

# $sl(2, \mathbb{C})$ as a complex Lie algebra and the associated non-Hermitian Hamiltonians with real eigenvalues

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## Abstract

The powerful group theoretical formalism of potential algebras is extended to non-Hermitian Hamiltonians with real eigenvalues by complexifying  $so(2,1)$ , thereby getting the complex algebra  $sl(2,\mathbb{C})$  or  $A_1$ . This leads to new types of both PT-symmetric and non-PT-symmetric Hamiltonians.

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# 1 Introduction

In recent times, the subject of quantum mechanics has been the focus of very active research. While on the one hand, new methods have been developed which provide deeper insights into the underpinnings of the theory [1], on the other, group theoretical methods have increasingly gained acceptance as an important means towards understanding the extremely rich structure of the underlying symmetries [2, 3].

Of late, a sharp increase of interest has been noticed in searching for non-Hermitian Hamiltonians [4, 5, 6, 7, 8, 9, 10, 11, 12]. Although the history of complex potentials is old [13] especially in relation to scattering problems, Bender and Boettcher [4], a few years ago, revived interest in complex potentials by restricting a non-Hermitian Hamiltonian to be PT-symmetric. In this way, they showed that it is possible to derive new infinite classes of PT-invariant systems whose spectrum is real. Subsequently the idea of PT symmetry has been pursued by several authors [5, 6, 7, 8, 9, 10, 11, 12], who have obtained different kinds of potentials with real eigenvalues. These include the quasi-solvable type [5, 11] and supersymmetry-inspired ones too [7, 9, 10].

In this letter we propose an  $\mathfrak{sl}(2, \mathbb{C})$  potential algebra as a complex Lie algebra for the Schrödinger equation to study non-Hermitian systems from a group theoretical point of view. Adopting a most general differential realization of the  $\mathfrak{sl}(2, \mathbb{C})$  algebra, we demonstrate how new complex potentials can be generated which are not necessarily PT symmetric but possess common real eigenvalues. However, a subclass of our potentials does turn out to respect PT symmetry. As with the case of  $\mathfrak{so}(2,1)$  [14, 15], here also we find possible to classify our results into various types of solutions. Indeed the main spirit of the realization of the potential algebra  $\mathfrak{so}(2,1)$  persists in our scheme in that a class of potentials is found to exist which share the same real energy eigenvalues and have their eigenfunctions derived from an application of the  $\mathfrak{sl}(2, \mathbb{C})$  generators on normalized states.

## 2 $\mathfrak{sl}(2, \mathbb{C})$ algebra and its realization

Let us begin by noting that the commutation relations of  $\mathfrak{sl}(2, \mathbb{C})$  (or  $A_1$  in Cartan's classification of simple complex Lie algebras), namely

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2J_0, \quad (1)$$

can be given a differential realization

$$J_0 = -i \frac{\partial}{\partial \phi}, \quad J_{\pm} = e^{\pm i \phi} \left[ \pm \frac{\partial}{\partial x} + \left( i \frac{\partial}{\partial \phi} \mp \frac{1}{2} \right) F(x) + G(x) \right], \quad (2)$$

where the auxiliary variable  $\phi$  ranges in  $0 \leq \phi < 2\pi$ ,  $x \in \mathbb{R}$ , and the two functions  $F(x)$ ,  $G(x) \in \mathbb{C}$  satisfy coupled differential equations

$$\frac{dF}{dx} = 1 - F^2, \quad \frac{dG}{dx} = -FG. \quad (3)$$

Note that since  $J_- \neq J_+^\dagger$ , we generate an  $\mathfrak{sl}(2, \mathbb{C})$  algebra rather than  $\mathfrak{so}(2, 1)$ , which is consistent with  $J_- = J_+^\dagger$ .

The Casimir operator corresponding to the above generators is

$$J^2 = J_0^2 \mp J_0 - J_{\pm} J_{\mp}. \quad (4)$$

In terms of  $F$  and  $G$ , it reads

$$J^2 = \frac{\partial^2}{\partial x^2} - \left( \frac{\partial^2}{\partial \phi^2} + \frac{1}{4} \right) F' + 2i \frac{\partial}{\partial \phi} G' - G^2 - \frac{1}{4}, \quad (5)$$

where a prime denotes derivative with respect to spatial variable  $x$ .

