

Heavy-Quark Potential from Gauge/Gravity Duality: A large D Analysis

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Abstract

The heavy-quark potential is calculated in the framework of gauge/gravity duality using the large- D approximation, where D is the number of dimensions transverse to the flux tube connecting a quark and an antiquark in a flat $D + 2$ -dimensional spacetime. We find that in the large- D limit the leading correction to the ground-state energy, as given by an effective Nambu-Goto string, arises not from the heavy modes but from the behavior of the massless modes in the vicinity of the quark and the antiquark. We estimate this correction and find that it should be visible in the near-future lattice QCD calculations of the heavy-quark potential.

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1 Introduction

There is strong numerical evidence that the dynamics of a flux tube formed between a heavy-quark and an antiquark is well described by the Nambu-Goto string in four spacetime dimensions (see, e.g., [1, 2, 3], and [4] for a review of the effective string approach to QCD flux tubes with a review of the results for the closed flux tubes). In particular, the ground-state energy of such a flux tube matches very well with the Arvis formula for the ground-state energy of an open Nambu-Goto string with fixed end points in four dimensions [5]. Yet one expects on general grounds that there should be corrections to Arvis formula, for after all, the QCD flux tube is not a one-dimensional object and has some “intrinsic” thickness that we expect to be related to the *finite* correlation length of the confining SU(3) gauge theory [6]. If one thinks of the fluctuations in the intrinsic thickness as an additional degree of freedom, then integrating out this degree of freedom would lead to an effective string action. Such an action, in general, should contain all possible terms consistent with reparameterization invariance. In the spirit of effective field theories, these terms can be organized into relevant, marginal and irrelevant terms (for a review of these ideas see, for e.g, Refs. [7, 8, 9].) The Nambu-Goto action is the only relevant term; then there is a marginal term—the so-called extrinsic-curvature term—and infinitely many irrelevant terms corresponding to higher and higher derivatives of the string coordinates. Further, since we are considering open strings, there could also be boundary terms. In fact, we know on very general grounds that the resulting effective string action cannot be just the Nambu-Goto action [10, 11].

To obtain an effective string action for QCD flux tubes—starting from the fundamental description in terms of quarks and gluons—we need to integrate out all the fluctuations of the gauge fields whose wavelength is less than the correlation length of the confining theory. This task is as difficult as deriving an effective chiral Lagrangian for pions starting from the QCD action. The gauge/gravity duality [12], inspired by the AdS/CFT correspondence [13, 14, 15, 16], provides us with an alternate—though approximate and often heuristic—way of analyzing the dynamics of

strongly coupled gauge theories. Specifically, it provides us with a geometrical way of exploring the consequences of one of the defining properties of $SU(N)$ gauge theories, namely the existence of a finite correlation length of the gauge fields. According to the gauge/gravity duality, a QCD flux tube should be thought of as a holographic projection of a fundamental string in five-dimensional curved space (that may have more compact directions but they will not play a role in the present investigation.) This picture then suggests that for confining gauge theories the intrinsic thickness of the flux tube can be related to the position of a fundamental string in the fifth dimension [17, 18]. It was also pointed out in Ref. [19] that this relationship was consistent with the expected behavior of the intrinsic thickness of a Nielsen-Olesen-like flux tube with its string tension [20].

The aim of the present investigation is to use the geometrical picture provided by the gauge/gravity duality to delineate the various sources of correction to the heavy-quark potential as compared to the ground-state energy of a Nambu-Goto string in flat four-dimensional spacetime. These corrections are induced by the fact that the fundamental QCD string lives in a five-dimensional curved space. We will analyze this situation using the large- D approximation, where $D + 2$ is the number of dimensions of the flat spacetime where the quark lives. D is therefore the number of directions transverse to a QCD flux tube. The main advantage that this approach offers is that it treats the massless modes of the string nonperturbatively.

In gauge/gravity duality, the confining property of the boundary gauge theory is reflected in the fact that a fundamental string in five dimensions can minimize its energy by staying at a fixed value of its fifth coordinate as shown in Fig. 1. We first calculate the large- D behavior of the fundamental string in the approximation where the fundamental string descends from its minimal energy value to the boundary right at $x = 0$ and $x = L$, where the quark and the antiquark are placed. This is shown in Fig. 2. The dynamics of the open string in this approximation is very much like that of an open string placed at a fixed value of the fifth coordinate and stretched between two D-branes. The leading term in the large- D analysis for this case turns out to be nothing but the Arvis formula. As we will see in Sec. 4, when one corrects for this approximation then the leading term in the large D expansion is no longer the Arvis formula alone. Rather, the Arvis formula gets modified by an overall factor that can be thought of as a length-dependent string tension.

It is important to note that there should be corrections to the Arvis formula even if the Nambu-Goto action by itself provided an exact description of QCD flux tubes [?]². These expected corrections should arise from quantizing the Nambu-Goto string in noncritical dimensions. From the point of view of the large D expansion such corrections should exist since the Arvis formula is

² I would like to thank Ofer Aharony for pointing this out to me.

only the leading term in the large- D expansion and there is no *a priori* reason to believe that the higher-order corrections should vanish. Though we will not explore these higher-order corrections in our large- D analysis, but it is interesting to note that these corrections should themselves be proportional to $D - 24$ as they should vanish for the critical dimension of $D + 2 = 26$. Thus, surprisingly the Arvis formula gives the exact ground-state energy of a Nambu-Goto string in flat spacetime for $D = 24$ and for $D \rightarrow \infty$.

In an alternative approach, pioneered in Refs. [21, 22], one starts with a derivative expansion for the action of a fundamental string stretched between two D-branes in five-dimensional confining background; then one perturbatively integrates out the fluctuations corresponding to the oscillations in the fifth dimension. These world-sheet fluctuations are massive, and after integrating them out one obtains an effective action for the massless transverse fluctuations. In Ref. [22] it was shown that this leads to a correction to the Arvis potential at the order of $1/L^4$, where L is the length of the fundamental string stretched between the two D-branes. This correction comes from a boundary term induced by integrating out the heavy modes. In another approach to boundary terms explored in Refs. [23, 24], one expands the boundary action in the derivatives of the massless transverse modes. The terms in this expansion are then constrained by the Lorentz invariance of the underlying theory.

