

\mathcal{PT} -symmetric quartic anharmonic oscillator and position-dependent mass in a perturbative approach

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Abstract

To lowest order of perturbation theory we show that an equivalence can be established between a \mathcal{PT} -symmetric generalized quartic anharmonic oscillator model and a Hermitian position-dependent mass Hamiltonian h . An important feature of h is that it reveals a domain of couplings where the quartic potential could be attractive, vanishing or repulsive. We also determine the associated physical quantities.

The interplay between pseudo-Hermitian \mathcal{PT} -symmetric Hamiltonians and their equivalent Hermitian representation is currently a matter of active research [1, 2, 3, 4, 5]. While several case studies already exist in the literature, Mostafazadeh, in particular, has also considered [1] the related issue of transition to the classical limit. Quite significantly, he has observed that the underlying classical Hamiltonian for the PT -symmetric cubic anharmonic oscillator (PTCAO), in presence of a harmonic term, reveals the characteristics of a point particle that is endowed with a position-dependent mass (PDM) interacting with a quartic anharmonic field.

Motivated by Mostafazadeh's work, we recently developed [6] an algorithm that affords a categorization of a whole class of quantal PDM Hamiltonians in terms of perturbatively equivalent \mathcal{PT} -symmetric counterparts with configuration space \mathbb{R} . The PTCAO is one such model, which was shown to have a Hermitian PDM partner Hamiltonian.

Keeping \mathcal{PT} symmetry intact, the PTCAO scheme can obviously be extended to include the one-parameter family of potentials $x^2(ix)^\delta$, δ real. It is curious to note that $\delta = 2$ implies quartic anharmonicity but with a wrong sign (see, e.g., [7]). However, as Bender and Boettcher argued [8] (see also [9]), such a feature facilitates its quasi-exact solvability.

In this paper, we take up the study of a generalized scheme [10, 11] of PT -symmetric quartic anharmonic oscillator (PTQAO) in the spirit of the perturbative analysis of [6] and examine when it should map to a Hermitian PDM Hamiltonian. We also estimate the fourth-order contribution to it and write down the physical position and momentum operators up to third order. Finally, we determine the classical limit.

Consider the harmonic oscillator perturbed by an imaginary cubic term to the first order and a $\delta = 2$ quartic anharmonic term to the second order as follows:

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon\mathcal{H}_1 + \varepsilon^2\mathcal{H}_2. \quad (1)$$

The various terms in the Hamiltonian \mathcal{H} are

$$\begin{aligned} \mathcal{H}_0 &= \frac{p^2}{2\mu_0} + av_1^{(r)}(x), & v_1^{(r)}(x) &= x^2, \\ \mathcal{H}_1 &= ibv^{(i)}(x), & v^{(i)}(x) &= x^3, \\ \mathcal{H}_2 &= -cv_2^{(r)}(x), & v_2^{(r)}(x) &= x^4, \end{aligned} \quad (2)$$

with $\varepsilon \ll 1$, $a, b (\neq 0)$ and c are positive coupling constants, $v_j^{(r)}(-x) = v_j^{(r)}(x) \in \mathbb{R}$, $j = 1, 2$, $v^{(i)}(-x) = -v^{(i)}(x) \in \mathbb{R}$.

Resorting to dimensionless variables proves convenient and requires the following transformations to be enforced:

$$\begin{aligned} X &= \ell^{-1}x, & P &= \ell\hbar^{-1}p, \\ \alpha &= \ell^4\hbar^{-2}\mu_0a, & \beta &= \ell^3\hbar^{-2}\mu_0b, & \gamma &= \ell^2\hbar^{-2}\mu_0c, & \epsilon &= \ell^2\varepsilon, \end{aligned} \quad (3)$$

where ℓ is the length scale.

The Hamiltonian \mathcal{H} then gets modified to

$$H = \nu^{-1}\mathcal{H} = H_0 + \epsilon H_1 + \epsilon^2 H_2, \quad \nu = \frac{\hbar^2}{\mu_0\ell^2}, \quad (4)$$

with

$$H_0 = \frac{P^2}{2} + V_1^{(r)}(X), \quad H_1 = iV^{(i)}(X), \quad H_2 = -V_2^{(i)}(X), \quad (5)$$

where $V_1^{(r)} = \alpha X^2$, $V^{(i)} = \beta X^3$ and $V_2^{(i)} = \gamma X^4$.