In the  $\mathfrak{so}(2, 1)$  case [2, 14, 15], one considers for bound states unitary irreducible representations of type  $D_k^+$ , for which

$$\begin{aligned} J_0 |km\rangle &= m |km\rangle, & m &= k, k+1, k+2, \dots, \\ J^2 |km\rangle &= k(k-1) |km\rangle, \end{aligned} \quad (6)$$

and  $k$  is positive (but not necessarily restricted to integers or half-integers as one only deals with the algebra and not with the whole group [16]). When going to the complex Lie

algebra  $\mathfrak{sl}(2, \mathbb{C})$ , Eq. (6) still defines irreducible representations. These are the only ones we shall consider here.

Hence, in the remainder of this paper, we are going to look for solutions of (6) given by

$$|km\rangle = \Psi_{km}(x, \phi) = \psi_{km}(x) \frac{e^{im\phi}}{\sqrt{2\pi}}, \quad (7)$$

where  $k > 0$  and  $m = k + n$ ,  $n = 0, 1, 2, \dots$ . From Eqs. (5)–(7), it follows that the coefficient functions  $\psi_{km}(x)$  obey the Schrödinger equation

$$-\psi_{km}'' + V_m \psi_{km} = -\left(k - \frac{1}{2}\right)^2 \psi_{km}. \quad (8)$$

In (8), the one-parameter family of potentials, denoted  $V_m$ , is represented by

$$V_m = \left(\frac{1}{4} - m^2\right) F' + 2mG' + G^2. \quad (9)$$

Provided they are normalizable, the functions  $\psi_{km}(x)$  are the  $(n + 1)$ th bound state wavefunctions for the potentials  $V_m$ , corresponding to the energy eigenvalues

$$E_n^{(m)} = -\left(m - n - \frac{1}{2}\right)^2. \quad (10)$$

The potentials  $V_m$ ,  $m = k, k + 1, k + 2, \dots$ , supporting the same eigenvalues  $-\left(k - \frac{1}{2}\right)^2$  produce a potential algebra:  $J_+$  and  $J_-$  connect among themselves all the wavefunctions corresponding to the same energy, but to different potentials  $V_m$ .

Before we embark upon our detailed study of  $\mathfrak{sl}(2, \mathbb{C})$ , let us make a few remarks on the use of (1) as an  $\mathfrak{so}(2, 1)$  potential algebra in the real domain. Wu and Alhassid [14] showed that Morse, Pöschl-Teller, and Rosen-Morse potentials emerge as particular solutions of (3). Later Englefield and Quesne [15] quite exhaustively identified three classes of  $\mathfrak{so}(2, 1)$  solutions for the coupled set of equations (3) according as  $F^2 < 1$ ,  $F^2 > 1$ , or  $F^2 = 1$ :

$$\begin{aligned} F^2 < 1 : & \quad F(x) = \tanh(x - c), \quad G(x) = b \operatorname{sech}(x - c), \\ F^2 > 1 : & \quad F(x) = \coth(x - c), \quad G(x) = b \operatorname{cosech}(x - c), \\ F^2 = 1 : & \quad F(x) = \pm 1, \quad G(x) = b e^{\mp x}, \end{aligned} \quad (11)$$

where  $b$  and  $c$  are real constants. When substituted in (9), the above possibilities for  $F$  and  $G$  lead to three distinct types of potentials. These are the (nonsingular) Scarf II or first Gendenshtein potential [17]  $V_1$ , given by

$$V_1(x) = \left(b^2 - m^2 + \frac{1}{4}\right) \operatorname{sech}^2 x - 2mb \operatorname{sech} x \tanh x, \quad (12)$$

the (singular) generalized Pöschl-Teller or second Gendenshtein potential [17]  $V_2$ , given by

$$V_2(x) = \left(b^2 + m^2 - \frac{1}{4}\right) \operatorname{cosech}^2 x - 2mb \operatorname{cosech} x \coth x, \quad (13)$$

and the Morse potential  $V_3$ , given by

$$V_3(x) = b^2 e^{-2x} - 2mbe^{-x}. \quad (14)$$

Eqs. (12) and (13) correspond to  $c = 0$ , while Eq. (14) corresponds to the upper signs in (11). These simplifications do not alter the physical significance of the results. Note that our convention of denoting the potentials differs slightly from Ref. [15], where the Morse potential is referred to as  $V_2$  and the generalized Pöschl-Teller or second Gendenshtein potential as  $V_3$ .

