Positivity properties of some special matrices

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Abstract

It is shown that for positive real numbers $0 < \lambda_1 < \cdots < \lambda_n$, $\left[\frac{1}{\beta(\lambda_i,\lambda_j)}\right]$, where $\beta(\cdot,\cdot)$ denotes the beta function, is infinitely divisible and totally positive. For $\left[\frac{1}{\beta(i,j)}\right]$, the Cholesky decomposition and successive elementary bidiagonal decomposition are computed. Let $\mathfrak{w}(n)$ be the nth Bell number. It is proved that $[\mathfrak{w}(i+j)]$ is a totally positive matrix but is infinitely divisible only upto order 4. It is also shown that the symmetrized Stirling matrices are totally positive.

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1 Introduction

Let $M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices. A matrix $A \in M_n(\mathbb{C})$ is said to be *positive semidefinite* if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$ and *positive definite* if $x^*Ax > 0$ for all $x \in \mathbb{C}^n, x \neq 0$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. In this paper, $1 \leq i, j \leq n$, unless otherwise specified. The *Hadamard product* or *the Schur product* of two matrices A and B is denoted by $A \circ B$, where $A \circ B = [a_{ij}b_{ij}]$. For a nonnegative real number r, $A^{\circ r} = [a_{ij}^r]$.

Let $A = [a_{ij}]$ be such that $a_{ij} \geq 0$. The matrix A is called *infinitely divisible* if $A^{\circ r}$ is positive semidefinite for every real number $r > 0$. For examples and properties of infinitely divisible matrices, see [3, 7, 15]. The matrix A is called *totally positive* or *totally nonnegative* if all its minors are positive or nonnegative respectively. For more results on these, see [12]. The main objective of this paper is to explore the above mentioned properties for a few matrices which are constructed from interesting functions. Many such matrices have been studied in [3, 7, 5, 6]. A famous example of such a matrix is the Hilbert matrix $\left[\frac{1}{i+j-1}\right]$. Another important example is the Pascal matrix $\mathcal{P} = \left[\binom{i+j}{i}\right]_{i,j=0}^n$. Both of these are known to be infinitely divisible and totally positive. In [11], *Cholesky decomposition* of $\mathcal{P}$ was given, that is, a lower triangular matrix L was obtained such that $\mathcal{P} = LL^*$.

Let $\mathcal{B} = [\frac{1}{\beta(i,j)}]$, where $\beta(\cdot,\cdot)$ is the *beta function*. We call $\mathcal{B}$ as the *beta matrix*. By definition,

$$\beta(i,j) = \frac{\Gamma(i)\Gamma(j)}{\Gamma(i+j)},$$

where $\Gamma(\cdot)$ is the Gamma function. Thus

$$\mathcal{B} = \left[\frac{(i+j-1)!}{(i-1)!(j-1)!}\right].$$

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Note that the entries of this matrix look similar to those of $\mathcal{P}$. Using the infinite divisibility and total positivity of $\mathcal{P}$, we show that $\mathcal{B}$ is infinitely divisible and totally positive, respectively. We also compute the Cholesky decomposition of $\mathcal{B}$.

Let $E_{i,j}$ be the matrix whose (i,j)th entry is 1 and others are zero. For any complex numbers s,t, let

$$L_i(s) = I + sE_{i,i-1} \text{ and } U_j(t) = I + tE_{j-1,j},$$

where $2 \leq i,j \leq n$. Matrices of the form $L_i(s)$ or $U_j(t)$ are called *elementary bidiagonal matrices*. For a vector $(d_1, d_2, \ldots, d_n)$, let $\operatorname{diag}([d_i])$ denote the diagonal matrix $\operatorname{diag}(d_1, \ldots, d_n)$. An $n \times n$ matrix is totally positive if and only if it can be written as $(L_n(l_k) L_{n-1}(l_{k-1}) \cdots L_2(l_{k-n+2}))$ $(L_n(l_{k-n+1}) L_{n-1}(l_{k-n}) \cdots L_3(l_{k-2n+4})) \cdots (L_n(l_1)) D (U_n(u_1))(U_{n-1}(u_2) U_n(u_3)) \cdots (U_2(u_{k-n+2}) \cdots U_{n-1}(u_{k-1}) U_n(u_k))$, where $k = \binom{n}{2}$, $l_i, u_j > 0$ for all $i,j \in \{1,2,\ldots,k\}$ and $D = \operatorname{diag}[(d_i)]$ is a diagonal matrix with all $d_i > 0$ [12, Corollary 2.2.3]. This particular factorization (with $l_i, u_j \geq 0$) is called *successive elementary bidiagonal (SEB) factorization* or *Neville factorization*. We obtain the SEB factorization for $\left[\frac{1}{\beta(i,j)}\right]$ explicitly.

