$\mathcal{PT}$-symmetric supersymmetry in a solvable short-range model

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Abstract

The simplest purely imaginary and piecewise constant $\mathcal{PT}$-symmetric potential located inside a larger box is studied. Unless its strength exceeds a certain critical value, all the spectrum of its bound states remains real and discrete. We interpret such a model as an initial element of the generalized non-Hermitian Witten's hierarchy of solvable Hamiltonians and construct its first supersymmetric (SUSY) partner in closed form.

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1 Introduction and summary

One of the most intriguing experimental puzzles encountered in contemporary physics is the evident absence of SUSY partners of elementary particles in nature. In the context of field theory this means that SUSY, if it exists, must be spontaneously broken. Witten [1] proposed a schematic model which, incidentally, failed to clarify this breakdown but, nonetheless, survived and found a number of applications within the so-called SUSY quantum mechanics (SUSYQM) [2].

In the latter formalism one introduces the so-called superpotential $W(x)$ and defines the two operators

$$\mathcal{A} = \frac{d}{dx} + W(x) \qquad \bar{\mathcal{A}} = -\frac{d}{dx} + W(x) \tag{1.1}$$

with the property that the two related *different* (so-called 'SUSY partner') potentials $V^{(\pm)} - E_0 = W^2 \mp W'$ may prove *both* exactly solvable at the same time. An easy explanation of this phenomenon lies in the fact that the related Hamiltonians

$$H^{(\pm)} = -\frac{d^2}{dx^2} + V^{(\pm)}(x) - E_0 \tag{1.2}$$

become inter-related, at a convenient auxiliary energy $E = E_0$, by the factorization rules $H^{(+)} = \bar{\mathcal{A}}\mathcal{A}$ and $H^{(-)} = \mathcal{A}\bar{\mathcal{A}}$. The spectra of $H^{(+)}$ and $H^{(-)}$ are then alike except possibly for the ground state. In the unbroken SUSY case, the ground state at vanishing energy is nondegenerate and, in the present notational set-up, it belongs to $H^{(+)}$. This means that

$$\mathcal{A}\,\psi_0^{(+)}(x) = 0 \tag{1.3}$$

where $\psi_n^{(+)}(x)$ (resp. $\psi_n^{(-)}(x)$), $n = 0, 1, 2, \ldots$, denote the wavefunctions of $H^{(+)}$ (resp. $H^{(-)}$). The (double) degeneracy of $\left(\psi_{n+1}^{(+)}(x), \psi_n^{(-)}(x)\right)$ for $n = 0, 1, 2, \ldots$ is implied by the intertwining relationships

$$\mathcal{A}\,H^{(+)} = H^{(-)}\mathcal{A} \qquad H^{(+)}\bar{\mathcal{A}} = \bar{\mathcal{A}}\,H^{(-)}. \tag{1.4}$$

In the conventional setting, the Hamiltonians (1.2) are assumed self-adjoint.

New horizons have been opened by the pioneering letter by Bender and Boettcher [3] who noticed, in a slightly different context, that the latter condition $H = H^\dagger$ might be relaxed as redundant and replaced by its suitable weakened forms. For our present purposes, we shall employ their proposal and in equation (1.2) allow complex potentials that are merely constrained by the requirement that their real and imaginary parts are spatially symmetric and antisymmetric, respectively [4].

It is not too difficult to show that the above SUSYQM factorization scheme remains unchanged under such a non-Hermitian generalization [5, 6, 7, 8, 9]. Of course, the relaxation of the usual condition $H = H^\dagger$ is by far not a trivial step. Formally, we may put $H^\dagger = \mathcal{T} H \mathcal{T}$ with an antilinear 'time-reversal' operator $\mathcal{T}$ [10]. In such a setting, Bender and Boettcher (loc. cit., cf. also some older studies [11] or newer developments [12]) merely replaced $\mathcal{T}$ by its product with parity $\mathcal{P}$ and conjectured that the above-mentioned and physically well-motivated weakening of Hermiticity could be most appropriately characterized as an antilinear 'symmetry' or '$\mathcal{PT}$-symmetry', $\mathcal{PT}H = H\mathcal{PT}$, of all the Hamiltonians in question. Equivalently [13] one may speak about the $\mathcal{P}$-pseudo-Hermiticity defined by the relation

$$H^\dagger = \mathcal{P}\, H\, \mathcal{P}^{-1}\,. \tag{1.5}$$

In this paper we intend to concentrate on implementing the resulting $\mathcal{PT}$-symmetric SUSYQM factorization scheme in the case of the 'simplest' model which remains 'realistic' and 'solvable' at the same time. This means that our 'initial' Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + V^r(x) + \mathrm{i}V^i(x)\right]\psi(x) = E\psi(x) \tag{1.6}$$

(where we dropped the superscript '(+)' as temporarily redundant) will contain just the most trivial infinitely deep square-well form

$$V^r(x) = \begin{cases} +\infty & x < -L \\ 0 & -L < x < L \\ +\infty & x > L \end{cases} \tag{1.7}$$

for the real part of the potential and the most elementary short-range one

$$V^i(x) = \begin{cases} 0 & x < -l \\ -g & -l < x < 0 \\ +g & 0 < x < l, \\ 0 & x > l \end{cases} \qquad l < L, \qquad g > 0 \tag{1.8}$$

for its imaginary part. As a consequence of (1.7), the wavefunctions will be defined on a finite interval $(-L, L)$ with a variable length $2L$, on which they satisfy the standard Dirichlet boundary conditions [14, 15]

$$\psi(\pm L) = 0. \tag{1.9}$$

Given the background of the result obtained in [16], we derive in section 2, an elegant trigonometric form of the standard matching conditions for wavefunctions at the discontinuities of the potential (subsection 2.1) and discuss the practical semi-numerical determination of the energies with arbitrary precision (subsection 2.2).