In our large- D analysis we do not see such a boundary term at least to the order of $1/D$, but it is worth recalling that for the physical case of interest D is equal to 2 and therefore even $1/D^2$ correction can be important. What we do see is that the holographic description of QCD flux tubes leads to additional terms to the Nambu-Goto action that are non-zero only in the immediate vicinity of the quark and the antiquark, and we estimate their contribution to the heavy-quark potential. Presumably, the effects of such terms can be approximated by introducing boundary terms in an effective string description of a QCD flux tube.

The outline of the paper is as follows. In Sec. 2, we consider a particular class of confining geometries and calculate the leading term in the large- D expansion under the simplifying assumption that we discussed above. Further, in this Sec. we recast the large- D analysis in terms of a “master-field,” which turns out to be very useful for a later analysis. In Sec. 3, within the same simplifying assumption, we calculate the $1/D$ correction arising from the fluctuation of the string in the fifth dimension that give rise to massive world-sheet modes. In Sec. 4, we make allowance for the fact that the fundamental string descends to the boundary smoothly over a transition region of finite length. This leads to an effective string description of the QCD flux tube in which the Nambu-Goto string action is supplemented by terms that have support only in the vicinity of the quark and the antiquark. We also estimate the contributions of these terms in the large- D limit.

In the final section we state our conclusions.

2 heavy-quark Potential from Confining Geometry in the Large D Limit

The gauge-invariant observable of an $SU(N)$ gauge theory that is directly related to the heavy-quark potential is the Wilson loop,

$$W[\Gamma] = \text{Tr} \hat{P} \left(\exp i \int_{\Gamma} A \right). \quad (2.1)$$

Here Γ is a closed curve in four-dimensional Euclidean space, and A is the vector potential of the $SU(N)$ gauge theory [25]. According to the assumed gauge/gravity duality, the expectation value of the Wilson loop can be written as a sum over the world-sheets of an open string whose end points terminate on the loop Γ . The open string itself lives in a five-dimensional curved space whose boundary is the the four-dimensional Euclidian space where Γ is located [16, 26]. Formally,

$$\langle W[\Gamma] \rangle_{YM} \equiv \int [dA] \exp \{-S[A]_{YM}\} = \int_{\partial X = \Gamma} [dX] \exp \{-S[X]_{NG}\}, \quad (2.2)$$

here $S[A]_{YM}$ is the Yang-Mills action, while $X(\tau, \sigma)$ represents the world sheet of a string in a curved five-dimensional space, and $S[X]_{NG}$ is the Nambu-Goto action. To evaluate the Nambu-Goto action for a given surface in this space we need to know its metric. Ideally, we would like a prescription that gives us the geometry of the five-dimensional space once the beta function (governing the running of the Yang-Mills coupling constant with energy) of the dual boundary quantum field theory is specified. In the absence of such a prescription we can look for simple modifications of AdS_5 space that leads to an area law for the expectation value of the Wilson loop. Following Ref. [18] we will consider one such class of modification, where the metric of the curved five-dimensional space in the coordinate system $\{t, x, \mathbf{X}, Y\}$ is given by

$$ds^2 = F(Y) (dt^2 + dx^2 + d\mathbf{X}^2 + dY^2). \quad (2.3)$$

In this coordinate system $Y = 0$ is the four-dimensional boundary³ where the gauge theory lives, \mathbf{X} denotes $D = 2$ directions transverse to the flux tube lying along the x axis, and t is the Euclidian

³ Strictly speaking for AdS_5 like space, $Y = 0$ is a conformal boundary [15]. $F(Y)$ diverges there, so in what follows we will assume that the boundary is at $Y = \epsilon$ and that corresponds to considering the boundary gauge-theory with a UV cutoff of the order $1/\epsilon$ [27]

time. The area law for the expectation value of the Wilson loop is implemented by requiring that $F(Y)$ has a minima at some fixed value $Y = Y^*$. For this class of metric we will evaluate (2.2) using a large- D expansion, where D is the number of dimensions transverse to the QCD flux tube.

In the static gauge, or a physical gauge, a world-sheet of the string connecting the quark and the antiquark is given by

$$X = (t, x, \mathbf{X}(t, x), Y(t, x)) = (\tau, \sigma, \mathbf{X}(\vec{\sigma}), Y(\vec{\sigma})), \quad (2.4)$$

$$\mathbf{X} = (X^1, X^2, \dots, X^D), \quad (2.5)$$

where we identify the world-sheet parameter τ with t and σ with x . A point on the world sheet with coordinates (τ, σ) will be represented by $\vec{\sigma}$. The world-sheet coordinates range from

$$-\frac{T}{2} \leq t = \tau \leq \frac{T}{2}; \quad 0 \leq x = \sigma \leq L, \quad (2.6)$$

corresponding to a Wilson loop made up of the worldlines of a quark at $x = 0$ and an antiquark at $x = L$, and we will be interested in the limit $T \rightarrow \infty$. Surfaces appearing in (2.2) satisfy the boundary conditions

$$\mathbf{X}(\tau, 0) = 0 = \mathbf{X}(\tau, L), \quad Y(\tau, 0) = 0 = Y(\tau, L). \quad (2.7)$$

The minimal surface corresponding to the metric (2.3), in the limit $T \rightarrow \infty$ depends only on the x coordinate,

$$X_c = (t, x, \mathbf{0}, Y_c(x)), \quad (2.8)$$

and is qualitatively depicted in Fig.(1). Since the warp factor $F(Y)$ has a minima at $Y = Y^*$, classically the string stays at $Y = Y^*$ except at the end points where it dips to connect to the quark and the antiquark that are located at $Y = 0$. The region of transition is marked d in Fig.(1). For the confining geometries d is expected to grow with L but with $d/L \rightarrow 0$ as $L \rightarrow \infty$ [28, 29], where one can define d , for example, as the distance for which

$$\frac{Y^* - Y_c(d)}{Y^*} = 10^{-3}. \quad (2.9)$$

To start with, in calculating (2.2) using the large- D expansion we will neglect the region d and