Let $(x)_0 = 1$ and for a positive integer k, let $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$. The *Stirling numbers of first kind* $s(n,k)$ [9, p. 213] and the *Stirling numbers of second kind* $S(n,k)$ [9, p. 207] are respectively defined as

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k.$$

The *unsigned Stirling matrix of first kind* $\mathfrak{s} = [\mathfrak{s}_{ij}]$ is defined as

$$\mathfrak{s}_{ij} = \begin{cases} (-1)^{i-j} s(i,j) & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

The *Stirling matrix of second kind* $\mathcal{S} = [\mathcal{S}_{ij}]$ is defined as

$$\mathcal{S}_{ij} = \begin{cases} S(i,j) & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

It is a well known fact that $\mathfrak{s}$ and $\mathcal{S}$ are totally nonnegative (see for example [13]). We consider the matrices $\mathfrak{s}\mathfrak{s}^*$ and $\mathcal{S}\mathcal{S}^*$ and call them *symmetrized unsigned Stirling matrix of first kind* and *symmetrized Stirling matrix of second kind*, respectively. By definition, these are positive semidefinite and totally nonnegative. We show that both these matrices are in fact totally positive.

Another matrix that we consider is formed by the well known *Bell numbers*. The sum $\mathfrak{w}(n) = \sum_{k=0}^{n} S(n,k)$ is the number of partitions of a set of n objects and is known as the nth Bell number [9, p. 210]. Consider the matrix $\mathfrak{B} = [\mathfrak{w}(i+j)]_{i,j=0}^{n-1}$. Let $X = (x_{ij})_{i,j=0}^{n-1}$ be the lower triangular matrix defined recursively by

$$x_{00} = 1,\ x_{0j} = 0 \text{ for } j > 0, \text{ and } x_{ij} = x_{i-1,j-1} + (j+1)x_{i-1,j} + (j+1)x_{i-1,j+1} \text{ for } i \geq 1.$$

(Here $x_{i,-1} = 0$ for every i and $x_{in} = 0$ for $i = 0, \ldots, n-2$.) This is known as the Bell triangle [8]. Lemma 2 in [1] gives the Cholesky decomposition of $\mathfrak{B}$ as LL^*, where $L = X \operatorname{diag}\left(\left[\sqrt{i!}\right]\right)_{i=0}^{n-1}$. It is also shown in [1] that $\det \mathfrak{B} = \prod_{i=0}^{n-1} i!$. It is a known fact that $\mathfrak{B}$ is totally nonnegative [18]. We show that $\mathfrak{B}$ is totally positive. We also show that $\mathfrak{B}$ is infinitely divisible only upto order 4.

In Section 2 we give our results for the beta matrix, namely, its infinite divisibility and total positivity, its Cholesky decomposition, its determinant, $\mathcal{B}^{-1}$, and its SEB factorization. We show that $\mathcal{B}^{\circ r}$ is in fact totally positive for all $r > 0$. We end the discussion on the beta matrix by proving that for positive real numbers $\lambda_1 < \cdots < \lambda_n$ and $\mu_1 < \cdots < \mu_n$, $\left[\frac{1}{\beta(\lambda_i,\lambda_j)}\right]$ is an infinitely divisible matrix and $\left[\frac{1}{\beta(\lambda_i,\mu_j)}\right]$ is a totally positive matrix. In section 3, we prove that symmetrized Stirling matrices and $\mathfrak{B}$ are totally positive. For the first kind, we give the SEB factorization for $\mathfrak{s}$. For the second kind, we show that $\mathcal{S}$ is *triangular totally positive* [12, p. 3]. We also show that $\mathfrak{B}$ is infinitely divisible if and only if its order is less than or equal to 4.

2 The beta matrix

The infinite divisibility and total positivity of $\mathcal{B}$ are easy consequences of the corresponding results for $\mathcal{P}$. For $1 \le i, j \le n$, let $A(i, j)$ denote the submatrix of A obtained by deleting ith row and jth column from A. Each $A(i, i)$ is infinitely divisible, if A is infinitely divisible, and each $A(i, j)$ is totally positive, if A is totally positive.

Theorem 2.1. The matrix $\mathcal{B} = \left[\frac{1}{\beta(i,j)}\right]$ is infinitely divisible and totally positive.

Proof. By definition,

$$\frac{1}{\beta(i,j)} = \frac{(i+j-1)!}{(i-1)!(j-1)!} = \frac{ij(i+j)!}{(i+j)i!j!} = \frac{1}{\frac{1}{i}+\frac{1}{j}}.\binom{i+j}{i}. \tag{1}$$

Thus $\mathcal{B} = C \circ \mathcal{P}(1,1)$, where $C = \left[\frac{1}{\frac{1}{i}+\frac{1}{j}}\right]$ is a Cauchy matrix. Both C and $\mathcal{P}(1,1)$ are infinitely divisible [3]. Since Hadamard product of infinitely divisible matrices is infinitely divisible, we get $\mathcal{B}$ is infinitely divisible.