In section 3 we address the key concern of our present paper, viz., the investigation of the problem in the context of SUSYQM. Here the non-Hermiticity and discontinuities create some specific features, which are dealt with in detail. After deriving the superpotential and the partner potential in subsection 3.1, we construct the eigenfunctions of the latter and analyze the discontinuities in subsections 3.2 and 3.3, respectively.

Some physical aspects of our results are finally discussed in more detail in section 4.

2 Trigonometric secular equation

2.1 $\mathcal{PT}$-symmetric square well inside a real one

Let us denote the four regions $-L < x < -l$, $-l < x < 0$, $0 < x < l$, $l < x < L$ by $L2$, $L1$, $R1$, $R2$, respectively. We shall henceforth append these symbols as subscripts to all quantities pertaining to such regions. The complex potential $V(x)$, defined in equations (1.7) and (1.8), may therefore be rewritten as

$$V_{L2}(x) = 0 \qquad V_{L1}(x) = -\mathrm{i}g \qquad V_{R1}(x) = \mathrm{i}g \qquad V_{R2}(x) = 0. \tag{2.1}$$

The general solution of (1.6) satisfying the conditions (1.9) can be written as

$$\psi(x) = \begin{cases} \psi_{L2}(x) = A_L \sin[k(L+x)] \\ \psi_{L1}(x) = B_L \cosh(\kappa^* x) + \mathrm{i}\frac{C_L}{\kappa^* l} \sinh(\kappa^* x) \\ \psi_{R1}(x) = B_R \cosh(\kappa x) + \mathrm{i}\frac{C_R}{\kappa l} \sinh(\kappa x) \\ \psi_{R2}(x) = A_R \sin[k(L-x)] \end{cases} \tag{2.2}$$

where

$$\kappa = s + \mathrm{i}t \qquad E = k^2 = t^2 - s^2 \qquad g = 2st. \tag{2.3}$$

Here s, t and k are real and, for the sake of definiteness, are assumed positive. A priori, A_L, B_L, C_L, A_R, B_R and C_R are some complex constants.

On assuming that $\mathcal{PT}$-symmetry is unbroken, we obtain the conditions

$$\psi^*_{L2}(-x) = \psi_{R2}(x) \qquad \psi^*_{L1}(-x) = \psi_{R1}(x) \tag{2.4}$$

from which we get

$$A^*_L = A_R \equiv A \qquad B^*_L = B_R \equiv B \qquad C^*_L = C_R \equiv C. \tag{2.5}$$

The derivative of (2.2), taking (2.5) into account, reads

$$\partial_x\psi(x) = \begin{cases} \partial_x\psi_{L2}(x) = kA^* \cos[k(L+x)] \\ \partial_x\psi_{L1}(x) = \kappa^* B^* \sinh(\kappa^* x) + \mathrm{i}\frac{C^*}{l} \cosh(\kappa^* x) \\ \partial_x\psi_{R1}(x) = \kappa B \sinh(\kappa x) + \mathrm{i}\frac{C}{l} \cosh(\kappa x) \\ \partial_x\psi_{R2}(x) = -kA \cos[k(L-x)] \end{cases}. \tag{2.6}$$

Let us now match the wavefunction and its derivative at $x = 0$ and impose $\mathcal{PT}$-symmetry in the neighbourhood of the origin:

$$\psi_{R1}(0) = \psi_{L1}(0) \in \mathbb{R} \qquad \partial_x\psi_{R1}(0) = \partial_x\psi_{L1}(0) \in \mathrm{i}\mathbb{R}. \tag{2.7}$$

This leads to

$$B, C \in \mathbb{R}. \tag{2.8}$$

It now remains to match ψ and $\partial_x\psi$ at $x = \pm l$. Since ψ is $\mathcal{PT}$-symmetric, it is enough to impose matching conditions at $x = l$:

$$\psi_{R2}(l) = \psi_{R1}(l) \qquad \partial_x\psi_{R2}(l) = \partial_x\psi_{R1}(l). \tag{2.9}$$

This yields

$$
\begin{aligned}
A\sin[k(L-l)] &= B\cosh(\kappa l) + \mathrm{i}\frac{C}{\kappa l}\sinh(\kappa l) &\quad (2.10)\\
-kA\cos[k(L-l)] &= \kappa B\sinh(\kappa l) + \mathrm{i}\frac{C}{l}\cosh(\kappa l). &\quad (2.11)
\end{aligned}
$$

We conclude that the final form of ψ is

$$
\psi(x) = \begin{cases} \psi_{L2}(x) = A^* \sin[k(L+x)] \\ \psi_{L1}(x) = B\cosh(\kappa^* x) + \mathrm{i}\frac{C}{\kappa^* l}\sinh(\kappa^* x) \\ \psi_{R1}(x) = B\cosh(\kappa x) + \mathrm{i}\frac{C}{\kappa l}\sinh(\kappa x) \\ \psi_{R2}(x) = A\sin[k(L-x)] \end{cases} \qquad (2.12)
$$

where the complex constant A is determined by one of the equations (2.10) and (2.11), while the real constants B and C have to satisfy a condition obtained by eliminating A between (2.10) and (2.11):

$$
\begin{aligned}
&\kappa l B\{k\cos[k(L-l)]\cosh(\kappa l) + \kappa\sin[k(L-l)]\sinh(\kappa l)\}\\
&+ \mathrm{i}C\{k\cos[k(L-l)]\sinh(\kappa l) + \kappa\sin[k(L-l)]\cosh(\kappa l)\} = 0. \qquad (2.13)
\end{aligned}
$$