### 3 General results for complex potentials associated with $\mathfrak{sl}(2, \mathbb{C})$

We now proceed to write down solutions of Eq. (3). This can be done either by splitting the functions  $F$  and  $G$  into their real and imaginary components and looking for solutions of the four real equations satisfied by them or, equivalently, solving for the first equation of (3) directly to find  $F$  and then substituting it in the second equation of (3) to determine  $G$ . For simplicity we choose the second approach. Our solutions are summarized as follows:

$$\begin{aligned} \text{I:} \quad & F(x) = \tanh(x - c - i\gamma), \quad G(x) = (b_R + ib_I) \operatorname{sech}(x - c - i\gamma), \\ \text{II:} \quad & F(x) = \coth(x - c - i\gamma), \quad G(x) = (b_R + ib_I) \operatorname{cosech}(x - c - i\gamma), \\ \text{III:} \quad & F(x) = \pm 1, \quad G(x) = (b_R + ib_I)e^{\mp x}, \end{aligned} \quad (15)$$

where  $b = b_R + ib_I$ ,  $b_R, b_I \in \mathbb{R}$ , and  $-\frac{\pi}{4} \leq \gamma < \frac{\pi}{4}$ .

The resulting potentials are given by

$$\begin{aligned} \text{I: } V_m &= \left[ (b_R + ib_I)^2 - m^2 + \frac{1}{4} \right] \text{sech}^2(x - c - i\gamma) \\ &\quad - 2m(b_R + ib_I) \text{sech}(x - c - i\gamma) \tanh(x - c - i\gamma), \end{aligned} \quad (16)$$

$$\begin{aligned} \text{II: } V_m &= \left[ (b_R + ib_I)^2 + m^2 - \frac{1}{4} \right] \text{cosech}^2(x - c - i\gamma) \\ &\quad - 2m(b_R + ib_I) \text{cosech}(x - c - i\gamma) \coth(x - c - i\gamma), \end{aligned} \quad (17)$$

$$\text{III: } V_m = (b_R + ib_I)^2 e^{\mp 2x} \mp 2m(b_R + ib_I) e^{\mp x}. \quad (18)$$

The potentials (16)–(18) can be looked upon as the complexified versions of the corresponding real ones (12)–(14).

By separating out the real and imaginary parts, these complex potentials can be expressed as