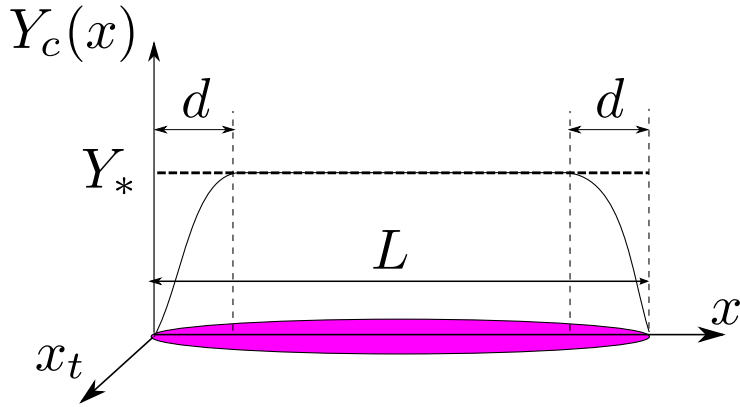
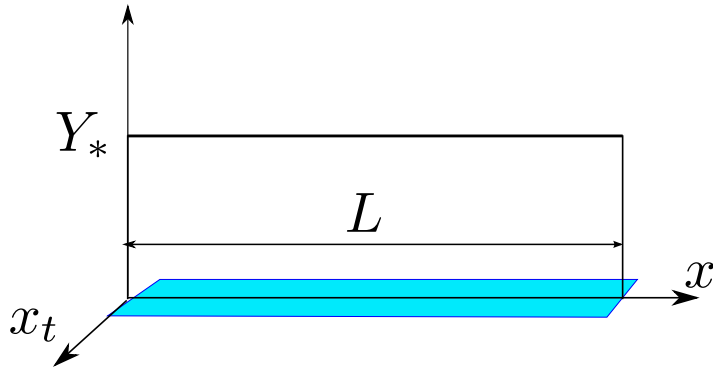


Fig. 1: The classical string configuration in a confining geometry

Fig. 2: Approximation to the classical string configuration used in the large D calculation

consider the fluctuation around the surface given by

$$X_c = (t, x, \mathbf{0}, Y^*), \quad (2.10)$$

as shown in Fig. 2. In Sec. 4 we will return to this approximation and make an estimate of the errors introduced by it. Within this approximation the fluctuations about the minimal surface are given by

$$X = (t, x, \mathbf{X}(t, x), Y^* + \phi(t, x)), \quad (2.11)$$

with $\phi(x, t)$ vanishing at $x = 0$ and at $x = L$. The Nambu-Goto action for such a surface is

$$S_{NG} = T_0 \int_{-T/2}^{T/2} dt \int_0^L dx F(Y^* + \phi) \sqrt{\det[\gamma_{ab}]}, \quad (2.12)$$

where T_0 is the bare string tension, and the induced world-sheet metric γ_{ab} (apart from the warp

factor) is

$$\gamma_{ab} = \delta_{ab} + \partial_a \mathbf{X} \cdot \partial_b \mathbf{X} + \partial_a \phi \partial_b \phi. \quad (2.13)$$

It will be convenient to write this metric in terms of two world-sheet matrices,

$$\mathbf{g} := g_{ab} = \delta_{ab} + \partial_a \mathbf{X} \cdot \partial_b \mathbf{X}, \quad (2.14)$$

and

$$\mathbf{f} := f_{ab} = \partial_a \phi \partial_b \phi. \quad (2.15)$$

Then the action for the world-sheet takes the form

$$S_{NG} = T_0 \int d^2 \sigma F(Y^* + \phi) \sqrt{\det g_{ab}} \sqrt{\det [\delta_{ab} + (g^{-1} f)_{ab}]}. \quad (2.16)$$

One can think of $T_0 F(Y^* + \phi)$ as a position-dependent string tension. Small fluctuations in the Y direction, $\phi(t, x) \neq 0$ increase the string tension. These fluctuations are suppressed compared to the fluctuations $\vec{X} \neq 0, \phi = 0$, for the latter fluctuations keep the string tension fixed at its minimum value. This suggests the following approximate way of evaluating (2.2): keep transverse fluctuations \vec{X} to all orders but keep only the quadratic fluctuations in ϕ . In this approximation the Nambu-Goto action (2.12) takes the following form

$$S_{NG5} = \frac{1}{l_s^2} \int d^2 \sigma \sqrt{\det g_{ab}} + \frac{1}{l_s^2} \int d^2 \sigma \sqrt{\det g_{ab}} \left(\frac{1}{2} \text{Tr} \mathbf{g}^{-1} \mathbf{f} + \frac{1}{2} M^2 \phi^2 \right) \quad (2.17)$$

where M^2 is defined via,

$$F(Y^* + \phi) = F(Y^*) \left(1 + \frac{1}{2} M^2 \phi^2 \right) + \mathcal{O}(\phi^3), \quad (2.18)$$

and the fundamental length scale for the problem, l_s , is given by

$$\frac{1}{l_s^2} = T_0 F(Y^*). \quad (2.19)$$

In this approximation, and according to our fundamental assumption (2.2), the expectation value of a Wilson loop is given by

$$\langle W[\Gamma] \rangle_{YM} = \int [d\mathbf{X}] \exp \left\{ -\frac{1}{l_s^2} \int d^2 \sigma \sqrt{\det g_{ab}} \right\} \mathbf{C}[g_{ab}], \quad (2.20)$$

where

$$\mathbf{C}[g_{ab}] = \int [d\phi] \exp \left\{ -\frac{1}{l_s^2} \int d^2\sigma \sqrt{\det g_{ab}} \left(\frac{1}{2} \text{Tr} \mathbf{g}^{-1} \mathbf{f} + \frac{1}{2} M^2 \phi^2 \right) \right\}. \quad (2.21)$$

Since the heavy-quark potential, $V[L]$, is obtained from the expectation value of a rectangular Wilson-Loop

$$\lim_{T \rightarrow \infty} \langle W[\square_{T \times L}] \rangle_{YM} = \mathcal{N} \exp \{-TV[L]\}, \quad (2.22)$$

we see that $\mathbf{C}[g_{ab}]$ contains the contribution to the heavy-quark potential due to the fluctuation of the fundamental string in the fifth dimension. This can be heuristically interpreted as the contribution to the heavy-quark potential due to the fluctuation of the intrinsic thickness of the QCD flux tube.

