PT-SYMMETRIC NON-POLYNOMIAL OSCILLATORS AND HYPERBOLIC POTENTIAL WITH TWO KNOWN REAL EIGENVALUES IN A SUSY FRAMEWORK

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Abstract

Extending the supersymmetric method proposed by Tkachuk to the complex domain, we obtain general expressions for superpotentials allowing generation of quasi-exactly solvable PT-symmetric potentials with two known real eigenvalues (the ground state and first-excited state energies). We construct examples, namely those of complexified non-polynomial oscillators and of a complexified hyperbolic potential, to demonstrate how our scheme works in practice. For the former we provide a connection with the sl(2) method, illustrating the comparative advantages of the supersymmetric one.

Running head: SUSY and PT-symmetric potentials

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1 Introduction

Non-Hermitian Hamiltonians, in particular the PT-symmetric ones, are of great current interest (see e.g. [1]–[12] and references quoted therein). The main reason for this is that PT invariance, in a number of cases, leads to energy eigenvalues that are real. In this regard, the early work of Bender and Boettcher [1] is noteworthy since it has sparked off some very interesting developments subsequently.

Some years ago, Tkachuk [13] proposed a supersymmetric (SUSY) method for generating quasi-exactly solvable (QES) potentials with two known eigenstates, which correspond to the wavefunctions of the ground state and the first excited state. A distinctive feature of this method is that in contrast with other ones, it does not require the knowledge of an initial QES potential for constructing a new one. Later on, the procedure was extended to deal with QES potentials with two arbitrary eigenstates [14] or with three eigenstates [15]. Quite recently, Brihaye *et al.* [16] established a connection between the Tkachuk approach and the Turbiner one, based upon the finite-dimensional representations of sl(2) [17].

In this letter, we pursue Tkachuk's ideas further by considering an extension of his results to the case of PT-symmetric potentials. As a consequence, we arrive at a pair of solutions, one of which is obtained by a straightforward and natural complexification of Tkachuk's results while the other is new. We demonstrate the applicability of our scheme by focussing on two specific potentials, both of which are PT-symmetric.

2 Procedure

2.1 The Hermitian case

Let us start with the Hermitian SUSY case, where the two known eigenstates are the ground state and the first excited state. As is well known [18], the SUSY partner Hamiltonians are given by

$$H^{(+)} = \bar{A}A = -\frac{d^2}{dx^2} + V^{(+)}(x), \qquad H^{(-)} = A\bar{A} = -\frac{d^2}{dx^2} + V^{(-)}(x), \tag{1}$$

for a vanishing factorization energy. Here A and $\bar{A}$ are taken to be first-derivative differential operators, namely

$$A = \frac{d}{dx} + W(x), \qquad \bar{A} = -\frac{d}{dx} + W(x), \tag{2}$$

where $W(x)$ is the underlying superpotential and $V^{(\pm)}(x)$ are the usual SUSY partner potentials

$$V^{(\pm)}(x) = W^2(x) \mp W'(x). \tag{3}$$

We assume SUSY to be unbroken with the ground state of the SUSY Hamiltonian $H_s \equiv \mathrm{diag}(H^{(+)}, H^{(-)})$ to belong to $H^{(+)}$:

$$H^{(+)}\psi_0^{(+)}(x) = 0, \qquad \psi_0^{(+)}(x) = C_0^{(+)} \exp\left(-\int^x W(t)dt\right), \tag{4}$$

$C_0^{(+)}$ being the normalization constant. Equation (4) implies

$$\mathrm{sgn}(W(\pm\infty)) = \pm 1. \tag{5}$$

In the above formulation of SUSY, the eigenvalues of $H^{(+)}$ and $H^{(-)}$ are related as

$$E_{n+1}^{(+)} = E_n^{(-)}, \qquad E_0^{(+)} = 0, \tag{6}$$

while the corresponding eigenfunctions are intertwined according to

$$\psi_{n+1}^{(+)}(x) = \frac{1}{\sqrt{E_n^{(-)}}} \bar{A}\psi_n^{(-)}(x), \qquad \psi_n^{(-)}(x) = \frac{1}{\sqrt{E_{n+1}^{(+)}}} A\psi_{n+1}^{(+)}(x). \tag{7}$$

Following Tkachuk [13], let us consider expressing $H^{(-)}$ in the form

$$H^{(-)} = H_1^{(+)} + \epsilon = A_1\bar{A}_1 + \epsilon, \tag{8}$$

where $\epsilon = E_1^{(+)} = E_0^{(-)}$ corresponds to the energy of the first excited state of $H^{(+)}$ or of the ground state of $H^{(-)}$ and the operators A_1 and $\bar{A}_1$ are such that relations analogous to (2) hold in terms of a new superpotential $W_1(x)$. Thus we can write

$$V^{(-)}(x) = V_1^{(+)}(x) + \epsilon \tag{9}$$

with $V_1^{(+)}(x) = W_1^2(x) - W_1'(x)$ from (3).