Again, note that

$$\frac{1}{\beta(i,j)} = \frac{(i+j-1)!}{(i-1)!(j-1)!} = j\frac{(i+j-1)!}{(i-1)!(j!)} = \binom{(i-1)+j}{i-1}j. \tag{2}$$

So $\mathcal{B}$ is the product of the totally positive matrix $\mathcal{P}(n+1,1)$ with the positive diagonal matrix diag$([i])$. Hence $\mathcal{B}$ is totally positive. □

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Remark 2.2. Since for each $r > 0$, $C^{\circ r}$ and $\mathcal{P}^{\circ r}$ are positive definite (see p. 183 of [4]), (1) gives that $\mathcal{B}^{\circ r}$ is positive definite. Let $\mathcal{G} = [(i+j)!]_{i,j=0}^n$. Then $\mathcal{G}$ is a Hankel matrix. Since $\mathcal{G}^{\circ r}$ is congruent to $\mathcal{P}^{\circ r}$ via the positive diagonal matrix diag$([i!^r])_{i=0}^n$, $\mathcal{G}^{\circ r}$ is positive definite. The matrix $\mathcal{G}^{\circ r}(n+1,1) = [(i+j-1)!^r]$ is congruent to $\mathcal{B}^{\circ r}$, via the positive diagonal matrix diag$[(i-1)!^r]$. So $\mathcal{G}^{\circ r}(n+1,1)$ is also positive definite. Hence $\mathcal{G}^{\circ r}$ is totally positive, by Theorem 4.4 in [19]. This shows that $\mathcal{P}^{\circ r}$ is totally positive. By (2), $\mathcal{B}^{\circ r}$ is also totally positive.

Remark 2.3. Another proof for infinite divisibility of $\mathcal{B}$ can be given as follows. For positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, the generalized Pascal matrix is the matrix $\left[\frac{\Gamma(\lambda_i+\lambda_j+1)}{\Gamma(\lambda_i+1)\Gamma(\lambda_j+1)}\right]$. The beta matrix $\mathcal{B}$ is congruent to it via a positive diagonal matrix, when $\lambda_i = i - 1/2$. The generalized Pascal matrix is infinitely divisible [3], and hence so is $\mathcal{B}$.

By Theorem 2.1, $\mathcal{B}$ is positive definite and so, can be written as LL^*. Our next theorem gives L explicitly.

Theorem 2.4. The matrix $[\frac{1}{\beta(i,j)}]$ has the Cholesky decomposition LL^*, where $L = [\binom{i}{j}\sqrt{j}]$.

Proof. We prove the result by the combinatorial method of two way counting. Consider a set of $(i+j-1)$ persons. The number of ways of choosing a committee of j people and a chairman of this committee is

$$j \cdot \binom{i+j-1}{j} = \frac{(i+j-1)!}{(i-1)!(j-1!)}.$$

Another way to count the same is to separate $(i+j-1)$ people into two groups of i and $(j-1)$ people each. The number of ways of choosing a committee of j people from these two groups of people and then a chairman of the committee is $j \cdot \sum_k \binom{i}{k}\binom{j-1}{j-k}$, where k varies from 1 to $\min\{i,j\}$. Rearranging the terms in this expression, we get

$$\begin{aligned} j \cdot \sum_k \binom{i}{k}\binom{j-1}{j-k} &= \sum_k j\binom{i}{k}\binom{j-1}{j-k} \\ &= \sum_k \binom{i}{k} j \frac{(j-1)!}{(j-k)!(k-1)!} \\ &= \sum_k \binom{i}{k} k \frac{j!}{(j-k)!k!} \\ &= \sum_k k\binom{i}{k}\binom{j}{k}. \end{aligned}$$

The last expression is the (i,j)th entry of the matrix LL^*, where $L = [\binom{i}{j}\sqrt{j}]$. □

Corollary 2.5. The determinant of $\mathcal{B}$ is equal to $n!$.

Corollary 2.6. The inverse of the matrix $\mathcal{B}$ has (i,j)th entry as $(-1)^{i+j}\sum_{k=1}^{n}\binom{k}{i}\binom{k}{j}\frac{1}{k}$.

Proof. By Theorem 2.4, $\mathcal{B} = LL^*$, where $L = \left[\binom{i}{j}\sqrt{j}\right]$. Let $Z = [\binom{i}{j}]$ and $D' = \mathrm{diag}([\sqrt{i}])$. Then $L = ZD'$. Since $\sum_{k=1}^{n}(-1)^{k+j}\binom{i}{k}\binom{k}{j} = \delta_{ij}$, we get that $Z^{-1} = [(-1)^{i+j}\binom{i}{j}]$. So $L^{-1} = D'^{-1}Z^{-1} = \left[(-1)^{i+j}\binom{i}{j}\frac{1}{\sqrt{i}}\right]$. Thus $\mathcal{B}^{-1} = L^{*^{-1}}L^{-1}$. This gives that the (i,j)th entry of $\mathcal{B}^{-1}$ is

$$\sum_{k=1}^{n}\left((-1)^{k+i}\binom{k}{i}\frac{1}{\sqrt{k}}\right)\left((-1)^{k+j}\binom{k}{j}\frac{1}{\sqrt{k}}\right)$$

which is equal to

$$(-1)^{i+j}\sum_{k=1}^{n}\binom{k}{i}\binom{k}{j}\frac{1}{k}.$$

□

Remark 2.7. The above theorems hold true if i,j are replaced by λ_i, λ_j, where $0 < \lambda_1 < \cdots < \lambda_n$ are positive integers. For $r > 0$, the matrix $\left[\frac{1}{\beta(\lambda_i,\lambda_j)^r}\right]$ is totally positive because it is a submatrix of the $\lambda_n \times \lambda_n$ matrix $\mathcal{B}^{\circ r}$. So $\left[\frac{1}{\beta(\lambda_i,\lambda_j)}\right]$ is also infinitely divisible. We have $\left[\frac{1}{\beta(\lambda_i,\lambda_j)}\right] = LL^*$, where L is the $n \times \lambda_n$ matrix $[\binom{\lambda_i}{j}\sqrt{j}]$. The proof is same as for Theorem 2.4.