Numerical calculations of the ground-state energy of a QCD flux tube, in lattice gauge theory matches very well with the ground-state energy of the Nambu-Goto string in four Euclidean dimensions as given by the Arvis formula. This suggests that the fluctuations of the string in the fifth dimension should only provide a small correction to the ground-state energy given by the Nambu-Goto string in the four flat dimensions. Therefore, setting $C[\mathbf{g}] = \text{constant}$ in (2.20) which corresponds to describing the QCD flux tube by a Nambu-Goto string in four Euclidean dimensions and ignoring the fluctuations in the intrinsic thickness—is a good approximation.

The above observations can be made more precise by evaluating (2.20) in the large- D expansion, where D is the number of transverse dimensions of the $D + 2$ -dimensional boundary. As we will see, in the large- D expansion of (2.20) the smallness of the correction due to $C[\mathbf{g}]$ will be the consequence of the fact that there is only one massive ϕ field while there are D massless transverse fields, X_i .

The large- D expansion will be implemented in the standard manner [30]. We start with the path integral

$$Z_5 = \int [d\mathbf{X}] \exp \left\{ -\frac{1}{l_s^2} \int d^2\sigma \sqrt{\det g_{ab}} \right\} \mathbf{C}[g_{ab}], \quad (2.23)$$

and introduce \mathbf{g} as an independent degree of freedom in the path integral but with the constraint (2.14). Next we implement the constraint using the Lagrangian multiplier field N . This converts the integration over \mathbf{X} fields into Gaussian integrals and one obtains

$$Z_5 = \mathcal{N}_1 \int [d\mathbf{g}] [d\mathbf{N}] \exp \{-S_{eff}[\mathbf{g}, \mathbf{N}]\} \mathbf{C}[\mathbf{g}], \quad (2.24)$$

where

$$S_{eff}[\mathbf{g}, \mathbf{N}] = \frac{1}{\bar{l}_s^2} \int d^2\sigma \left\{ \sqrt{\det \mathbf{g}} + \frac{1}{2} N^{ab} (\delta_{ab} - g_{ab}) \right\} + \frac{D}{2} \text{Tr} [\log (-\partial_a N^{ab} \partial_b)], \quad (2.25)$$

and \mathcal{N}_1 is a normalization constant. Further, we can integrate over the ϕ field in (2.21) and write

$$\mathbf{C}[\mathbf{g}] = \mathcal{N}_2 \exp \left\{ -\frac{1}{2} \text{Tr} [\log (\hat{\mathbf{A}})] \right\}, \quad (2.26)$$

where

$$\hat{\mathbf{A}}(\mathbf{g}) = -\partial_a \sqrt{\det \mathbf{g}} g_{ab}^{-1} \partial_b + \sqrt{\det \mathbf{g}} M^2, \quad (2.27)$$

and \mathcal{N}_2 is another normalization constant.

We would like to evaluate (2.23) in the limit $D \rightarrow \infty$ while keeping $D\bar{l}_s^2$ constant. In other words we are interested in the limit

$$D \rightarrow \infty; \quad \bar{l}_s^2 = \text{constant}, \quad (2.28)$$

where \bar{l}_s^2 is defined as

$$\bar{l}_s^2 = D\bar{l}_s^2. \quad (2.29)$$

The path integral (2.23) can now be written in the following form

$$Z_5 = \mathcal{N} \int [d\mathbf{g}] [d\mathbf{N}] \exp \{-D S[\mathbf{g}, \mathbf{N}]\}, \quad (2.30)$$

where $S[\mathbf{g}, \mathbf{N}]$ has two parts,

$$S[\mathbf{g}, \mathbf{N}] = \left(S_1[\mathbf{g}, \mathbf{N}] + \frac{1}{D} S_2[\mathbf{g}] \right), \quad (2.31)$$

with

$$S_1[\mathbf{g}, \mathbf{N}] = \frac{1}{\bar{l}_s^2} \int d^2\sigma \left\{ \sqrt{\det \mathbf{g}} + \frac{1}{2} N^{ab} (\delta_{ab} - g_{ab}) \right\} + \frac{1}{2} \text{Tr} [\log (-\partial_a N^{ab} \partial_b)], \quad (2.32)$$

and

$$S_2[\mathbf{g}] = \frac{1}{2} \text{Tr} [\log (\hat{\mathbf{A}}(\mathbf{g}))]. \quad (2.33)$$

Therefore in the large- D limit, the contribution of the fluctuations of the ϕ field is suppressed by a

factor of $1/D$ as compared to the contributions of the fluctuations of the transverse coordinates X_i . From the point of view of the large- D expansion, the success of the four-dimensional Nambu-Goto string in reproducing the heavy-quark potential is due to the fact that the corrections to it are suppressed by a factor of $1/D$.

The leading term in the large- D expansion of (2.30) is then given by

$$Z_5 = \mathcal{N}_5 \exp \left\{ -D S_1 [\bar{\mathbf{g}}, \bar{\mathbf{N}}] \right\}, \quad (2.34)$$

where the $\bar{\mathbf{g}}$ and $\bar{\mathbf{N}}$ are the fields that minimize the action S_1 alone,

$$\frac{\delta S_1}{\delta \bar{\mathbf{g}}} = 0, \quad \frac{\delta S_1}{\delta \bar{\mathbf{N}}} = 0. \quad (2.35)$$