We may therefore express both constants A and C in terms of B as

$$
\begin{aligned}
A &= B\,\frac{\kappa\csc[k(L-l)]\operatorname{csch}(\kappa l)}{k\cot[k(L-l)] + \kappa\coth(\kappa l)} &\quad (2.14)\\
C &= \mathrm{i}\kappa l B\,\frac{k\cot[k(L-l)]\coth(\kappa l) + \kappa}{k\cot[k(L-l)] + \kappa\coth(\kappa l)}. &\quad (2.15)
\end{aligned}
$$

Since, from (2.8), the left-hand side of equation (2.15) is real, the same should be true for the right-hand one. The resulting condition can be written as

$$
\begin{aligned}
&k^2\cot^2[k(L-l)][\kappa\coth(\kappa l) + \kappa^*\coth(\kappa^* l)]\\
&+ k\cot[k(L-l)][\kappa^2 + 2\kappa\kappa^*\coth(\kappa l)\coth(\kappa^* l) + \kappa^{*2}]\\
&+ \kappa\kappa^*[\kappa\coth(\kappa^* l) + \kappa^*\coth(\kappa l)] = 0. \qquad (2.16)
\end{aligned}
$$

On expressing k^2, κ and κ^* in terms of s and t through equation (2.3) and using some elementary trigonometric identities, condition (2.16) is easily transformed into

$$
\begin{aligned}
&k\sin[2k(L-l)][s^2\cosh(2sl) + t^2\cos(2tl)]\\
&-\cos[2k(L-l)][s^3\sinh(2sl) - t^3\sin(2tl)]\\
&+ st^2\sinh(2sl) - s^2 t\sin(2tl) = 0 \qquad (2.17)
\end{aligned}
$$

where $k = \sqrt{t^2 - s^2}$.

2.2 Graphical and numerical determination of the energies

The transcendental equation (2.17) has to be complemented by the constraint (2.3),

$$st = \frac{1}{2}g. \tag{2.18}$$

The couples of roots (s_n, t_n), $n = 0,\ 1,\ 2,\ \ldots$, of this pair of equations define all the bound-state energies E_n by the elementary formula

$$E_n = t_n^2 - s_n^2 \qquad n = 0, 1, 2, \ldots. \tag{2.19}$$

In practice, the (s_n, t_n) values may be obtained as the intersection points in the (s, t) plane of the curves representing the roots of the transcendental equation (2.17) with the hyperbola (2.18).

Before proceeding to discuss the graphical and numerical determination of E_n in general, it is worth reviewing three interesting limiting cases of equation (2.17). The first one corresponds to the limit $l \to L$, wherein the present square well with three matching points reduces to the one with a single discontinuity. Equation (2.17) then simply becomes

$$s \sinh(2sL) + t \sin(2tL) = 0 \tag{2.20}$$

which coincides with equation (9) of [14] (where g is denoted by Z and $L = 1$).

The second limiting case corresponds to $l \to 0$ and gives back the real square well. Since the constraint (2.18) then disappears, we are only left with equation (2.17) acquiring the simple form

$$\sin(2kL) = 0. \tag{2.21}$$

Its solutions are provided by the hyperbolas $t^2 - s^2 = \left(\frac{n\pi}{2L}\right)^2$, $n = 1,\ 2,\ \ldots$, where the $n = 0$ value is discarded because no acceptable wavefunction can be associated with it. We therefore arrive at the well-known quadratic spectrum $E_n^2 = \left(\frac{n\pi}{2L}\right)^2$, $n = 1$, $2,\ \ldots$, of the real square well.

The existence of the third special limiting regime is connected with the bounded nature of our imaginary barrier (1.8). In the language of perturbation theory this means [15] that the influence of this barrier on the values of the energies (2.19) weakens quickly with the growth of the quantum number n. At the higher excitations, as a consequence, the $n-$dependence of the energies will not deviate too much from the $l \to 0$ rule $E_n \sim n^2 \gg 1$. In the other words, the growth of n will imply the growth of $t_n \sim n \gg 1$ and the decrease and smallness of the roots $s_n = g/(2t_n) \ll 1$. In this regime, we may imagine that $k = t\sqrt{1 - s^2/t^2} = t - s^2/(2t) + \mathcal{O}(s^4/t^3) = t - g^2/(8t^3) + \mathcal{O}(1/n^7)$ so that the six components of our quantization condition (2.17), *viz.*,

$$\begin{aligned} &s^2 k \sin[2k(L-l)] \cosh(2sl) + t^2 k \sin[2k(L-l)] \cos(2tl) \\ &- s^3 \cos[2k(L-l)] \sinh(2sl) + t^3 \cos[2k(L-l)] \sin(2tl) \\ &+ st^2 \sinh(2sl) - s^2 t \sin(2tl) = 0 \end{aligned}$$

may be characterized by their asymptotic sizes $\mathcal{O}(1/n)$, $\mathcal{O}(n^3)$, $\mathcal{O}(1/n^4)$, $\mathcal{O}(n^3)$, $\mathcal{O}(n^0)$ and $\mathcal{O}(1/n)$, respectively. Once we omit all the negligible $\mathcal{O}(1/n)$ terms and insert $s = g/(2t)$ whenever necessary, we arrive at the thoroughly simplified approximate secular equation

$$\sin(2kL) + \frac{g^2 l}{2k^3} + \mathcal{O}\left(\frac{1}{k^4}\right) = 0. \tag{2.22}$$

Its roots are easily found,

$$k = k_n = \frac{\pi\, n}{2L} + (-1)^{n+1} \frac{2g^2 l L^2}{\pi^3 n^3} + \mathcal{O}\left(\frac{1}{n^4}\right), \tag{2.23}$$

and give

$$E_n = k_n^2 = \left(\frac{\pi\, n}{2L}\right)^2 + (-1)^{n+1} \frac{2g^2 l L}{\pi^2 n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \tag{2.24}$$

i.e., a nice and elementary approximate energy formula for all the highly excited states.