The next theorem gives the SEB factorization for the matrix $\mathcal{B}$, which also gives another proof for $\mathcal{B}$ to be totally positive.

Theorem 2.8. The matrix $\mathcal{B} = \left[\frac{1}{\beta(i,j)}\right]$ can be written as

$$\left(L_n\left(\frac{n}{n-1}\right)L_{n-1}\left(\frac{n-1}{n-2}\right)\cdots L_2(2)\right)\left(L_n\left(\frac{n}{n-1}\right)\cdots L_3\left(\frac{3}{2}\right)\right)\cdots\left(L_n\left(\frac{n}{n-1}\right)\right)D$$
$$\left(U_n\left(\frac{n}{n-1}\right)\right)\cdots\left(U_3\left(\frac{3}{2}\right)\cdots U_n\left(\frac{n}{n-1}\right)\right)\left(U_2(2)\cdots U_{n-1}\left(\frac{n-1}{n-2}\right)U_n\left(\frac{n}{n-1}\right)\right),$$

where $D = \operatorname{diag}([i])$.

To prove this theorem, we first need a lemma.

Lemma 2.9. For $1 \le k \le n-1$, let $Y_k = \left[y_{ij}^{(k)}\right]$ be the $n \times n$ lower triangular matrix where

$$y_{ij}^{(k)} = \begin{cases} \frac{i}{j}\binom{i-(n-k)}{j-(n-k)} & \text{if } n-k \le j < i \le n, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$Y_k = \left(L_n\left(\frac{n}{n-1}\right)\cdots L_{n-(k-1)}\left(\frac{n-k+1}{n-k}\right)\right)\left(L_n\left(\frac{n}{n-1}\right)\cdots L_{n-(k-2)}\left(\frac{n-k+2}{n-k+1}\right)\right)\cdots\left(L_n\left(\frac{n}{n-1}\right)\right). \tag{3}$$

Proof. For $k = 1$, the right hand side is $L_n\left(\frac{n}{n-1}\right)$, which is same as Y_1. We show below that for $1 \le k \le n-2$,

$$(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k = Y_{k+1}, \tag{4}$$

where $\mathcal{L}_p$ denotes $L_p\left(\frac{p}{p-1}\right)$. This will show that (3) is true for $k = 1, 2, \ldots, n-1$. We also keep note of the fact that multiplying $\mathcal{L}_p$ on the left of a matrix A is applying the elementary row operation row $p \to$ row $p + \left(\frac{p}{p-1}\right) \times$ row $(p-1)$ on A, which we will use in the cases $2, 3$ and 4. We also note that for $k = n-2$, only cases 1 and 4 are relevant.

Case 1: Let $1 \le i \le j \le n$. Since $\mathcal{L}_p$ and Y_k are lower triangular matrices with diagonal entries 1, so is $(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k$. So the (i,j)th entry of both the matrices in (4) is same.

Case 2: Let $1 \le j \le n-k-2$ and $j+1 \le i \le n-k-1$. In this case, $y_{ij}^{(k)} = 0$. Since multiplying Y_k on the left by $\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k}$ will keep its rows $j+1, \ldots, n-k-1$ unchanged, we get that the (i,j)th entry of $(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k$ is zero.

Case 3: Let $1 \le j \le n-k-2$ and $n-k \le i \le n$. Multiplying Y_k on the left by $\mathcal{L}_{n-k}, \mathcal{L}_{n-(k-1)}, \ldots, \mathcal{L}_n$ successively, we get that the (i,j)th entry of $(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k$ is given by

$$\begin{aligned} & y_{ij}^{(k)} + \frac{i}{i-1}\left[y_{i-1,j}^{(k)} + \frac{i-1}{i-2}\left[y_{i-2,j}^{(k)} + \cdots + \frac{n-k+1}{n-k}\left[y_{n-k,j}^{(k)} + \frac{n-k}{n-k-1}y_{n-k-1,j}^{(k)}\right]\right]\cdots\right] \\ &= y_{ij}^{(k)} + \frac{i}{i-1}y_{i-1,j}^{(k)} + \frac{i}{i-2}y_{i-2,j}^{(k)} + \cdots + \frac{i}{n-k-1}y_{n-k-1,j}^{(k)}. \end{aligned} \tag{5}$$

Now $y_{pj}^{(k)} = 0$ for all $1 \le j \le n-k-2$ and $p \ne j$. So the (i,j)th entry of $(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k$ is zero.