$$\begin{aligned} \text{I: } V_m &= \frac{2}{[\cosh 2(x - c) + \cos 2\gamma]^2} \left\{ \left( b_R^2 - b_I^2 - m^2 + \frac{1}{4} \right) [1 + \cosh 2(x - c) \cos 2\gamma] \right. \\ &\quad - 2b_R b_I \sinh 2(x - c) \sin 2\gamma \\ &\quad - 2m [b_R \sinh(x - c) \cos \gamma (\cosh 2(x - c) - \cos 2\gamma + 2) \\ &\quad \left. - b_I \cosh(x - c) \sin \gamma (\cosh 2(x - c) - \cos 2\gamma - 2)] \right\} \\ &\quad + \frac{2i}{[\cosh 2(x - c) + \cos 2\gamma]^2} \left\{ \left( b_R^2 - b_I^2 - m^2 + \frac{1}{4} \right) \sinh 2(x - c) \sin 2\gamma \right. \\ &\quad + 2b_R b_I [1 + \cosh 2(x - c) \cos 2\gamma] \\ &\quad - 2m [b_R \cosh(x - c) \sin \gamma (\cosh 2(x - c) - \cos 2\gamma - 2) \\ &\quad \left. + b_I \sinh(x - c) \cos \gamma (\cosh 2(x - c) - \cos 2\gamma + 2)] \right\} \\ \text{II: } V_m &= \frac{2}{[\cosh 2(x - c) - \cos 2\gamma]^2} \left\{ \left( b_R^2 - b_I^2 + m^2 - \frac{1}{4} \right) [-1 + \cosh 2(x - c) \cos 2\gamma] \right. \\ &\quad - 2b_R b_I \sinh 2(x - c) \sin 2\gamma \\ &\quad - 2m [b_R \cosh(x - c) \cos \gamma (\cosh 2(x - c) + \cos 2\gamma - 2) \\ &\quad \left. - b_I \sinh(x - c) \sin \gamma (\cosh 2(x - c) + \cos 2\gamma + 2)] \right\} \\ &\quad + \frac{2i}{[\cosh 2(x - c) - \cos 2\gamma]^2} \left\{ \left( b_R^2 - b_I^2 + m^2 - \frac{1}{4} \right) \sinh 2(x - c) \sin 2\gamma \right. \\ &\quad + 2b_R b_I [-1 + \cosh 2(x - c) \cos 2\gamma] \\ &\quad \left. - 2m [b_R \sinh(x - c) \sin \gamma (\cosh 2(x - c) + \cos 2\gamma + 2) \right. \end{aligned}$$

$$\begin{aligned}
& + b_I \cosh(x - c) \cos \gamma \left( \cosh 2(x - c) + \cos 2\gamma - 2 \right) \Big] \Big\} \\
\text{III: } V_m & = \left( b_R^2 - b_I^2 \right) e^{\mp 2x} \mp 2mb_R e^{\mp x} + ib_I \left( 2b_R e^{\mp 2x} \mp 2m e^{\mp x} \right). \tag{19}
\end{aligned}$$

The above categories of potentials are displayed in their most general forms and give a quite complete realization of  $\text{sl}(2, \mathbb{C})$  algebra corresponding to the representation (2). It should be remarked that although  $V_m$  of (II) can be apparently obtained from  $V_m$  of (I) by the transformation  $\gamma \rightarrow \gamma \pm \frac{\pi}{2}$ ,  $b_R \rightarrow \pm b_I$ ,  $b_I \rightarrow \mp b_R$ , the range of the parameter  $\gamma$  is not left invariant. So the potentials (I) and (II) are indeed different and quite independent of one another. Note further that the special case  $\gamma = b_I = 0$  reduces the three potentials of (19) to their real forms (12)–(14), as it should be. On the other hand, the case  $\gamma = b_R = 0$  (that is  $F \in \mathbb{R}$ ,  $G \in i\mathbb{R}$ ) gives

$$\text{I: } V_m = \left( -b_I^2 - m^2 + \frac{1}{4} \right) \text{sech}^2(x - c) - 2imb_I \text{sech}(x - c) \tanh(x - c), \tag{20}$$

$$\text{II: } V_m = \left( -b_I^2 + m^2 - \frac{1}{4} \right) \text{cosech}^2(x - c) - 2imb_I \text{cosech}(x - c) \coth(x - c), \tag{21}$$

$$\text{III: } V_m = -b_I^2 e^{\mp 2x} \mp 2imb_I e^{\mp x}. \tag{22}$$

Obviously PT symmetry holds for I, but not for the other two.

## 4 Analysis of some special cases

### 4.1 Complexification of the Scarf II potential

In the literature [18, 19], the real Scarf II potential is usually given in the form

$$V^{(S)}(x) = \left[ B^2 - A(A + 1) \right] \text{sech}^2 x + B(2A + 1) \text{sech} x \tanh x. \tag{23}$$

This potential is well known to be exactly solvable. For  $A > 0$ , the associated eigenfunctions and eigenvalues are

$$\psi_n(x) = N_n (\text{sech } x)^A \exp[-B \arctan(\sinh x)] P_n^{(-iB-A-\frac{1}{2}, iB-A-\frac{1}{2})}(i \sinh x), \tag{24}$$

$$E_n = -(A - n)^2, \quad n = 0, 1, \dots, n_{\max} < A, \tag{25}$$

where  $N_n$  is a normalization constant, and  $P_n^{(\alpha, \beta)}$  is a Jacobi polynomial.

