{r^2}. \quad (69)$$

and the two Hamiltonians H_{\pm} are

$$H_- = CB = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} \quad (70)$$

$$H_+ = BC = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell-1)}{r^2} \quad (71)$$

The operators B and C ,

$$B = \frac{d}{dr} + \frac{\ell+1}{r}, \quad C = \frac{d}{dr} + 2r \quad (72)$$

intertwine the solutions of the two Hamiltonians H_{\pm} . Thus if $\psi_{\ell}^{(-)}(r)$ is an eigenfunction of H_- , then

$$\psi_{\ell}^{(+)}(r) = B\psi_{\ell}^{(-)}(r) \quad (73)$$

$$= r^{-(\ell+1)} \frac{d}{dr} \left(r^{(\ell+1)} \psi_{\ell}^{(-)}(r) \right), \quad (74)$$

is an eigenfunction of H_+ . This is just the relation between the spherical Bessel functions $j_{\ell}(r), n_{\ell}(r), h_{\ell}(r)$ which are the solutions for H_- and $j_{\ell-1}(r), n_{\ell-1}(r), h_{\ell-1}(r)$, the solutions for H_+ .

4 Summary and Concluding Remarks

In this paper we have analysed the shape invariance requirement and presented an approach using a simple ansatz to reproduce all the known SI potentials in one dimension. This approach to SI relates the superpotentials

to solutions of free particle Schrödinger equation and works equally well in higher dimensions. Going beyond the simple ansatz of (14), we arrive at the isospectral shift deformation and to the SIPs related to the EOPs [29]. Further generalisations of the SI property are possible and are as follows.

Generalizing isospectral shift The first generalization consists in replacing the constant, such as that coming in shape invariance and isospectral shift conditions, by functions of x . Suppose we are given a potential $V(x)$. Solve QHJ and construct superpotential $W(x)$ for some energy E . Then define

$$V^{(\pm)} = W(x)^2 - W'(x). \quad (75)$$

While carrying out isospectral shift we set $\tilde{W} = W(x) + \phi(x)$ and try to find $\phi(x)$ by demanding that

$$\tilde{V}^{(+)}(x) = V^{(+)}(x) + R. \quad (76)$$

The equation for $\phi(x)$ can be transformed into a second order Schrödinger like equation. If that equation can be solved, this procedure will explicitly give a potential which is exactly solvable and has its spectrum shifted by the constant R . This procedure can be repeated by replacing the constant R by any function of x . So for example we can take

$$\tilde{V}^{(+)}(x) = V^{(+)}(x) + V_0(x) \quad (77)$$

and set up an equation for $\phi(x)$. We can look for suitable $V_0(x)$ for which the equation for $\phi(x)$ can be solved. This generalization of SI property, though known in literature, does not seem to have been investigated. Using tools from QHJ and SUSYQM, an investigation of applications of this extension is under study and will be reported elsewhere.

Generalizing shape invariance The second generalisation is indicated by re-examining shape invariance condition

$$V^{(+)}(x, \lambda) = V^{(-)}(x, \rho(\lambda)) + K. \quad (78)$$

We solved this equation by taking ansatz $w(x) = \lambda f(x)$. We ended up with free particle equation. For each solution of the free particle equation, we then find a $W(x)$ and a shape invariant potential.

Again, if we replace the constant K in the above condition by a function $V_0(x)$, the same ansatz will lead to the Schrödinger equation for potential $V_0(x)$. For each solution of Schrödinger equation for this potential $V_0(x)$,

we will be able to construct a potential which satisfies a generalized shape invariance condition

$$V^{(+)}(x, \lambda) = V^{(-)}(x, \rho(\lambda)) + V_0(x). \quad (79)$$

The second generalization of both the above processes to potentials in higher dimensions, can be obviously carried out without any difficulty.

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