Minimizing the action S_1 alone is nothing but the large- D limit of the Nambu-Goto string in a flat Euclidean space [30] and the required solutions are:

$$\bar{\mathbf{g}} = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} = \begin{bmatrix} \frac{1-\lambda}{1-2\lambda} & 0 \\ 0 & 1-\lambda \end{bmatrix}, \quad (2.36)$$

$$\bar{\mathbf{N}} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} = \begin{bmatrix} \sqrt{1-2\lambda} & 0 \\ 0 & \frac{1}{\sqrt{1-2\lambda}} \end{bmatrix}, \quad (2.37)$$

where λ depends only on the distance between the quark and the antiquark, L , and is given by

$$\lambda(L) = \frac{\pi}{24} \frac{\bar{l}_s^2}{L^2} = \frac{\pi D}{24} \frac{l_s^2}{L^2}. \quad (2.38)$$

A fact that will be important for us later—when we calculate the $1/D$ corrections—is that the metric $\bar{\mathbf{g}}$ has discontinuities at the boundaries— $\sigma = x = 0$ and $\sigma = x = L$, which are the worldlines of the quark and the antiquark (see Appendix of Ref. [30].) The metric at the boundary is given by

$$\bar{\mathbf{g}}_b = \begin{bmatrix} g_{b1} & 0 \\ 0 & g_{b2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1-2\lambda \end{bmatrix}, \quad (2.39)$$

but, interestingly, N is continuous,

$$\bar{\mathbf{N}} = \begin{bmatrix} \sqrt{\frac{g_{b2}}{g_{b1}}} & 0 \\ 0 & \sqrt{\frac{g_{b1}}{g_{b2}}} \end{bmatrix} = \begin{bmatrix} \sqrt{1-2\lambda} & 0 \\ 0 & \frac{1}{\sqrt{1-2\lambda}} \end{bmatrix}. \quad (2.40)$$

Using the above results, one readily obtains the ground-state energy of a Nambu-Goto string in the large- D limit [30], which turns out to be the Arvis formula for the ground-state energy of a Nambu-Goto string [5],

$$DS_1[\bar{\mathbf{g}}, \bar{\mathbf{N}}] = TV_A[L], \quad (2.41)$$

$$V_A[L] = \frac{L}{l_s^2} \sqrt{1 - 2\lambda(L)}. \quad (2.42)$$

The large- D expansion bears a close resemblance to the large- N expansion of multi-component quantum field theories (see, e.g., Ref. [31] for a review). It is also similar in spirit to the mean-field approximation (for a mean-field analysis of the Nambu-Goto string see [32]). Continuing with the analogy with the large- N expansion further, one can ask if there is a single surface that saturates the partition function of the Nambu-Goto string in the $D \rightarrow \infty$ limit. Such a surface would be the analog of the master field of the large- N gauge theories. In fact there is such a fictitious mathematical surface—whose metric is given by the boundary metric $\bar{\mathbf{g}}_b$ (2.39),

$$\begin{aligned} Z_{NG} &= \int [d\mathbf{X}] \exp \left\{ -\frac{1}{l_s^2} \int d^2\sigma \sqrt{\det g_{ab}} \right\} = \mathcal{N}_{NG} \exp \left\{ -\frac{1}{l_s^2} \int_{-T/2}^{T/2} dt \int_0^L dx \sqrt{\det \bar{\mathbf{g}}_b} \right\}, \\ &= \mathcal{N}_{NG} \exp \left\{ -\frac{TL}{l_s^2} \sqrt{1 - 2\lambda(L)} \right\}. \end{aligned} \quad (2.43)$$

This observation will be of great use to us in evaluating the $1/D$ corrections in the next Sec. . The metric $\bar{\mathbf{g}}_b$ should not be thought of as an induced metric, but as an independent metric defining the “master surface”. It is worth noting that g_{b1} coincides with the induced metric on the quark and antiquark worldline (apart from the overall warp factor that is absorbed in our definition of l_s).

3 The $1/D$ Corrections from Heavy Modes

We will evaluate the $1/D$ corrections arising from the oscillation of the string along the fifth dimension using the observation made in the last section , namely that in the large- D limit the partition function of the Nambu-Goto string is given by a single “master surface” with metric given by (2.39). The partition function of the fundamental string in five dimensions, (2.23), can

be written as

$$Z_5 = Z_{NG} \langle \mathbf{C}[\mathbf{g}] \rangle_{NG} = \mathcal{N}_2 Z_{NG} \langle \exp \left\{ -\frac{1}{2} \text{Tr} \left[\log \left(\hat{\mathbf{A}}[\mathbf{g}] \right) \right] \right\} \rangle_{NG}. \quad (3.1)$$

In the large- D limit, this average in turn can be evaluated as

$$Z_5 = \mathcal{N}_5 \exp \left\{ -\frac{TL}{l_s^2} \sqrt{1-2\lambda(L)} \right\} \exp \left\{ -\frac{1}{2} \text{Tr} \left[\log \left(\hat{\mathbf{A}}[\bar{\mathbf{g}}_b] \right) \right] \right\}, \quad (3.2)$$

where, using (2.27) and (2.39),

$$\hat{\mathbf{A}}[\bar{\mathbf{g}}_b] = -\sqrt{\frac{g_{b2}}{g_{b1}}} \partial_1^2 - \sqrt{\frac{g_{b1}}{g_{b2}}} \partial_2^2 + \sqrt{\det \bar{\mathbf{g}}_b} M^2 \quad (3.3)$$

The trace can be calculated using zeta-function regularization in a standard manner (see, e.g., Ref. [33]),

$$-\frac{1}{2} \text{Tr} \left[\log \left(\hat{\mathbf{A}}[\bar{\mathbf{g}}_b] \right) \right] = -TV_M(L), \quad (3.4)$$

where

$$V_M(L) = \left(-\frac{1}{4} g_{b1}^{1/4} M + \frac{1}{4\pi} \sqrt{\frac{g_{b1}}{g_{b2}}} \int_0^\infty d\omega \log \left(1 - e^{-2L\sqrt{\omega^2 + \omega_0^2}} \right) \right), \quad (3.5)$$

and

$$\omega_0 = g_{b2}^{1/2} M. \quad (3.6)$$

Using (2.39) we get

$$V_M(L) = -\frac{1}{4} M + V_h(L), \quad (3.7)$$

where

$$V_h(L) = \frac{1}{4\pi} \frac{1}{\sqrt{1-2\lambda}} \int_0^\infty d\omega \log \left(1 - e^{-2L\sqrt{\omega^2 + (1-2\lambda)M^2}} \right). \quad (3.8)$$

For $LM \gg 1$, the integral leads to an exponential decaying term, while for $LM \ll 1$ one would obtain an additional $1/L$ correction to the heavy-quark potential. Such a term is of course precluded by both theoretical arguments [23] and numerical results. In any case, for $LM \ll 1$ our large- D analysis fails, for then the approximation of replacing $F[Y_c]$ by $F[Y^*]$ in the Nambu-Goto action is no longer valid, equivalently, the approximation $d = 0$ in Fig. 1 is no longer valid. Therefore, the only region where the above correction is of significance is when $LM \sim 1$, which we can evaluate numerically.