Replacing  $B$  by  $iB$  leads to the PT-symmetric form of the potential (23):

$$V^{(CS)}(x) = - \left[ B^2 + A(A+1) \right] \operatorname{sech}^2 x + iB(2A+1) \operatorname{sech} x \tanh x, \quad (26)$$

where the real and imaginary parts have no singularity on the real axis. Further, they are invariant under the exchange  $A + \frac{1}{2} \leftrightarrow B$ . Without loss of generality, we may assume  $A + \frac{1}{2} > 0$  along with  $B > 0$ , since replacing  $B$  by  $-B$  only changes  $V$  into  $V^*$  with both  $V$  and  $V^*$  bearing the same real  $E_n$  corresponding to the wavefunctions  $\psi_n(x)$  and  $\psi_n^*(x)$ , respectively.

Comparing (26) with (20) obtained from the  $\mathfrak{sl}(2, \mathbb{C})$  algebra, we find  $c = 0$  and

$$b_I^2 + m^2 - \frac{1}{4} = B^2 + A(A+1), \quad (27)$$

$$-2mb_I = B(2A+1). \quad (28)$$

While (28) gives  $m = -B(2A+1)/(2b_I)$ , using it in (27) gives two solutions for  $b_I^2$  as

$$b_I^2 = B^2, \quad b_I^2 = \left( A + \frac{1}{2} \right)^2. \quad (29)$$

If  $b_I^2 = B^2$ , then  $b_I = \epsilon B$  and  $m = -\epsilon \left( A + \frac{1}{2} \right)$ , where  $\epsilon = \pm 1$ . Since by assumption  $m > 0$  (see discussion in Section 2), we have to choose  $\epsilon = -1$  (to be consistent with  $A + \frac{1}{2} > 0$ ) implying  $b_I = -B$ ,  $m = A + \frac{1}{2}$ . On the other hand, if  $b_I^2 = \left( A + \frac{1}{2} \right)^2$ , then  $b_I = \epsilon \left( A + \frac{1}{2} \right)$  and  $m = -\epsilon B$ , where we have again to choose  $\epsilon = -1$  (to be consistent with  $B > 0$ ), so that  $b_I = -\left( A + \frac{1}{2} \right)$ ,  $m = B$ . We thus get two (noncommuting)  $\mathfrak{sl}(2, \mathbb{C})$  algebras, which can be mapped onto each other by effecting a transformation  $A + \frac{1}{2} \leftrightarrow B$ . Let us denote their generators by  $J_0^{(i)}$ ,  $J_+^{(i)}$ ,  $J_-^{(i)}$ ,  $i = 1, 2$ , where  $i = 1$  (resp. 2) corresponds to  $(m, b_I) = \left( A + \frac{1}{2}, -B \right)$  [resp.  $\left( B, -A - \frac{1}{2} \right)$ ].

The eigenfunctions corresponding to (26) can be arrived at by using in turn both algebras. For the algebra labelled by  $i$ , the wavefunction corresponding to  $n = 0$  follows from  $J_-^{(i)} \Psi_{kk}^{(i)} = 0$ , while those for  $n \neq 0$  are obtained from the latter by employing the property  $J_+^{(i)} \Psi_{km}^{(i)} = \alpha_{km} \Psi_{k, m+1}^{(i)}$ , where  $\alpha_{km}$  are some constants. For conciseness, we do not give the details of our calculations, which will be presented elsewhere. We just state our results, which are

$$\psi_n^{(m)}(x) = N_n^{(m)} (\operatorname{sech} x)^{m-\frac{1}{2}} \exp[ib_I \arctan(\sinh x)] P_n^{(-b_I-m, b_I-m)}(i \sinh x), \quad (30)$$



where  $(m, b_I) = (A + \frac{1}{2}, -B)$  or  $(B, -A - \frac{1}{2})$ . The corresponding eigenvalues are given by (10).