The ground state wave function of $H_1^{(+)}$ (or $H^{(-)}$) can be read off from (4) as

$$\psi_0^{(-)}(x) = C_0^{(-)} \exp\left(-\int^x W_1(t)dt\right), \tag{10}$$

where $C_0^{(-)}$ is the normalization constant and

$$\operatorname{sgn}(W_1(\pm\infty)) = \pm 1. \tag{11}$$

On the other hand, the wave function of the first excited state of $H^{(+)}$ is obtained in the form

$$\psi_1^{(+)}(x) = C_1^{(+)} \bar{A} \exp\left(-\int^x W_1(t)dt\right), \qquad C_1^{(+)} = \frac{C_0^{(-)}}{\sqrt{\epsilon}}, \tag{12}$$

as guided by (7) and (10).

Using (3) and (9), it is clear that the superpotentials $W(x)$ and $W_1(x)$ have to satisfy a constraint

$$W^2(x) + W'(x) = W_1^2(x) - W_1'(x) + \epsilon. \tag{13}$$

The problem therefore amounts to finding a set of functions $W(x)$ and $W_1(x)$ satisfying (13) for some $\epsilon > 0$, along with the conditions (5) and (11). For this purpose, Tkachuk introduced the following combinations

$$W_\pm(x) = W_1(x) \pm W(x), \tag{14}$$

which transform (13) into

$$W_+'(x) = W_-(x)W_+(x) + \epsilon. \tag{15}$$

An advantage with Eq. (15) is that it readily gives $W_-(x)$ in terms of $W_+(x)$:

$$W_-(x) = \frac{W_+'(x) - \epsilon}{W_+(x)}. \tag{16}$$

Consequently the representations

$$W(x) = \frac{1}{2}\left[W_+(x) - \frac{W_+'(x) - \epsilon}{W_+(x)}\right], \qquad W_1(x) = \frac{1}{2}\left[W_+(x) + \frac{W_+'(x) - \epsilon}{W_+(x)}\right] \tag{17}$$

satisfy Eq. (13). In (17), $W_+(x)$ is some function for which $W(x)$ and $W_1(x)$ fulfil conditions (5) and (11). This means

$$\operatorname{sgn}(W_+(\pm\infty)) = \pm 1. \tag{18}$$

Restricting to continuous functions $W_+(x)$, the above condition reflects that $W_+(x)$ must have at least one zero. Then from (16) and (17), $W_-(x)$, $W(x)$, and $W_1(x)$ may have poles. Tkachuk considered the case when $W_+(x)$ has only one simple zero at $x = x_0$. In the neighbourhood of x_0, one gets

$$\frac{W'_+(x) - \epsilon}{W_+(x)} \simeq \frac{W'_+(x_0) - \epsilon + (x - x_0)W''_+(x_0) + \cdots}{(x - x_0)W'_+(x_0) + \cdots}, \tag{19}$$

so that the superpotentials will be free of singularity if one chooses

$$\epsilon = W'_+(x_0). \tag{20}$$

One is then led to

$$W(x) = \frac{1}{2}\left[W_+(x) - \frac{W'_+(x) - W'_+(x_0)}{W_+(x)}\right], \quad W_1(x) = \frac{1}{2}\left[W_+(x) + \frac{W'_+(x) - W'_+(x_0)}{W_+(x)}\right]. \tag{21}$$

To summarize, provided the continuous function $W_+(x)$ with a single pole at $x = x_0$ is such that $W(x)$ and $W_1(x)$, given by (21), satisfy conditions (5) and (11), the Hamiltonian $H^{(+)}$ has two known eigenvalues 0 and ϵ, given by (20), with corresponding eigenfunctions (4) and

$$\psi_1^{(+)}(x) = C_1^{(+)} W_+(x) \exp\left(-\int^x W_1(t)dt\right). \tag{22}$$

In deriving (22), use is made of (2), (12), and (14).

2.2 The non-Hermitian case

In the non-Hermitian case, we have to deal with complex potentials. Consider the decomposition [3]

$$W(x) = f(x) + \mathrm{i}g(x), \qquad V^{(\pm)}(x) = V_R^{(\pm)}(x) + \mathrm{i}V_I^{(\pm)}(x), \tag{23}$$

where f, g, $V_R^{(\pm)}$, $V_I^{(\pm)} \in \mathbb{R}$ and

$$V_R^{(\pm)} = f^2 - g^2 \mp f', \qquad V_I^{(\pm)} = 2fg \mp g'. \tag{24}$$

If $V^{(\pm)}(x)$ are PT-symmetric, then $f(x)$ and $g(x)$ are odd and even functions, respectively.

It may be noted that Eqs. (4) – (13) remain true for some real and positive ϵ. Employing the separation

$$W_1(x) = f_1(x) + \mathrm{i} g_1(x), \qquad V_1^{(+)}(x) = V_{1R}^{(+)}(x) + \mathrm{i} V_{1I}^{(+)}(x) \tag{25}$$

with f_1, g_1, $V_{1R}^{(+)}$, $V_{1I}^{(+)} \in \mathbb{R}$, the behaviour of $f_1(x)$ and $g_1(x)$ also turns out to be odd and even, respectively, should $V^{(+)}(x)$ be PT-symmetric.