As in [6], we can set up, corresponding to (1), an equivalent Hermitian Hamiltonian $h(x, p)$ which reduces to some PDM Hamiltonian to lowest order in ε . Noting that $h(X, P) = \nu^{-1}h(x, p)$, we therefore write

$$h(X, P) = H_0(X, P) + \epsilon^2 h^{(2)}(X, P) + \epsilon^4 h^{(4)}(X, P) + O(\epsilon^6), \quad (6)$$

where $H_0(X, P) + \epsilon^2 h^{(2)}(X, P) = P \frac{1}{2m(X)} P + V_{\text{eff}}(X)$, $1/m(X) = 1 + \epsilon^2 M^{(2)}(X)$, and $V_{\text{eff}}(X) = V_1^{(r)}(X) + \epsilon^2 V_{\text{eff}}^{(2)}(X)$. Comparing terms of $O(\epsilon^2)$ produces

$$h^{(2)}(X, P) = \frac{1}{2} P M^{(2)}(X) P + V_{\text{eff}}^{(2)}(X). \quad (7)$$

In a pseudo-Hermitian theory, for the spectrum of a diagonalizable Hamiltonian to be real, it is necessary [2] that such an operator be Hermitian with respect to a positive-definite inner product $\langle \cdot, \cdot \rangle_+$. The latter may be expressed in terms of the defining inner product $\langle \cdot, \cdot \rangle$ as

$$\langle \cdot, \cdot \rangle_+ = \langle \cdot, \eta_+ \cdot \rangle, \quad (8)$$

where the positive-definite metric operator $\eta_+ : \mathcal{L} \rightarrow \mathcal{L}$ of the reference Hilbert space \mathcal{L} , in which the Hamiltonian acts, belongs to the set of all Hermitian invertible operators. The Hilbert space equipped with the inner product (8) may be identified as the physical Hilbert space $\mathcal{L}_{\text{phys}}$.

Pseudo-Hermiticity of Hamiltonian \mathcal{H} with respect to η_+ is defined by

$$\mathcal{H}^\dagger = \eta_+ \mathcal{H} \eta_+^{-1} \quad (9)$$

and serves as a necessary and sufficient condition for \mathcal{H} to possess a real spectrum. Furthermore, given \mathcal{H} , the equivalent Hermitian operator would be

$$h(x, p) = \rho \mathcal{H} \rho^{-1}, \quad (10)$$

with $\rho = \sqrt{\eta_+}$.

The metric η_+ can be represented by

$$\eta_+ = e^{-Q(X, P)}, \quad Q(X, P) = \epsilon Q_1(X, P) + \epsilon^3 Q_3(X, P) + \dots, \quad (11)$$

where every $Q_j(X, P)$, $j = 1, 3, \dots$, is Hermitian, symmetric in X and antisymmetric in P . Using (11), the following relations emerge when equation (4) along with (5) are substituted in the dimensionless counterpart of (9):

$$\left[\frac{P^2}{2} + V_1^{(r)}, Q_1 \right] = -2iV^{(i)}, \quad (12)$$

$$\left[\frac{P^2}{2} + V_1^{(r)}, Q_3 \right] = -\frac{1}{6} [Q_1, [Q_1, iV^{(i)}]] - [Q_1, V_2^{(r)}]. \quad (13)$$

With $h(X, P)$ prescribed by (6), it is easy to deduce from the dimensionless counterpart of (10) and the Baker-Campbell-Hausdorff identity that

$$h^{(2)}(X, P) = H_2 + \frac{1}{4}[H_1, Q_1], \quad (14)$$

$$h^{(4)}(X, P) = \frac{1}{4}[H_1, Q_3] - \frac{1}{192}[[[H_1, Q_1], Q_1], Q_1]. \quad (15)$$

Comparison of (14) with (7) then yields

$$\frac{1}{2}PM^{(2)}P + V_{\text{eff}}^{(2)} = -V_2^{(r)} + \frac{1}{4}[iV^{(i)}, Q_1]. \quad (16)$$

In Ref. [6], it was noted that the operator Q_1 admits of a normal form

$$Q_1 = -i \sum_{k=0}^{\infty} S_k(X) \frac{d^k}{dX^k}, \quad (17)$$

where

$$\begin{aligned} S_{2k} &= \sum_{j=k}^{\infty} (-1)^j \binom{2j+1}{2k} \frac{d^{2j-2k+1} R_j}{dX^{2j-2k+1}}, \\ S_{2k+1} &= \sum_{j=k}^{\infty} (-1)^j (1 + \delta_{j,k}) \binom{2j+1}{2k+1} \frac{d^{2j-2k} R_j}{dX^{2j-2k}}, \end{aligned} \quad (18)$$

and R_j 's are appropriate even functions of X . A similar expansion can be carried out for the operator Q_3 .