The functions (30) being normalizable on the real line, we conclude that there is in general a doubling of energy levels when going from the real to the complex Scarf II potential. The latter indeed has two series of levels associated with the two algebras: the first ones,

$$E_n^{(A+\frac{1}{2})} = -(A - n)^2, \quad n = 0, 1, \dots, n_{\max}^{(A+\frac{1}{2})} < A, \quad (31)$$

coincide with the levels (25) of the real potential, with the corresponding wavefunctions obtained from (24) by the substitution  $B \rightarrow iB$ , while the second ones,

$$E_n^{(B)} = -\left(B - n - \frac{1}{2}\right)^2, \quad n = 0, 1, \dots, n_{\max}^{(B)} < B - \frac{1}{2}, \quad (32)$$

have no counterparts in the real case. Note that only the first set of wavefunctions was mentioned in the brief account of the complexified Scarf II potential made in Ref. [10].

Whenever  $|A + \frac{1}{2} - B|$  approaches an integer  $m$ , some levels corresponding to (31) and (32) become quasi-degenerate. It can be checked that for  $|A + \frac{1}{2} - B| = m$ , the corresponding wavefunctions (30) become proportional, so that we again observe the phenomenon of unavoided level crossings without degeneracy, previously encountered for the PT-symmetric harmonic oscillator [8].

It is of interest to consider the special case  $B = 1$ . We obtain from (26)

$$V = -\left[\left(A + \frac{1}{2}\right)^2 + \frac{3}{4}\right] \operatorname{sech}^2 x + 2i\left(A + \frac{1}{2}\right) \operatorname{sech} x \tanh x. \quad (33)$$

It can be immediately seen that by setting  $A + \frac{1}{2} = -\lambda$ , ( $\lambda < 0$ ), (33) reduces to the potential  $V^{(1)} - \frac{1}{4}$  of Ref. [9] for  $\mu = 1$ . The energies (31) obtained from the first  $\mathfrak{sl}(2, \mathbb{C})$  algebra become  $E_n^{(-\lambda)} = -\left(\lambda + n + \frac{1}{2}\right)^2$  and coincide with  $E_n^{(2)} - \frac{1}{4}$  of Eq. (6) in [9]. The second algebra leads to a single energy level corresponding to (32) for  $B = 1$  and  $n = 0$ , that is  $E_0^{(1)} = -\frac{1}{4}$ , which is consistent with the zero-energy state of [9].

## 4.2 Complexification of the generalized Pöschl-Teller potential

The wavefunctions and energy levels of the complexified generalized Pöschl-Teller potential (17) with  $b_I = 0$  can be found out in a manner similar to the one of (16) with  $\gamma = b_R = 0$ .

Again we reserve the details of our calculations for a future communication. Let us however mention two interesting aspects of the potential.

Writing it in a form similar to its real counterpart [18, 19], we obtain

$$V^{(CGPT)}(x) = [B^2 + A(A + 1)] \operatorname{cosech}^2(x - i\gamma) - B(2A + 1) \operatorname{cosech}(x - i\gamma) \coth(x - i\gamma). \quad (34)$$

The real potential, corresponding to  $\gamma = 0$ , being singular must be confined to the semi-axis  $(0, +\infty)$ . The complexified potential (34) gets regularized by the complex shift  $x \rightarrow x - i\gamma$  and may be considered on the whole real line. Using now a complex analogue of the point canonical coordinate transformation known to relate the generalized Pöschl-Teller and Pöschl-Teller II potentials [18, 19], the potential (34) can be changed into the complexified Pöschl-Teller II potential

$$V^{(CPT)}(t) = \frac{(B - A)(B - A - 1)}{\sinh^2(t - i\epsilon)} - \frac{(A + B)(A + B + 1)}{\cosh^2(t - i\epsilon)}, \quad (35)$$

where  $t = x/2$  and  $\epsilon = \gamma/2$ . In Ref. [12], the real spectrum and corresponding wavefunctions of the potential (35) were found by solving the Schrödinger equation. It is remarkable that they can also be derived algebraically by applying the above-mentioned complex point canonical coordinate transformation to the  $\mathfrak{sl}(2, \mathbb{C})$  results for the potential (34).