In Fig. 3 we show the contribution of the difference between $V_A(L)$ and $V_h(L)$ as compared to

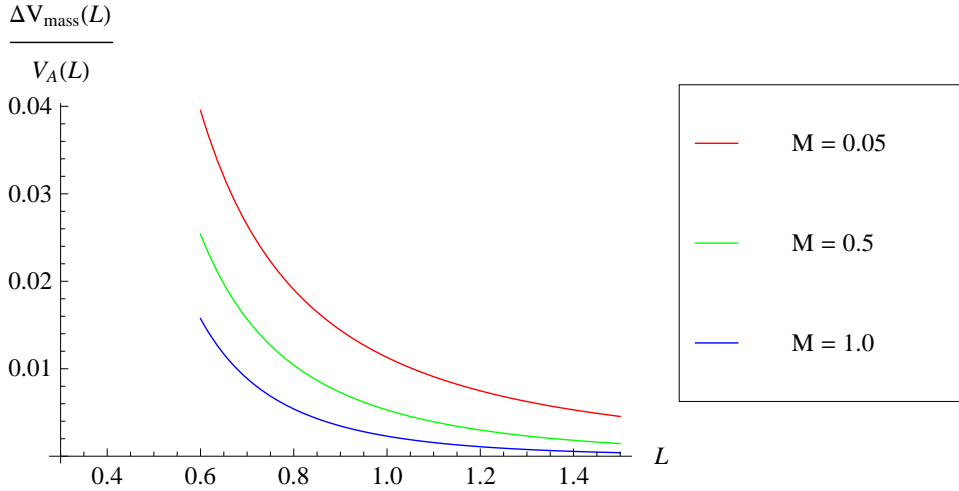


Fig. 3: Effect of a massive mode on the heavy-quark potential, where $\Delta V_{\text{mass}} \equiv |V_A - V_h|$

the Arvis potential $V_A(L)$ for representative values of M in the units of inverse fermi using $l_s = 0.4$ fermi. Therefore, the main qualitative contribution of the heavy modes is to modify the Arvis potential by the addition of a constant term, $-\frac{1}{4}M$.

4 Massless Modes Near the Boundary

Our large- D calculation has been done under the approximation in which the minimal surface of the fundamental string is taken to be (2.10), $X_c = (t, x, \vec{\mathbf{0}}, Y^*)$, that amounts to taking $d = 0$ in Fig.(1). As a result of this approximation, in evaluating (2.2) the fluctuations in the string world sheets are incorrectly weighted near the vicinity of the quark and the antiquark. If we relax the approximation $d = 0$ [see Fig. 1 and Fig. 2], then the minimal surface is given by (2.8), $X_c = (t, x, \vec{\mathbf{0}}, Y_c(x))$. We would like to consider the fluctuations around the exact minimal surface; in particular we will consider the fluctuations around it of the form

$$X = (t, x, \mathbf{X}(t, x), Y_c(x)), \quad (4.1)$$

that is we will be ignoring the fluctuations of the string along the fifth dimension that are massive, but will include the massless fluctuation around the exact minimal surface. In the static gauge, the action for the above fluctuations is given by

$$S[\mathbf{X}; F[Y_c(x)]] = T_0 \int dt dx F[Y_c(x)] \sqrt{\det g_{ab}} \cdot (1 + g_{22}^{-1} (\partial_x Y_c)^2)^{1/2}, \quad (4.2)$$

where the world-sheet metric g_{ab} is given by (2.14) and T_0 is the bare fundamental string tension. To delineate the effects of the boundary one can rewrite the above action as

$$S[\mathbf{X}; F[Y_c(x)]] = T_0 F(Y^*) \int_{-T/2}^{T/2} dt \int_0^L dx \sqrt{\det g_{ab}} \\ + T_0 \int_{-T/2}^{T/2} dt \int_{x \in \mathbf{B}} dx \sqrt{\det g_{ab}} \left\{ F[Y_c(x)] (1 + g_{22}^{-1} (\partial_x Y_c)^2)^{1/2} - F(Y^*) \right\}, \quad (4.3)$$

where we have used the fact that in the interval $(d, L-d)$, to a very good approximation $Y_c(x) = Y^*$. Also we have represented the boundary regions, $(0, d)$ and $(L-d, L)$, by \mathbf{B} .

Since the warp factor $F[Y_c(x)]$ differs from $F[Y^*]$ only near the vicinity of the quark and the antiquark, we rewrite it as

$$F[Y_c(x)] = F(Y^*) (1 + f_c(x)), \quad f_c(x) = \frac{F_c[Y_c(x)]}{F[Y^*]} - 1, \quad (4.4)$$

and the action for the massless fluctuations becomes

$$S[\mathbf{X}; F[Y_c(x)]] = \frac{1}{l_s^2} \int_{-T/2}^{T/2} dt \int_0^L dx \sqrt{\det g_{ab}} \\ + \frac{1}{l_s^2} \int_{-T/2}^{T/2} dt \int_{x \in \mathbf{B}} dx \sqrt{\det g} \left\{ (1 + g_{22}^{-1} (\partial_x Y_c)^2)^{1/2} - 1 \right\} \\ + \frac{1}{l_s^2} \int_{-T/2}^{T/2} dt \int_{x \in \mathbf{B}} dx \sqrt{\det g_{ab}} f_c(x) (1 + g_{22}^{-1} (\partial_x Y_c)^2)^{1/2}, \quad (4.5)$$

where we have used $T_0 F(Y^*) = 1/l_s^2$.