Consider Q_1 first. Substituting (17) in equation (12), we find after a little algebra

$$\begin{aligned} \frac{1}{2} \frac{d^2 S_0}{dX^2} + \sum_{j=1}^{\infty} S_j \frac{d^j V_1^{(r)}}{dX^j} &= -2V^{(i)}, \\ \frac{1}{2} \frac{d^2 S_k}{dX^2} + \frac{dS_{k-1}}{dX} + \sum_{j=k+1}^{\infty} \binom{j}{k} S_j \frac{d^{j-k} V_1^{(r)}}{dX^{j-k}} &= 0, \quad k = 1, 2, \dots \end{aligned} \quad (19)$$

On the other hand, equation (16) leads to

$$-4 \left(V_{\text{eff}}^{(2)} + V_2^{(r)} \right) = W_0, \quad 2 \frac{dM^{(2)}}{dX} = W_1, \quad 2M^{(2)} = W_2, \quad W_k = 0 \quad k = 3, 4, \dots, \quad (20)$$

with

$$W_k \equiv \sum_{j=k+1}^{\infty} \binom{j}{k} S_j \frac{d^{j-k} V^{(i)}}{dX^{j-k}}. \quad (21)$$

For the problem of PTQAO at hand, it is evident that Q_1 contains up to cubic power in P only, so that the sum over k in (17) is restricted from 0 to 3. This implies [6] $R_k = 0$, $k = 2, 3, \dots$, and $S_k = 0$, $k = 4, 5, \dots$. Our results, corresponding to $V_1^{(r)}$, $V^{(i)}$ and $V_2^{(r)}$ given by (5), are summarized below:

$$\begin{aligned} S_0 &= -\frac{\beta}{\alpha} X, & S_1 &= -\frac{\beta}{\alpha} X^2, & S_3 &= \frac{\beta}{3\alpha^2}, \\ V_{\text{eff}}^{(2)} &= \frac{1}{4\alpha} (3\beta^2 - 4\alpha\gamma) X^4 - \frac{\beta^2}{2\alpha^2}, & M^{(2)} &= \frac{3\beta^2}{2\alpha^2} X^2. \end{aligned} \quad (22)$$

On the other hand, the operator Q_3 contains up to fifth power in P only. The coefficient functions can be calculated by solving equation (13) and the results used to determine $h^{(4)}(X, P)$ through equation (15).

From (6) we find, on going back to variables and operators with dimensions, the following form of the equivalent Hermitian PDM Hamiltonian to PTQAO:

$$h(x, p) = \frac{1}{2\mu_0}p \left(1 + \frac{3\varepsilon^2 b^2}{2a^2}x^2 \right) p + ax^2 - \frac{\hbar^2 \varepsilon^2 b^2}{2\mu_0 a^2} + \frac{\varepsilon^2}{4a}(3b^2 - 4ac)x^4 + \varepsilon^4 h^{(4)}(x, p) + O(\varepsilon^6), \quad (23)$$

where

$$h^{(4)}(x, p) = \frac{b^4}{32a^6} \left(\frac{p^6}{\mu_0^3} - \frac{18a}{\mu_0^2} \{x^2, p^4\} - \frac{51a^2}{2\mu_0} \{x^4, p^2\} - 14a^3 x^6 - \frac{81\hbar^2 a}{\mu_0^2} p^2 - \frac{138\hbar^2 a^2}{\mu_0} x^2 \right) + \frac{3b^2 c}{2a^4} \left(\frac{1}{2\mu_0^2} \{x^2, p^4\} + \frac{3a}{2\mu_0} \{x^4, p^2\} + a^2 x^6 + \frac{2\hbar^2}{\mu_0^2} p^2 + \frac{8\hbar^2 a}{\mu_0} x^2 \right) \quad (24)$$

has been written in terms of anticommutators. It is evident that inclusion of higher-order corrections will make the structure of $h(x, p)$ more complicated.