On the other hand, some time ago Andrianov *et al.* [7] wrote down a transparent complex potential in the form

$$V^{(1)}(y) = \frac{2\epsilon_R}{\cosh^2[\sqrt{-\epsilon_R}(y + b) + i\rho]}, \quad (36)$$

where  $\epsilon_R < 0$ ,  $b \in \mathbb{R}$ ,  $\rho = \pm \frac{1}{2} \arctan(2\sqrt{-\epsilon_R}/a) \neq \frac{\pi}{2}(2n + 1)$ , and  $a \in \mathbb{R}_0$ .  $V^{(1)}(y)$  is invariant under PT with its imaginary part being P odd. It vanishes at infinity and has a bound state with energy  $\epsilon_R$ . Furthermore it can be obtained from the real transparent potential  $\operatorname{sech}^2 x$  by a complex shift of the coordinate  $x$ .

The potential (36) compares well with (34) with  $B = A + 1$  (and  $x \rightarrow x - c$ ):

$$V^{(CGPT)}(x) = -\frac{1}{2}(A + 1)(2A + 1) \operatorname{sech}^2 \frac{1}{2}(x - c - i\gamma). \quad (37)$$

Indeed under a change of variable  $y = x/(2\sqrt{-\epsilon_R})$ , the Hamiltonian corresponding to (36) reads

$$H = -4\epsilon_R \left[ -\frac{d^2}{dx^2} - \frac{1}{2} \operatorname{sech}^2 \frac{1}{2}(x - c - i\gamma) \right], \quad (38)$$

where we have put  $c = -2\sqrt{-\epsilon_R}b$  and  $\gamma = -2\rho$ . Apart from a multiplicative constant, the above Hamiltonian is a particular case of

$$H^{(CGPT)} = -\frac{d^2}{dx^2} - \frac{1}{2}(A+1)(2A+1) \operatorname{sech}^2 \frac{1}{2}(x-c-i\gamma) \quad (39)$$

[obtained from (37)] for  $A = 0$ . From the algebraic results for the general potential (34), it follows that the Hamiltonian (39) with  $A = 0$  has a single bound state with energy  $-\frac{1}{4}$ , thus giving a single bound state with energy  $\epsilon_R$  for the Hamiltonian (38), as it should be.

### 4.3 Complexification of the Morse potential

As a last example, we consider the complexified Morse potential

$$V^{(CM)}(x) = (B_R + iB_I)^2 e^{-2x} - (B_R + iB_I)(2A+1)e^{-x}, \quad (40)$$

corresponding to the potential (18) and giving back the real one [18, 19] for  $B_I = 0$ . It is straightforward to show that for  $A$  and  $B_R$  positive, the potential (40) has the same real eigenvalues as its real counterpart (but of course different wavefunctions). This provides a very simple example of non-PT-symmetric complex potential with real eigenvalues. Other examples of such potentials are already known (see e.g. [6]).

## 5 Conclusion

In this letter, we showed that the powerful group theoretical formalism of potential algebras can be extended to non-Hermitian Hamiltonians by a simple complexification of the real algebras considered for Hermitian Hamiltonians. This provides us with a simple method of constructing both PT-symmetric and non-PT-symmetric Hamiltonians with real eigenvalues. We considered here the case of the  $\mathfrak{sl}(2, \mathbb{C})$  (or  $A_1$ ) potential algebra, corresponding to the complexification of  $\mathfrak{so}(2,1)$ .

Our construction method of new non-Hermitian Hamiltonians with real eigenvalues may be extended in two ways. First, instead of complexifying  $\mathfrak{so}(2,1)$ , we may try to complexify the larger algebra  $\mathfrak{so}(2,2) \simeq \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ , which is a potential algebra for the whole class of real Natanzon potentials [2, 14]. Second, to all the potentials constructed by

using either  $\mathfrak{sl}(2, \mathbb{C})$  or some generalization thereof, we may apply complex analogues of the transformations used to inter-relate real exactly solvable potentials [18, 19], as we did in Section 4.2. It should be clear that both of these extensions will give rise to a whole menagerie of new Hamiltonians with real eigenvalues.

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