Case 4: Let $i > j \ge n-k-1$. Again, the (i,j)th entry of $(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k$ is given by $y_{ij}^{(k)} + \frac{i}{i-1}y_{i-1,j}^{(k)} + \frac{i}{i-2}y_{i-2,j}^{(k)} + \cdots + \frac{i}{j}y_{jj}^{(k)} + \cdots + \frac{i}{n-k-1}y_{n-k-1,j}^{(k)}$. For $j = n-k-1$, $y_{pj}^{(k)} = 0$ for $p \ne j$. So the $(i, n-k-1)$th entry of $(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k$ is $\frac{i}{n-k-1}$. For $j \ge n-k$, $y_{pj}^{(k)} = 0$ for $p < j$ and $y_{pj}^{(k)} = \frac{p}{j}\binom{p-n+k}{j-n+k}$ for $p \ge j$. So we obtain that the (i,j)th entry of $(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k$ is

$$\begin{aligned} &= \frac{i}{j}\binom{i-n+k}{j-n+k} + \frac{i}{i-1}\cdot\frac{i-1}{j}\binom{(i-1)-n+k}{j-n+k} + \frac{i}{i-2}\cdot\frac{i-2}{j}\binom{(i-2)-n+k}{j-n+k} + \cdots + \frac{i}{j} \\ &= \frac{i}{j}\left[\sum_{p=j}^{i}\binom{p-n+k}{j-n+k}\right]. \end{aligned} \tag{6}$$

Since $\sum_{k=0}^{n} \binom{m+k}{m} = \binom{m+n+1}{m+1}$, the expression in (6) is equal to $\frac{i}{j}\binom{i-n+k+1}{j-n+k+1}$. In all the above four cases, the (i,j)th entry of $(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_{n-k})Y_k$ is the same as that of Y_{k+1}. Hence we are done. ◻

Proof of Theorem 2.8. By Theorem 2.4 we have that $\mathcal{B} = \left[\binom{i}{j}\right] \; D \; \left[\binom{i}{j}\right]^*$. So it is enough to show that

$$(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_2)(\mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_3)\cdots(\mathcal{L}_n\mathcal{L}_{n-1})(\mathcal{L}_n) = \left[\binom{i}{j}\right]. \tag{7}$$

This is easily obtained by putting $k = n-1$ in (3).

◻

Remark 2.10. Let $p_0, \ldots, p_{n-1}$ be functions from a set $\mathfrak{X}$ to a field and $\lambda_1, \ldots, \lambda_m \in \mathfrak{X}$. Then the $m \times n$ matrix defined by $[p_{j-1}(\lambda_i)]_{1\le i\le m, 1\le j\le n}$ is called an *alternant matrix* [2, p. 112]. Let $\langle x\rangle_n = x(x+1)\cdots(x+n-1)$. For positive integers $0 < \lambda_1 < \cdots < \lambda_n$, we have

$$\frac{1}{\beta(\lambda_i,\lambda_j)} = \frac{(\lambda_i+\lambda_j-1)!}{(\lambda_i-1)!(\lambda_j-1)!} = \frac{\langle\lambda_j\rangle_{\lambda_i}}{(\lambda_i-1)!}.$$

Thus with $p_{j-1}(x) = \frac{\langle\lambda_j\rangle_x}{(x-1)!}$, $\left[\frac{1}{\beta(\lambda_i,\lambda_j)}\right]$ is an alternant matrix.

We now consider the more general matrix $\left[\frac{1}{\beta(\lambda_i,\lambda_j)}\right]$ for positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. The (i,j)th entry of this matrix is given by $\frac{\Gamma(\lambda_i+\lambda_j)}{\Gamma(\lambda_i)\Gamma(\lambda_j)}$. The proof for infinite divisibility of generalized Pascal matrix $\left[\frac{\Gamma(\lambda_i+\lambda_j+1)}{\Gamma(\lambda_i+1)\Gamma(\lambda_j+1)}\right]$ is given in [3]. Infinite divisibility of $\left[\frac{1}{\beta(\lambda_i,\lambda_j)}\right]$ follows by a similar argument. Alternatively, one can also observe that $\left[\frac{1}{\beta(\lambda_i,\lambda_j)}\right] = \left[\frac{\Gamma(\lambda_i+\lambda_j+1)}{\Gamma(\lambda_i+1)\Gamma(\lambda_j+1)}\right] \circ \left[\frac{1}{\frac{1}{\lambda_i}+\frac{1}{\lambda_j}}\right]$ and deduce its infinite divisibility. This is also same as saying that $\frac{1}{\beta(\cdot,\cdot)}$ is an *infinitely divisible kernel* [15] on $\mathbb{R}^+ \times \mathbb{R}^+$.

We observe that $\frac{1}{\beta(\cdot,\cdot)}$ is also a *totally positive kernel* [17] on $\mathbb{R}^+ \times \mathbb{R}^+$. For that we first show that $[\Gamma(\lambda_i+\mu_j)]$ is totally positive, where $0 < \lambda_1 < \cdots < \lambda_n$ and $0 < \mu_1 < \cdots < \mu_n$. The proof of this was guided to us by Abdelmalek Abdesselam and Mateusz Kwaśnicki [1].

Theorem 2.11. Let $0 < \lambda_1 < \cdots < \lambda_n$ and $0 < \mu_1 < \cdots < \mu_n$ be positive real numbers. Then $[\Gamma(\lambda_i+\mu_j)]$ is totally positive.