In the general case, the bound-state energies (2.19) of our model are determined from the simultaneous solutions of equations (2.17) and (2.18). Although the former is transcendental, one of its roots is quite obvious, namely $s = t$. When we realize that this implies $k = 0$ and substitute the solution into equations (2.12) – (2.15), we obtain a vanishing wavefunction. This is in accordance with an insight provided by the Hermitian limit $g \to 0$ or $l \to 0$.

The other solutions of (2.17) can be found numerically and graphically. As we can see in figure 1 where we work with re-scaled length units in which $L = 1$, they form semi-ovals in (s, t) plane. We can observe the absence of robustly real energy levels, i.e., levels remaining real for any value of g, which played their role in [16].

The locally decreasing character of the semi-oval maxima could cause a complexification of higher energy pairs while the lower pairs would remain real. In other words, the semi-oval maxima might be decreasing faster then the hyperbola (2.18). This race in decrease can be judged easily when we use a hyperbolic coordinate system. As shown in figure 2, in this setting, the maxima prove to increase monotonically while the hyperbola is represented by a horizontal straight line. Consequently, our model preserves a sequential merging of the energy levels. The critical value g_c of the coupling constant g, for which the two lowest energy levels merge together, is of high importance. It is the boundary of exact $\mathcal{PT}$-symmetry, which we consider to be physically relevant and assumed in deriving equation (2.17). For a higher value of g, the wavefunction $\mathcal{PT}$-symmetry would be broken.

We found g_c for various values of the parameter l. Since g_c rises rapidly as $l \to 0$, we present its values in combination of graph and table (see figure 3 and table 1). As the parameter l approaches zero, g_c tends to infinity and the semi-oval maxima run to infinity as well. As explained in subsection 2.1, equation (2.17) then provides the bound-state energies of the real square well. On the other hand, for $l \to L = 1$, we get back the critical coupling $g_c \simeq 4.4753$, previously obtained for the square well in [14] and [17].

3 The SUSY partner potential

The purpose of the present section is to construct and study the SUSY partner $H^{(-)}$ of the square-well Hamiltonian $H^{(+)}$, defined in equation (2.1), in the physically-relevant unbroken $\mathcal{PT}$-symmetry regime, corresponding to $g < g_c$.

3.1 Determination of the parameters

Identifying $V^{(+)}$ with the square-well potential (2.1), i.e., $V_{L2}^{(+)}(x) = 0$, $V_{L1}^{(+)}(x) = -\mathrm{i}g$, $V_{R1}^{(+)}(x) = \mathrm{i}g$, $V_{R2}^{(+)}(x) = 0$ and $E_0 = k_0^2 = t_0^2 - s_0^2 = -\kappa_0^2 + \mathrm{i}g$, we obtain for the superpotential and the partner potential the results

$$W(x) = \begin{cases} W_{L2}(x) = k_0 \tan[k_0(x + x_{L2})] \\ W_{L1}(x) = -\kappa_0^* \tanh[\kappa_0^*(x + x_{L1})] \\ W_{R1}(x) = -\kappa_0 \tanh[\kappa_0(x - x_{R1})] \\ W_{R2}(x) = k_0 \tan[k_0(x - x_{R2})] \end{cases} \tag{3.1}$$

and

$$V^{(-)}(x) = \begin{cases} V_{L2}^{(-)}(x) = 2k_0^2 \sec^2[k_0(x + x_{L2})] \\ V_{L1}^{(-)}(x) = -2\kappa_0^{*2} \operatorname{sech}^2[\kappa_0^*(x + x_{L1})] - \mathrm{i}g \\ V_{R1}^{(-)}(x) = -2\kappa_0^2 \operatorname{sech}^2[\kappa_0(x - x_{R1})] + \mathrm{i}g \\ V_{R2}^{(-)}(x) = 2k_0^2 \sec^2[k_0(x - x_{R2})] \end{cases} \tag{3.2}$$

respectively. Here x_{L2}, x_{L1}, x_{R1} and x_{R2} denote four integration constants.

We now choose x_{L2} and x_{R2} as

$$x_{L2} = L + \frac{\pi}{2k_0} \qquad x_{R2} = L - \frac{\pi}{2k_0} \tag{3.3}$$

to ensure that $V_{L2}^{(-)}$ and $V_{R2}^{(-)}$ blow up at the end points $x = -L$ and $x = L$. This is in tune with [9]. We thus get

$$V_{L2}^{(-)}(x) = 2k_0^2 \csc^2[k_0(x + L)] \qquad V_{R2}^{(-)}(x) = 2k_0^2 \csc^2[k_0(x - L)]. \tag{3.4}$$

Observe that for the superpotential, $W_{L2}(x)$ and $W_{R2}(x)$ also blow up at these points:

$$W_{L2}(x) = -k_0 \cot[k_0(x + L)] \qquad W_{R2}(x) = -k_0 \cot[k_0(x - L)]. \tag{3.5}$$