Using the similarity transformation induced by ρ , it is straightforward to obtain the physical position and momentum operators [2]

$$x_{\text{phys}} = x + \frac{i\varepsilon b}{2\mu_0 a^2} (p^2 + \mu_0 a x^2) + \frac{\varepsilon^2 b^2}{8\mu_0 a^3} (\{x, p^2\} - 2\mu_0 a x^3) - \frac{i\varepsilon^3 b^3}{8\mu_0^2 a^5} [5p^4 + 6\mu_0 a \{x^2, p^2\} + \mu_0 a (5\mu_0 a x^4 + 3\hbar^2)] + \frac{i\varepsilon^3 b c}{2\mu_0^2 a^4} [2p^4 + 3\mu_0 a \{x^2, p^2\} + 2\mu_0 a (\mu_0 a x^4 + \hbar^2)], \quad (25)$$

$$p_{\text{phys}} = p - \frac{i\varepsilon b}{2a} \{x, p\} + \frac{\varepsilon^2 b^2}{8\mu_0 a^3} (2p^3 - \mu_0 a \{x^2, p\}) + \frac{i\varepsilon^3 b^3}{4\mu_0 a^4} (\{x, p^3\} + 4\mu_0 a \{x^3, p\}) - \frac{i\varepsilon^3 b c}{\mu_0 a^3} (\{x, p^3\} + 2\mu_0 a \{x^3, p\}), \quad (26)$$

where the results have been written up to third order.

Finally, the classical Hamiltonian $\mathcal{H}_c(x_c, p_c)$ can be derived by replacing x and p in $h(x, p)$ by the classical variables x_c and p_c and proceeding to the limit $\hbar \rightarrow 0$ (assuming that the limit exists), i.e., $\mathcal{H}_c(x_c, p_c) = \lim_{\hbar \rightarrow 0} h(x_c, p_c)$. Our result is

$$\begin{aligned} \mathcal{H}_c = & \frac{p_c^2}{2\mu_0} + ax_c^2 + \varepsilon^2 \left[\frac{3b^2}{4a^2} \left(\frac{1}{\mu_0} x_c^2 p_c^2 + ax_c^4 \right) - cx_c^4 \right] \\ & + \varepsilon^4 \left[\frac{b^4}{32a^6} \left(\frac{p_c^6}{\mu_0^3} - \frac{36a}{\mu_0^2} x_c^2 p_c^4 - \frac{51a^2}{\mu_0} x_c^4 p_c^2 - 14a^3 x_c^6 \right) \right. \\ & \left. + \frac{3b^2 c}{2a^4} \left(\frac{1}{\mu_0^2} x_c^2 p_c^4 + \frac{3a}{\mu_0} x_c^4 p_c^2 + a^2 x_c^6 \right) \right] + O(\varepsilon^6). \end{aligned} \quad (27)$$

It is clear from (27) that, for sufficiently small ε so that terms of order ε^4 and higher can be neglected, \mathcal{H}_c describes a point particle with a PDM

$$\frac{\mu_0}{1 + 3\varepsilon^2 b^2 x_c^2 / (2a^2)} \simeq \mu_0 \left(1 - \frac{3\varepsilon^2 b^2}{2a^2} x_c^2 \right), \quad (28)$$

interacting with a quartic anharmonic potential of strength $\varepsilon^2(3b^2 - 4ac)/(4a)$. It is worth noting that the PDM (28) is the same as that obtained for the PTQAO [1]. Also, an important point to observe is that the quartic potential may be attractive, null or repulsive according to whether $3b^2 - 4ac$ is positive, vanishing or negative.

We should point out that employing x_{phys} and p_{phys} would enable one to obtain their classical counterparts. However, this would necessitate a formulation of classical mechanics having a complex phase-space structure [12, 13]. This is beyond the scope of the present paper.

To conclude, we have explored the relationship between the PTQAO model and the corresponding Hermitian PDM Hamiltonian in the framework of perturbation theory. We have also constructed the nonlocal physical position and momentum operators and the classical PDM Hamiltonian associated with PTQAO.

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