Proof. Since all the minors of $[\Gamma(\lambda_i+\mu_j)]$ are also of the same form, it is enough to show that $\det([\Gamma(\lambda_i+\mu_j)]) > 0$. Let $K_1(x,y) = x^y$ and $K_2(x,y) = y^x$. For any $\lambda, \mu \in \mathbb{R}^+$,

$$\begin{aligned}
\Gamma(\lambda+\mu) &= \int_0^\infty e^{-t}t^{\lambda+\mu-1}dt \\
&= \int_0^\infty t^{\lambda+\mu}\left(\frac{e^{-t}}{t}\right)dt \\
&= \int_0^\infty t^\lambda t^\mu \sigma(dt), \quad \text{where } \sigma(dt) = \frac{e^{-t}}{t}dt \\
&= \int_0^\infty K_2(\lambda,t)K_1(t,\mu)\sigma(dt).
\end{aligned} \tag{8}$$

[1] https://mathoverflow.net/questions/306366/

Let $0 < t_1 < \cdots < t_n$. Then by (8) and the *basic composition formula* [17, p. 17] ,

$$\det([\Gamma(\lambda_i+\mu_j)]) = \int_{t_1=0}^{\infty} \cdots \int_{t_n=0}^{\infty} \det([K_2(\lambda_i,t_j)]) \times \det([K_1(t_i,\mu_j)])\, \sigma(dt_1)\cdots\sigma(dt_n).$$

Since K_1 and K_2 are totally positive kernels on $\mathbb{R}^+ \times \mathbb{R}^+$ (see [19, p. 90]), $\det([K_2(\lambda_i,t_j)])$ and $\det([K_1(t_i,\mu_j)])$ are positive functions of $t_1,\ldots,t_n$. So $\det([\Gamma(\lambda_i+\mu_j)]) > 0$. □

Corollary 2.12. For positive real numbers $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $0 < \mu_1 < \mu_2 < \cdots < \mu_n$, the matrix $\left[\frac{1}{\beta(\lambda_i,\mu_j)}\right]$ is totally positive.

Proof. Since $\mathcal{B} = \operatorname{diag}\left(\left[\frac{1}{\Gamma(\lambda_i)}\right]\right)\ [\Gamma(\lambda_i+\mu_j)]\operatorname{diag}\left(\left[\frac{1}{\Gamma(\mu_i)}\right]\right)$, we obtain the required result. □

3 Combinatorial matrices

The unsigned Stirling matrix of first kind $\mathfrak{s}$ is totally nonnegative as well as invertible. Theorem 2.2.2 in [12] says that every invertible totally nonnegative matrix can be written as $(L_n(l_k)\,L_{n-1}(l_{k-1})\cdots L_2(l_{k-n+2}))$ $(L_n(l_{k-n+1})\,L_{n-1}(l_{k-n})\cdots L_3(l_{k-2n+4}))\cdots(L_n(l_1))D(U_n(u_1))(U_{n-1}(u_2)\,U_n(u_3))\cdots(U_2(u_{k-n+2})\cdots U_{n-1}(u_{k-1})\,U_n(u_k))$, where $k=\binom{n}{2}$, $l_i,u_j \geq 0$ for all $i,j \in \{1,2,\ldots,k\}$ and $D=\operatorname{diag}([d_i])$ is a diagonal matrix with all $d_i > 0$. The below proposition gives that $u_j = 0$ for $\mathfrak{s}$, which is not surprising in view of [16, Theorem 7].

Proposition 3.1. The $n\times n$ unsigned Stirling matrix of first kind $\mathfrak{s}$ can be factorized as

$$\mathfrak{s} = (L_n(n-1)L_{n-1}(n-2)\cdots L_2(1))(L_n(n-2)L_{n-1}(n-3)\cdots L_3(1))\cdots(L_n(2)L_{n-1}(1))(L_n(1)). \quad (9)$$

Proof. Since $(L_i(s))^{-1} = L_i(-s)$, so it is enough to show that

$$(L_n(-1))\ (L_{n-1}(-1)L_n(-2))\cdots(L_3(-1)\cdots L_{n-1}(-(n-3))L_n(-(n-2)))\,(L_2(-1)\cdots L_{n-1}(-(n-2))L_n(-(n-1)))\,\mathfrak{s} = I_n, \quad (10)$$

where I_n is the $n\times n$ identity matrix. We shall prove (10) by induction on n. For clarity, we shall denote the $n\times n$ matrices $\mathfrak{s}$ and $L_i(s)$, by $\mathfrak{s}_n$ and $L_i(s)_{(n)}$, respectively. For $n=2$, $\mathfrak{s}_2 = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, which is clearly equal to $L_2(1)_{(2)}$. Let us assume that (10) holds for n. We have the following recurrence relation [14, p. 166] for $\mathfrak{s}_{ij}$:

$$\begin{aligned}&\mathfrak{s}_{00} = 1, \mathfrak{s}_{0j} = 0, \mathfrak{s}_{i0} = 0\\ &\mathfrak{s}_{i+1,j} = \mathfrak{s}_{i,j-1} + i\ \mathfrak{s}_{ij}.\end{aligned} \quad (11)$$