The first term in the above equation is nothing but the Nambu-Goto action for a QCD flux tube. The rest of the terms, which originate from the holographic description of QCD flux tube, are corrections to the Nambu-Goto action. These terms have support only in the vicinity of the quark and the antiquark, a region that we have denoted by \mathbf{B} . Let us first consider the last term. Near the boundary, $Y_c(x) \rightarrow 0$, the string experiences AdS₅ like geometry (that ensures that at very short distances the heavy-quark potential is coulomb like); as a result the warp factor $f_c[Y_c(x)]$ behaves like

$$\lim_{Y_c(x) \rightarrow 0} f_c[Y_c(x)] \approx \frac{1}{F(Y^*)} \frac{R^2}{Y_c(x)^2} \approx \frac{1}{F(Y^*)} \frac{R^2}{(mx)^2}, \quad (4.6)$$

where we have focused on the $x = 0$ end of the flux tube. R is the radius of curvature of the AdS₅ space near the boundary and $m = \partial_x Y_c(0)$. As a result the last term in (4.5) is divergent. To

understand the nature of this divergence, let us put the boundary at $Y = m\epsilon$ and evaluate this term for the classical configuration (2.8)

$$\lim_{\epsilon \rightarrow 0} T_0 \int dt \int_{\epsilon}^d dx \frac{R^2}{x^2} (1 + (\partial_x Y_c)^2)^{1/2} \approx T_0 \int dt \frac{R^2 \mathbf{C}(0)}{\epsilon}, \quad (4.7)$$

where $\mathbf{C}(x) = (1 + (\partial_x Y_c)^2)^{1/2}$ is regular at $x = 0$. Placing the boundary of the five-dimensional space at $Y = m\epsilon$ instead of at $Y = 0$ is equivalent to introducing an ultraviolet cutoff in the boundary theory and the divergence of the last term corresponds to the linearly divergent contribution to the self-energy of the quark [27],

$$\delta M = T_0 \frac{R^2 \mathbf{C}(0)}{\epsilon}. \quad (4.8)$$

From the flux tube point of view the divergence arises because near the quark and the antiquark the intrinsic thickness is going to zero and correspondingly the effective string tension diverges [19]. Thus, the main effect of the last term in (4.5) is just to renormalize the mass of the quark and the antiquark on which the string terminates, and we will therefore ignore it in the discussion below.

The second term in (4.5) is finite and represents a correction to the Nambu-Goto string in four dimensions. We can estimate its effect on the heavy-quark potential by treating it as a perturbation to the Nambu-Goto action,

$$\begin{aligned} Z_d &\equiv \int [d\mathbf{X}] \exp \left\{ -\frac{1}{l_s^2} \int_{-T/2}^{T/2} dt \int_0^L dx \sqrt{\det g_{ab}} - \frac{1}{l_s^2} \int_{-T/2}^{T/2} dt \int_{x \in \mathbf{B}} dx \mathbf{b}(t, x) \right\} \\ &\cong Z_{NG} \left\langle \exp \left(-\frac{1}{l_s^2} \int_{-T/2}^{T/2} dt \int_{x \in \mathbf{B}} dx \mathbf{b}(t, x) \right) \right\rangle_{NG}, \end{aligned} \quad (4.9)$$

where we have defined,

$$\mathbf{b}(t, x) \equiv \sqrt{\det g} \left\{ (1 + g_{22}^{-1} (\partial_x Y_c)^2)^{1/2} - 1 \right\}. \quad (4.10)$$

We can evaluate (4.9) in the large- D limit using (2.43) and by approximating $\partial_x Y_c(x) \approx \partial_x Y_c(0)$, to obtain the heavy-quark potential as

$$V_d[L] = (1 + h(L)) V_A[L], \quad (4.11)$$

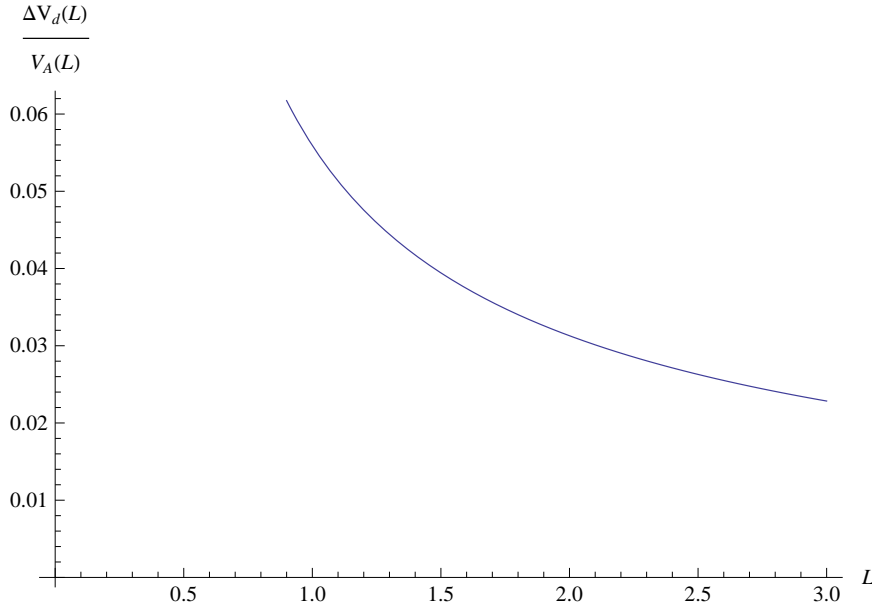


Fig. 4: The effect of massless modes near the boundaries on the heavy-quark potential, where $\Delta V_d \equiv |V_A - V_d|$

where $V_A[L]$ is the Arvis potential (2.42) and

$$h(L) = \frac{2d}{L} \left\{ \left(1 + \frac{(\partial_x Y_c(0))^2}{1 - 2\lambda(L)} \right)^{1/2} - 1 \right\}. \quad (4.12)$$

In [29] it was argued that for confining geometries d is a function of L and grows with L either as a power law or logarithmically, but with $d/L \rightarrow 0$ as $L \rightarrow \infty$. With this in mind, and to get a qualitative feeling for the corrections to the Arvis potential, we parametrize

$$2d = d_0 \left(\frac{L}{l_s} \right)^\beta, \quad (4.13)$$

where $0 < \beta < 1$ and take d_0 to be the inverse glueball mass, which is a measure of the correlation length of the gauge field. To plot these corrections, we will arbitrarily take the value of $\beta = 1/2$, and approximate the classical string configuration near the boundary by a straight line, $(\partial_x Y_c(0))^2 \sim 1$, and take $m_{\text{glueball}} \approx 2\text{GeV}$ and $l_s = 0.4$ fermi. With these parameters the relative deviation from the Arvis potential is plotted in Fig. (4) and is, at least for our choice of parameters, significantly more than the corresponding correction due to heavy modes, Fig. (3).