Let us next consider the unbroken SUSY condition (1.3), where according to (2.12) the ground-state wavefunction of $H^{(+)}$ is given by

$$\psi_{0R2}^{(+)}(x) = \psi_{0L2}^{(+)*}(-x) = A_0^{(+)} \sin[k_0(L - x)] \tag{3.6}$$

$$\psi_{0R1}^{(+)}(x) = \psi_{0L1}^{(+)*}(-x) = B_0^{(+)} \cosh(\kappa_0 x) + \mathrm{i}\frac{C_0^{(+)}}{\kappa_0 l} \sinh(\kappa_0 x). \tag{3.7}$$

Note that the superscript '(+)' is appended to the wavefunction and the coefficients to signify that we are dealing with Hamiltonian $H^{(+)}$. It is straightforward to see that equation (1.3) is automatically satisfied in the regions $R2$ and $L2$ due to the choice made for the integration constants x_{R2}, x_{L2} in equation (3.3). On the other hand, in the region $R1$ we find a condition fixing the value of x_{R1},

$$\tanh(\kappa_0 x_{R1}) = -\frac{\mathrm{i}C_0^{(+)}}{\kappa_0 l B_0^{(+)}} = \frac{k_0 \cot[k_0(L-l)]\coth(\kappa_0 l) + \kappa_0}{k_0 \cot[k_0(L-l)] + \kappa_0 \coth(\kappa_0 l)} \tag{3.8}$$

where in the last step we used equation (2.15). A similar relation applies in $L1$, thus leading to the result

$$x_{L1} = x_{R1}^{*}. \tag{3.9}$$

Note that in contrast with the real integration constants x_{R2}, x_{L2}, the constants x_{R1} and x_{L1} are complex. Separating both sides of equation (3.8) into a real and an imaginary part, we obtain the two equations

$$\frac{\sinh X \cosh X}{\cosh^2 X \cos^2 Y + \sinh^2 X \sin^2 Y} = \frac{N^r}{D} \tag{3.10}$$

$$\frac{\sin Y \cos Y}{\cosh^2 X \cos^2 Y + \sinh^2 X \sin^2 Y} = \frac{N^i}{D} \tag{3.11}$$

where we have used the decompositions $\kappa_0 = s_0 + \mathrm{i}t_0$, $x_{R1} = x_{R1}^r + \mathrm{i}x_{R1}^i$, $\kappa_0 x_{R1} = X + \mathrm{i}Y$, implying that

$$X = s_0 x_{R1}^r - t_0 x_{R1}^i \qquad Y = t_0 x_{R1}^r + s_0 x_{R1}^i \tag{3.12}$$

and we have defined

$$N^r = \{-s_0^2 \cos[2k_0(L-l)] + t_0^2\}\sinh(2s_0 l) + k_0 s_0 \sin[2k_0(L-l)]\cosh(2s_0 l) \tag{3.13}$$

$$N^i = \{s_0^2 - t_0^2 \cos[2k_0(L-l)]\}\sin(2t_0 l) - k_0 t_0 \sin[2k_0(L-l)]\cos(2t_0 l) \tag{3.14}$$

$$\begin{aligned} D \;=\; & \{-s_0^2 \cos[2k_0(L-l)] + t_0^2\}\cosh(2s_0 l) + \{s_0^2 - t_0^2 \cos[2k_0(L-l)]\}\cos(2t_0 l) \\ & + k_0 \sin[2k_0(L-l)][s_0 \sinh(2s_0 l) + t_0 \sin(2t_0 l)]. \end{aligned} \tag{3.15}$$

Equations (3.10) and (3.11), when solved numerically, furnish the values of both the parameters x_{R1}^r and x_{R1}^i.

One may also observe that the resulting superpotential $W(-x) = -W^*(x)$ and partner potential $V^{(-)}(-x) = V^{(-)*}(x)$ are $\mathcal{PT}$-antisymmetric and $\mathcal{PT}$-symmetric, respectively.

3.2 Eigenfunctions in the partner potential

On exploiting the first intertwining relation in (1.4), the eigenfunctions $\psi_n^{(-)}(x)$, $n = 0$, 1, 2, ..., of $H^{(-)}$ can be obtained by acting with $\mathcal{A}$ on $\psi_{n+1}^{(+)}(x)$, subject to the preservation of the boundary and continuity conditions

$$\psi_{nL2}^{(-)}(-L) = 0 \qquad \psi_{nR2}^{(-)}(L) = 0 \tag{3.16}$$

$$\psi_{nL2}^{(-)}(-l) = \psi_{nL1}^{(-)}(-l) \qquad \partial_x\psi_{nL2}^{(-)}(-l) = \partial_x\psi_{nL1}^{(-)}(-l) \tag{3.17}$$

$$\psi_{nL1}^{(-)}(0) = \psi_{nR1}^{(-)}(0) \qquad \partial_x\psi_{nL1}^{(-)}(0) = \partial_x\psi_{nR1}^{(-)}(0) \tag{3.18}$$

$$\psi_{nR1}^{(-)}(l) = \psi_{nR2}^{(-)}(l) \qquad \partial_x\psi_{nR1}^{(-)}(l) = \partial_x\psi_{nR2}^{(-)}(l). \tag{3.19}$$