So for $1 \leq k \leq n$, multiplying $\mathfrak{s}_{n+1}$ on the left by $(L_{k+1}(-k))_{(n+1)}$ replaces its $(k+1)$th row by the row whose first element is 0 and jth element is the $(j-1)$th element of the previous row. So we get

$$\left(L_2(-1)_{(n+1)}\cdots L_n(-(n-1))_{(n+1)}L_{n+1}(-n)_{(n+1)}\right)\mathfrak{s}_{n+1} = \begin{bmatrix}1 & 0\\ 0 & \mathfrak{s}_n\end{bmatrix}. \quad (12)$$

It is easy to see that

$$L_i(s)_{(n+1)} = \begin{bmatrix}1 & 0\\ 0 & L_{i-1}(s)_{(n)}\end{bmatrix}.$$

Hence

$$\begin{aligned}&\left(L_{n+1}(-1)_{(n+1)}\right)\left(L_n(-1)_{(n+1)}L_{n+1}(-2)_{(n+1)}\right)\cdots\left(L_3(-1)_{(n+1)}\cdots L_n(-(n-2))_{(n+1)}L_{n+1}(-(n-1))_{(n+1)}\right)\\ &= \begin{bmatrix}1 & 0\\ 0 & \left(L_n(-1)_{(n)}\right)\left(L_{n-1}(-1)_{(n)}L_n(-2)_{(n)}\right)\cdots\left(L_2(-1)_{(n)}\cdots L_{n-1}(-(n-2))_{(n)}L_n(-(n-1))_{(n)}\right)\end{bmatrix}.\end{aligned}$$

Using induction hypothesis and (12), we obtain

$$\left(L_{n+1}(-1)_{(n+1)}\right)\left(L_n(-1)_{(n+1)}L_{n+1}(-2)_{(n+1)}\right)\cdots\left(L_3(-1)_{(n+1)}\cdots L_n(-(n-2))_{(n+1)}L_{n+1}(-(n-1))_{(n+1)}\right)$$
$$\left(L_2(-1)_{(n+1)}\cdots L_n(-(n-1))_{(n+1)}L_{n+1}(-n)_{(n+1)}\right)\mathfrak{s}_{n+1}=I_{n+1}.$$

□

As an immediate consequence, we obtain the following.

Theorem 3.2. The symmetrized unsigned Stirling matrix of first kind $\mathfrak{s}\mathfrak{s}^*$ is totally positive.

Proof. This follows from the above Proposition 3.1 and Corollary 2.2.3 of [12]. □

We now show that symmetrized Stirling matrix of second kind $\mathcal{S}\mathcal{S}^*$ is totally positive. For that we first prove that $\mathcal{S}$ is triangular totally positive. For $\alpha=\{\alpha_1,\ldots,\alpha_p\}$, $\gamma=\{\gamma_1,\ldots,\gamma_p\}$ with $1\le\alpha_1<\cdots<\alpha_p\le n$ and $1\le\gamma_1<\cdots<\gamma_p\le n$, let $A[\alpha,\gamma]$ denotes the submatrix of A obtained by picking rows $\alpha_1,\ldots,\alpha_p$ and columns $\gamma_1,\ldots,\gamma_p$ of A. The *dispersion* of α, denoted by $d(\alpha)$, is defined as $d(\alpha)=\alpha_p-\alpha_1-(p-1)$. Note that $d(\alpha)=0$ if and only if $\alpha_1,\alpha_2,\ldots,\alpha_p$ are consecutive p numbers. We denote by α' the set $\{\alpha_1+1,\alpha_2+1,\ldots,\alpha_p+1\}$, and by $\sigma^{(p)}$ the set $\{1,\ldots,p\}$.

Proposition 3.3. The Stirling matrix of second kind $\mathcal{S}$ is triangular totally positive.

Proof. By Theorem 3.1 of [10], $\mathcal{S}$ is triangular totally positive if and only if $\det(\mathcal{S}[\alpha,\sigma^{(p)}])>0$ for all $1\le p\le n$ and for all $\alpha=\{\alpha_1,\ldots,\alpha_p\}$ satisfying $1\le\alpha_1<\cdots<\alpha_p\le n$ and $d(\alpha)=0$. For $p=n$, this is obviously true. Let $1\le p<n$. If $\alpha_1=1$ and $d(\alpha)=0$, then $\alpha=\sigma^p$ and $\det(\mathcal{S}[\sigma^{(p)},\sigma^{(p)}])=1>0$. Next we show that if $\det(\mathcal{S}[\alpha,\sigma^{(p)}])>0$, where $d(\alpha)=0$, then $\det(\mathcal{S}[\alpha',\sigma^{(p)}])>0$ (and $d(\alpha')=0$).