Finally, let us take note of the fluctuations like the one depicted in Fig. 5. These are the

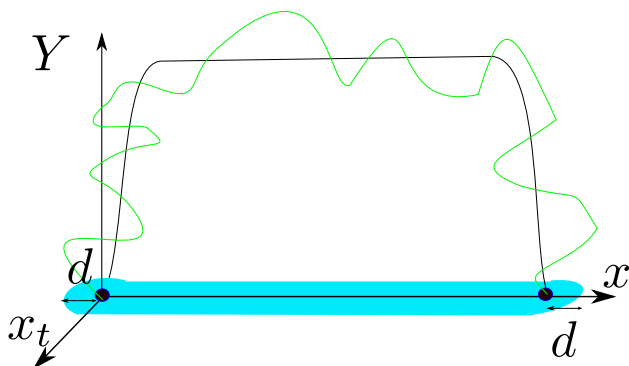


Fig. 5: Fluctuation of string in the extra dimension about the classical solution

massive fluctuations of the string in the extra dimension about the classical solution $Y_c(x)$. These fluctuations differ from the fluctuations that we have considered in Sec. [3] only near the boundary. It is worth noting that these fluctuations are most conveniently described using a normal gauge (see, e.g., [29]) rather than in the (t, x) gauge that we have used, for when x is close to the boundary even a small fluctuation $\phi(t, x)$ in the Y coordinate can become a multiple-valued function of x . In the holographic description of QCD flux tube these fluctuations are suggestive of the *longitudinal* fluctuations causing the flux tube to extend beyond the quark and the anti quark. In a future work we hope to probe their effects in confining geometries.

5 Summary and Conclusions

The key features of the heavy-quark potential as suggested by the gauge/gravity duality in the large- D limit can be summarized as

$$V[L] = (1 + h(L)) \frac{L}{l_s^2} \sqrt{\left(1 - \frac{2\pi l_s^2}{12 L^2}\right)} + \left(-\frac{1}{4}M + V_h[L]\right), \quad (5.1)$$

where $h(L)$ is given by (4.12) and arises from the behavior of the massless modes in the vicinity of the quark and the antiquark. The contribution from the heavy modes is contained in $V_h[L]$, which is given by (3.8) and its relative effect on the heavy-quark potential is shown in Fig. 3. The leading contribution to the heavy-quark potential in the large- D expansion is contained in the first term, while the last parentheses contains the $1/D$ corrections arising from the oscillations of the fundamental string in the fifth dimension. As we have noted in the Introduction, there are possible $1/D$ and higher corrections arising from the massless modes in noncritical dimensions,

which we have ignored in our analysis. To understand the manner in which they vanish for the critical dimension is an important and interesting problem to which we hope to come to in the future.

There are two key qualitative features of $V[L]$. Firstly, there is the correction to the Arvis potential by the factor $h(L)$. Even though in our investigation we cannot fix the precise form of $h(L)$, we can clearly see the reason for its existence. It arises from the simultaneous requirement that the fundamental string wants to lie at a fixed value $Y = Y^*$, to minimize its energy, and the requirement that the fundamental string has to terminate on the boundary, $Y = 0$, where the quark and the antiquark are placed. Thus, the origin of $h(L)$ is in the behavior of the flux tube in the vicinity of the quark and the antiquark. Though the present numerical results are consistent with $h(L) = 0$, but our analysis strongly suggests that there should be corrections to the Arvis potential, especially around $L \sim 1$ fermi and they should become visible in future calculations of the heavy-quark potential on the lattice. The other interesting feature of (5.1) is the existence of the constant term in the potential, $-M/4$. At first sight the existence of the constant term in the potential is of no consequence, but—as has been emphasized in Refs. [34, 35]—once the renormalization prescription for the expectation value of the Wilson loop is appropriately fixed, the constant arises only from the corrections to the Nambu-Goto string⁴. It is worth recalling that there is no constant term in the ground-state energy of a Nambu-Goto string as given by the Arvis formula.

The main motivation behind calculating the heavy-quark potential using gauge/gravity duality is to compare its predication with that of lattice QCD at all scales; that is, from the scale where the potential is Coulombic and the coupling constant is weak, to the strong-coupling scale where the QCD flux tubes are formed. The hope is that in this manner we can delineate the geometry of the five-dimensional curved space that incorporates both asymptotic freedom and confinement. Having found such a geometry one can hope to calculate more interesting quantities, like the hadron masses and their parton distributions.

Acknowledgments

I would like to thank Ofer Aharony for his very useful comments on a preliminary version of this paper. I have also greatly benefited from my discussions with the participants of the ECT* workshop on “Confining flux tubes and strings” in Trento, 2010, and would like to thank them all. This work was started while I was visiting the Institute of Theoretical Physics, at the University

⁴ I would like to thank R. D. Pisarski for pointing this out to me.

of Regensburg. I am grateful to the members of the Institute for their hospitality, particularly to Gunnar Bali for his invitation to visit and DAAD for their financial support. Part of this work was done subsequently while I was visiting the Raman Research Institute, Bangalore and I would like to thank the members of the theoretical physics group at RRI for their hospitality and Madhavan Varadarajan for his kind invitation. I would also like to thank Janaki Abraham for her help in proof reading this paper.

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