Application of $\mathcal{A}$ leads to the forms

$$\begin{aligned}\psi_{nL2}^{(-)}(x) &= C_{nL2}^{(-)}\, A_{n+1}^{(+)*} \sin[k_{n+1}(L+x)] \\ &\quad \times \{k_{n+1}\cot[k_{n+1}(L+x)] - k_0\cot[k_0(L+x)]\}\end{aligned} \tag{3.20}$$

$$\begin{aligned}\psi_{nL1}^{(-)}(x) &= C_{nL1}^{(-)}\, B_{n+1}^{(+)} \sinh(\kappa_{n+1}^* x)\{\kappa_{n+1}^* - \kappa_0^* \tanh[\kappa_0^*(x+x_{R1}^*)]\coth(\kappa_{n+1}^* x)\} \\ &\quad + C_{nL1}^{(-)} \frac{\mathrm{i}C_{n+1}^{(+)}}{\kappa_{n+1}^* l} \sinh(\kappa_{n+1}^* x) \\ &\quad \times \{\kappa_{n+1}^* \coth(\kappa_{n+1}^* x) - \kappa_0^* \tanh[\kappa_0^*(x+x_{R1}^*)]\}\end{aligned} \tag{3.21}$$

$$\begin{aligned}\psi_{nR1}^{(-)}(x) &= C_{nR1}^{(-)}\, B_{n+1}^{(+)} \sinh(\kappa_{n+1} x)\{\kappa_{n+1} - \kappa_0 \tanh[\kappa_0(x-x_{R1})]\coth(\kappa_{n+1} x)\} \\ &\quad + C_{nR1}^{(-)} \frac{\mathrm{i}C_{n+1}^{(+)}}{\kappa_{n+1} l} \sinh(\kappa_{n+1} x) \\ &\quad \times \{\kappa_{n+1} \coth(\kappa_{n+1} x) - \kappa_0 \tanh[\kappa_0(x-x_{R1})]\}\end{aligned} \tag{3.22}$$

$$\begin{aligned}\psi_{nR2}^{(-)}(x) &= C_{nR2}^{(-)}\, A_{n+1}^{(+)} \sin[k_{n+1}(L-x)] \\ &\quad \times \{-k_{n+1}\cot[k_{n+1}(L-x)] + k_0\cot[k_0(L-x)]\}\end{aligned} \tag{3.23}$$

where $C_{nL2}^{(-)}$, $C_{nL1}^{(-)}$, $C_{nR1}^{(-)}$, $C_{nR2}^{(-)}$ denote some complex constants and equation (3.9) has been used. It can be easily checked that the boundary conditions (3.16) are automatically satisfied by these eigenfunctions. It therefore remains to impose the continuity conditions (3.17) – (3.19).

Let us first match the regions $L1$ and $R1$ at $x = 0$. The continuity conditions

(3.18) yield the two relations

$$C_{nR1}^{(-)}\left[B_{n+1}^{(+)}\kappa_0\tanh(\kappa_0 x_{R1})+\frac{\mathrm{i}C_{n+1}^{(+)}}{l}\right]=C_{nL1}^{(-)}\left[-B_{n+1}^{(+)}\kappa_0^*\tanh(\kappa_0^* x_{R1}^*)+\frac{\mathrm{i}C_{n+1}^{(+)}}{l}\right] \tag{3.24}$$

$$\begin{aligned}&C_{nR1}^{(-)}\left\{B_{n+1}^{(+)}[\kappa_{n+1}^2-\kappa_0^2\,\mathrm{sech}^2(\kappa_0 x_{R1})]+\frac{\mathrm{i}C_{n+1}^{(+)}}{l}\kappa_0\tanh(\kappa_0 x_{R1})\right\}\\&=C_{nL1}^{(-)}\left\{B_{n+1}^{(+)}[\kappa_{n+1}^{*2}-\kappa_0^{*2}\,\mathrm{sech}^2(\kappa_0^* x_{R1}^*)]-\frac{\mathrm{i}C_{n+1}^{(+)}}{l}\kappa_0^*\tanh(\kappa_0^* x_{R1}^*)\right\}.\end{aligned} \tag{3.25}$$

Since equations (3.8) and (2.3) provide the two constraints

$$\kappa_0\tanh(\kappa_0 x_{R1}) = -\kappa_0^*\tanh(\kappa_0^* x_{R1}^*) \tag{3.26}$$

$$\kappa_{n+1}^{*2}-\kappa_{n+1}^2 = \kappa_0^{*2}-\kappa_0^2=-2g \tag{3.27}$$

equations (3.24) and (3.25) are compatible and lead to the condition

$$C_{nR1}^{(-)}=C_{nL1}^{(-)}. \tag{3.28}$$

Considering next the matching between $R1$ and $R2$ at $x=l$, we obtain from equation (3.19) the two conditions

$$\begin{aligned}&C_{nR1}^{(-)}\{k_{n+1}\cot[k_{n+1}(L-l)]+\kappa_0\tanh[\kappa_0(l-x_{R1})]\}\\&=C_{nR2}^{(-)}\{k_{n+1}\cot[k_{n+1}(L-l)]-k_0\cot[k_0(L-l)]\}\end{aligned} \tag{3.29}$$

$$\begin{aligned}&C_{nR1}^{(-)}\Big(\kappa_{n+1}^2-\kappa_0^2+\kappa_0\tanh[\kappa_0(l-x_{R1})]\{k_{n+1}\cot[k_{n+1}(L-l)]\\&\quad+\kappa_0\tanh[\kappa_0(l-x_{R1})]\}\Big)\\&=C_{nR2}^{(-)}\Big(k_0^2-k_{n+1}^2-k_0\cot[k_0(L-l)]\{k_{n+1}\cot[k_{n+1}(L-l)]\\&\quad-k_0\cot[k_0(L-l)]\}\Big)\end{aligned} \tag{3.30}$$

after making use of equations (2.14) and (2.15) to eliminate $A_{n+1}^{(+)}$, $B_{n+1}^{(+)}$ and $C_{n+1}^{(+)}$. Equations (3.29) and (3.30) both yield the same result

$$C_{nR1}^{(-)}=C_{nR2}^{(-)} \tag{3.31}$$

due to the two relations

$$\kappa_0 \tanh[\kappa_0(l - x_{R1})] = -k_0 \cot[k_0(L - l)] \tag{3.32}$$

and

$$\kappa_{n+1}^2 - \kappa_0^2 = k_0^2 - k_{n+1}^2 \tag{3.33}$$

deriving from (3.8) and (2.3), respectively.