Let $T=[t_{ij}]$ be defined as $t_{ij}=\begin{cases} i & \text{if } i=j\\ 1 & \text{if } j-i=1\\ 0 & \text{otherwise}\end{cases}$. We prove that

$$\mathcal{S}[\alpha',\sigma^{(p)}]=\mathcal{S}[\alpha,\sigma^{(p)}]T \text{ for } 1\le p<n. \tag{13}$$

The (i,j)th entry of $\mathcal{S}[\alpha,\sigma^{(p)}]T$ is $j\,S(\alpha_i,j)+S(\alpha_i,j-1)$. The Stirling numbers of second kind satisfy the following recurrence relation:

$$S(0,0)=1;$$
$$\text{for } \ell,m\ge1,\ S(0,m)=0=S(\ell,0),\ S(\ell,m)=m\,S(\ell-1,m)+S(\ell-1,m-1).$$

Thus the (i,j)th entry of $\mathcal{S}[\alpha,\sigma^{(p)}]T$ is $=S(\alpha_i+1,j)$, which is also the (i,j)th entry of $\mathcal{S}[\alpha',\sigma^{(p)}]$. Hence (13) holds, which gives that $\det(\mathcal{S}[\alpha',\sigma^{(p)}])=p!\det(\mathcal{S}[\alpha,\sigma^{(p)}])>0$. □

Theorem 3.4. The symmetrized Stirling matrix of second kind $\mathcal{S}\mathcal{S}^*$ is totally positive.

Proof. This follows from Proposition 3.3 and Corollary 2.4.2 of [12]. □

Next, we show that $\mathfrak{B}=[\mathfrak{w}(i+j)]$ is totally positive. Let $Y=(y_{ij})_{i,j=0}^{n-2}$ be the lower triangular matrix defined recursively by $y_{00}=1$, $y_{0j}=0$ for $j>0$, and $y_{ij}=y_{i-1,j-1}+(j+2)y_{i-1,j}+(j+1)y_{i-1,j+1}$ for $i\ge1$, where $y_{i,-1}=0$ for every i and $y_{in}=0$ for $0\le i\le n-3$.

Theorem 3.5. The matrix $\mathfrak{B}=[\mathfrak{w}(i+j)]$ is totally positive.

Proof. Let $\mathfrak{B}(n,1)=[\mathfrak{w}(i+j+1)]_{i,j=0}^{n-2}$ be the matrix obtained from $\mathfrak{B}$ by deleting its first column and nth row. From the proof of the Theorem in [1], $\mathfrak{B}=LL^*$, where $L=X\ \mathrm{diag}\left(\left[\sqrt{i!}\right]\right)_{i=0}^{n-1}$, and $\mathfrak{B}(n,1)=L'L'^*$, where $L'=Y\ \mathrm{diag}\left(\left[\sqrt{i!)}\right]\right)_{i=0}^{n-2}$. Hence $\mathfrak{B}$ and $\mathfrak{B}(n,1)$ are positive semidefinite. The Theorem in [1] also shows that both $\mathfrak{B}$ and $\mathfrak{B}(n,1)$ are nonsingular, hence they are positive definite. Since $\mathfrak{B}$ is a Hankel matrix, the result now follows from Theorem 4.4 of [19].

□

Now we show that $\mathfrak{B}$ is infinitely divisible only upto order 4. For $A = [a_{ij}]$, let $\log A = [\log(a_{ij})]$. Let ΔA denote the $(n-1) \times (n-1)$ matrix $[a_{ij} + a_{i+1,j+1} - a_{i+1,j} - a_{i,j+1}]_{i,j=1}^{n-1}$.

Theorem 3.6. The $n \times n$ matrix $\mathfrak{B}$ is infinitely divisible if and only if $n \leq 4$.

Proof. We denote the $n \times n$ matrix $\mathfrak{B}$ by $\mathfrak{B}_n$. Since $\mathfrak{B}_n$ is a principal submatrix of $\mathfrak{B}_{n+1}$, it is enough to show that $\mathfrak{B}_4$ is infinitely divisible but $\mathfrak{B}_5$ is not infinitely divisible.

By Corollary 1.6 and Theorem 1.10 of [15], to prove infinite divisibility of $\mathfrak{B}_4$, it is enough to prove that $\Delta \log \mathfrak{B}_4$ is positive definite. Now

$$\mathfrak{B}_4 = \begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 15 \\ 2 & 5 & 15 & 52 \\ 5 & 15 & 52 & 203 \end{bmatrix}, \ \log \mathfrak{B}_4 = \begin{bmatrix} 0 & 0 & \log 2 & \log 5 \\ 0 & \log 2 & \log 5 & \log 15 \\ \log 2 & \log 5 & \log 15 & \log 52 \\ \log 5 & \log 15 & \log 52 & \log 203 \end{bmatrix},$$

$$\text{and } \Delta \log \mathfrak{B}_4 = \begin{bmatrix} \log 2 & \log(5/4) & \log(6/5) \\ \log(5/4) & \log(6/5) & \log(52/45) \\ \log(6/5) & \log(52/45) & \log(3045/2704) \end{bmatrix}.$$

Since all the leading principal minors of $\Delta \log \mathfrak{B}_4$ are positive, we get that $\Delta \log \mathfrak{B}_4$ is positive definite. Hence $\mathfrak{B}_4$ is infinitely divisible. Since $\det\left(\mathfrak{B}_5^{\circ(\frac{1}{4})}\right) = -1.62352 \times 10^{-9} < 0$, $\mathfrak{B}_5$ is not infinitely divisible. □

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