Since a result similar to (3.31) applies at the interface between regions $L2$ and $L1$, we conclude that the partner potential eigenfunctions are given by equations (3.20) – (3.23) with

$$C_{nL2}^{(-)} = C_{nL1}^{(-)} = C_{nR1}^{(-)} = C_{nR2}^{(-)} \equiv C_n^{(-)}. \tag{3.34}$$

Such eigenfunctions are $\mathcal{PT}$-symmetric provided we choose $C_n^{(-)}$ imaginary:

$$C_n^{(-)*} = -C_n^{(-)}. \tag{3.35}$$

3.3 Discontinuities in the partner potential

In subsection 3.1, we have constructed the SUSY partner $V^{(-)}(x)$ of a piece-wise potential with three discontinuities at $x = -l$, 0 and l. We may now ask the following question: does the former have the same discontinuities as the latter or could the discontinuity number decrease? We plan to prove here that the second alternative can be ruled out.

For such a purpose, we will examine successively under which conditions $V^{(-)}(x)$ could be continuous at $x = l$ or at $x = 0$ and we will show that such restrictions would not be compatible with some relations deriving from the unbroken-SUSY assumption (1.3). Observe that we do not have to study continuity at $x = -l$ separately, since $V^{(-)}(x)$ being $\mathcal{PT}$-symmetric must be simultaneously continuous or discontinuous at $x = -l$ and $x = l$.

Let us start with the point $x = l$. Matching there $V_{R1}^{(-)}(x)$ and $V_{R2}^{(-)}(x)$, given in equations (3.2) and (3.4), respectively, leads to the relation

$$-2\kappa_0^2 \operatorname{sech}^2[\kappa_0(l - x_{R1})] + \mathrm{i}g = 2k_0^2 \csc^2[k_0(L - l)]. \tag{3.36}$$

On using (3.32) and some simple trigonometric identities, such a relation can be transformed into $k_0^2 = -\kappa_0^2 + \frac{1}{2}\mathrm{i}g$, which manifestly contradicts equation (2.3). Hence continuity of $V^{(-)}(x)$ at $x = l$ is ruled out.

Consider next the point $x = 0$. On equating $V_{R1}^{(-)}(0)$ with $V_{L1}^{(-)}(0)$ and employing (3.2) and (3.9), we obtain the condition

$$-2\kappa_0^2 \operatorname{sech}^2(\kappa_0 x_{R1}) + \mathrm{i}g = -2\kappa_0^{*2} \operatorname{sech}^2(\kappa_0^* x_{R1}^*) - \mathrm{i}g. \tag{3.37}$$

Equations (2.8) and (3.8) then yield the relation $-\kappa_0^2 + \frac{1}{2}\mathrm{i}g = -\kappa_0^{*2} - \frac{1}{2}\mathrm{i}g$, which contradicts equation (2.3) again. Continuity of $V^{(-)}(x)$ at $x = 0$ is therefore excluded too.

We conclude that under the simplest assumption of unbroken SUSY with a factorization energy equal to the ground-state energy of $H^{(+)}$, the partner potential $V^{(-)}(x)$ has the same three discontinuities at $x = -l$, 0 and l as $V^{(+)}(x)$.

4 Discussion

Among all the $\mathcal{PT}$-symmetric models, field-theoretical background explains the lasting interest in the purely imaginary long-range model $V(x) = \mathrm{i}x^3$ [18, 19] and its generalizations $V(x) = x^2(\mathrm{i}x)^\delta$ with the imaginary part $V^i(x)$ exhibiting, at any $\delta \in [0, 2)$, a characteristic 'strongly non-Hermitian' (SNH) long-range growth in 'coordinate' $x \in \mathbb{R}$. Up to the harmonic oscillator at $\delta = 0$, all of the latter SNH $\mathcal{PT}$-symmetric models are only solvable by approximate methods. Still, rigorous proofs exist showing that their spectra are all real [19].

By rigorous means, the reality of the spectrum has also been shown for many other $\mathcal{PT}$-symmetric potentials V. Some of them turn out to be exactly solvable [20, 21, 22], and those for which $V^i(\pm\infty) = 0$ may be called 'weakly non-Hermitian' (WNH). Their WNH character is reflected not only by a less explicit influence of the imaginary part of the potential upon the spectrum, but also by the existence of SUSY partners [8, 21, 23] which, in some special cases, may be real and Hermitian [5, 21].

In the light of similar observation one might feel tempted to perceive WNH models as 'partially compatible' with our intuitive expectations. This impression may be further enhanced by noticing that another exactly solvable model, viz., the typical WNH spiked form of the $\delta = 0$ harmonic oscillator, as described in [24], proved of particular interest in the SUSYQM context as well [6, 23].

Potentials $V(x)$ with shapes that are piece-wise constant may be considered equally exceptional. All of these square-well-type models with forces located inside a finite interval $(-L, L)$ may be easily classified by the number of their discontinuities.

The simplest nontrivial non-Hermitian square-well potential must have at least one discontinuity (= matching point at $x = 0$). While the real part of this V is just a trivial shift of the energy scale, it may be kept equal to zero. Then, the non-zero strength Z of the spatially antisymmetric and purely imaginary V is the only free (real) parameter of the whole model with SNH features [9, 14]. Its $\mathcal{PT}$-symmetry remains unbroken in an interval of $Z \in (-Z_{crit}, Z_{crit})$ while its ground-state energy becomes complex beyond $Z_{crit} \approx 4.48$ (in standard units $\hbar = 2m = 1$ [14, 17]).

It is known that some of these features are generic [15]. Quantitatively, their occurrence has also been confirmed for the twice-constant SNH model V with two discontinuities [16]. Qualitatively, all of these observations facilitate the applicability and physical interpretation of the piece-wise constant models significantly [25], especially because the numerical values of the maximal allowed couplings prove to be, in general, quite large. This allows us to guarantee the (necessary) reality of the energies by keeping simply our choice of Z safely below this maximum.

The family of WNH square-well models may only start at the piece-wise potential with three discontinuities. In our present study of such a model it was important to demonstrate the parallelism of its properties with the exact solutions of the *smooth* complex potentials of similar shapes [6].

The most obvious parallel lies in the observation that a key formal feature of the SUSY partners $H^{(\pm)}$ is that they may remain both non-Hermitian and $\mathcal{PT}$-symmetric. Of course, the parity $\mathcal{P}$ cannot define the positive-definite norm [12, 15, 26, 27]. A consistent physical interpretation of the similar non-Hermitian models was recently agreed (cf., e.g., [28]) to lie in the existence of *a new* metric-like operator $\mathcal{P}_{(+)} > 0$ which is positive definite. This Hermitian operator may be assumed to play the role of the 'physical' metric [29]. This means that once our equation (1.5) is satisfied by the old Hamiltonian and by the new, *positive-definite* metric $\mathcal{P}_{(+)}$, we may declare the underlying quantum Hamiltonian quasi-Hermitian, leading to the standard probabilistic interpretation of the theory (cf. the recent discussions of some related subtleties in [30]). Against this background our attention has been concentrated upon the feasibility of bound-state construction in a model with a phenomenologically appealing shape of the potential.

A couple of consequences may be expected. Our model may open the way towards addressing one of the most difficult problems encountered in $\mathcal{PT}$-symmetric quantum mechanics [27], viz., the control of a possible instability of the spectrum reality [16, 31]. Indeed, due to the pseudo-Hermiticity property (1.5) of our Hamiltonians H, the energies need not be real (i.e., observable) in principle [13].

Our WNH model may be also characterized by the simplicity of the bound-state wavefunctions. This allowed us to construct the superpotential yielding access, rather easily, to the Witten-type SUSY hierarchy. In this regard the compact form of our trigonometric secular equation was welcome and particularly important, especially for any future projects trying to connect the mathematical $\mathcal{PT}$-symmetry with physical phenomenology.

In such a perspective, the most challenging mathematical problems attached to the non-Hermitian models descend from the reality of their exceptional points [32]. The simplest solvable models of the square-well type seem to offer a transparent laboratory for their study since the indeterminate auxiliary pseudo-metric $\mathcal{P}$ coincides with the common parity.

In the context of physics, the phenomenological appeal of all the piece-wise constant analogues of the purely imaginary cubic force represented a strong motivation for the systematic constructions of the positive-definite metric operators of [29] (cf. also [12, 13, 25]). In particular, the highly appealing factorized form $\mathcal{P}_{(+)} = \mathcal{CP} > 0$ of these metric operators has been used and, for physical reasons, the factor $\mathcal{C}$ itself has been called 'charge' (cf. [28]). For all the models with relevance in field theory (like $V \sim ix^3$), the constructions of $\mathcal{C}$ were shown feasible by WKB and perturbative methods [33].

In comparison, the solvability of all the simpler models facilitates the construction of $\mathcal{C}$ (called, usually, quasi-parity in this context [20, 24, 26, 34]). An interesting energy-shift interpretation of the quasi-parity (which is a new symmetry of the Hamiltonian) emerged in the strongly spiked short-range model considered in [35].

After we return to the square-well models, the quasi-parity or charge operator $\mathcal{C}$ may be constructed in the specific form which differs sufficiently significantly from the unit operator just in a finite-dimensional subspace of the Hilbert space [15, 16, 25]. This is one of the most important merits of this class of models. It seems to open a new inspiration for a direct physical applicability of non-Hermitian models whenever their spectrum remains real.

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Figure captions

Figure 1: Solutions of (2.17) form the semi-ovals. Their intersections with the hyperbola $2st = g$ determine energy levels $E = k^2 = t^2 - s^2$ of the system. Here $g = 650$ and $l = 0.04$.

Figure 2: The previous picture (Fig.1) in $[ts, k]$ plane, where $k = \sqrt{t^2 - s^2}$. We set $g = 650$ and $l = 0.04$ again.

Figure 3: Fifty values of critical couplings g_c, increasing rapidly as l decreases, $l \to 0$.

Table captions

Table 1: Numerical values of g_c in dependence on the parameter l. The table suggests that the critical coupling grows faster than $1/l$ for small l.

Table 1

l	1.00	0.70	0.50	0.40	0.30	0.20	0.10	0.01	0.001
$g_c \sim$	4.4753	4.8129	6.4364	8.6011	13.426	27.273	95.832	9895.4	486950

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