In the non-Hermitian case, the conditions (5) and (11) are to be replaced by

$$\mathrm{sgn}(f(\pm\infty)) = \mathrm{sgn}(f_1(\pm\infty)) = \pm 1. \tag{26}$$

These conditions are actually compatible with the odd character of f and f_1.

On introducing the first relations of (23) and (25) into (13) and splitting into real and imaginary parts, we get the system of two equations

$$f^2 - g^2 + f' = f_1^2 - g_1^2 - f_1' + \epsilon, \qquad 2fg + g' = 2f_1 g_1 - g_1'. \tag{27}$$

The problem now amounts to finding a pair of odd functions f, f_1 and a pair of even functions g, g_1, satisfying Eq. (27) for some $\epsilon > 0$, as well as the conditions (26).

To this end, we introduce the linear combinations

$$f_\pm(x) = f_1(x) \pm f(x), \qquad g_\pm(x) = g_1(x) \pm g(x), \tag{28}$$

which replace the constraints (27) by

$$f_+' = f_- f_+ - g_- g_+ + \epsilon, \qquad g_+' = f_+ g_- + f_- g_+. \tag{29}$$

Solving for f_- and g_- in terms of f_+ and g_+, we get

$$f_- = \frac{(f_+' - \epsilon) f_+ + g_+' g_+}{f_+^2 + g_+^2}, \qquad g_- = \frac{-(f_+' - \epsilon) g_+ + g_+' f_+}{f_+^2 + g_+^2}. \tag{30}$$

Hence the functions

$$\begin{aligned} f &= \frac{1}{2}\left[f_+ - \frac{(f_+' - \epsilon) f_+ + g_+' g_+}{f_+^2 + g_+^2}\right], \qquad g = \frac{1}{2}\left[g_+ + \frac{(f_+' - \epsilon) g_+ - g_+' f_+}{f_+^2 + g_+^2}\right], \\ f_1 &= \frac{1}{2}\left[f_+ + \frac{(f_+' - \epsilon) f_+ + g_+' g_+}{f_+^2 + g_+^2}\right], \qquad g_1 = \frac{1}{2}\left[g_+ - \frac{(f_+' - \epsilon) g_+ - g_+' f_+}{f_+^2 + g_+^2}\right] \end{aligned} \tag{31}$$

satisfy the coupled equations (27). In (31), f_+ must be such that the conditions (26) are fulfilled. These suggest

$$\mathrm{sgn}(f_+(\pm\infty)) = \pm 1. \tag{32}$$

If we restrict ourselves to continuous functions $f_+(x)$ and $g_+(x)$, the condition (32) shows that $f_+(x)$ must have at least one zero. For simplicity's sake, we assume that $f_+(x)$ has only one simple zero at $x = x_0$. This means that the parity operation is defined with respect to a mirror placed at $x = x_0$. Thus in the neighbourhood of x_0, we get

$$\frac{(f'_+ - \epsilon) f_+ + g'_+ g_+}{f_+^2 + g_+^2} \tag{33}$$

$$\simeq (x - x_0) \frac{[f'_+(x_0) - \epsilon] f'_+(x_0) + g_+(x_0) g''_+(x_0)}{[g_+(x_0)]^2} + \cdots \qquad \text{if } g_+(x_0) \neq 0$$

$$\simeq \frac{1}{x - x_0} \frac{f'_+(x_0) - \epsilon}{f'_+(x_0)} + \cdots \qquad \text{if } g_+(x_0) = 0, \tag{34}$$

$$\frac{(f'_+ - \epsilon) g_+ - g'_+ f_+}{f_+^2 + g_+^2} \tag{35}$$

$$\simeq \frac{f'_+(x_0) - \epsilon}{g_+(x_0)} + \cdots \qquad \text{if } g_+(x_0) \neq 0$$

$$\simeq -\frac{[f'_+(x_0) + \epsilon] g''_+(x_0)}{2[f'_+(x_0)]^2} + \cdots \qquad \text{if } g_+(x_0) = 0. \tag{36}$$

Hence the superpotentials will be free of singularity if either $g_+(x_0) \neq 0$ and ϵ arbitrary or $g_+(x_0) = 0$ and $\epsilon = f'_+(x_0)$ (or $\epsilon = W'_+(x_0)$ since $g'_+(x_0) = 0$).

The superpotentials may therefore be written as

$$W(x) = \frac{1}{2} \left[W_+(x) - \frac{W'_+(x) - \epsilon}{W_+(x)} \right], \qquad W_1(x) = \frac{1}{2} \left[W_+(x) + \frac{W'_+(x) - \epsilon}{W_+(x)} \right], \tag{37}$$

if $W_+(x_0) = \mathrm{i} g_+(x_0) \neq 0$ and $\epsilon > 0$, or

$$W(x) = \frac{1}{2} \left[W_+(x) - \frac{W'_+(x) - W'_+(x_0)}{W_+(x)} \right], \qquad W_1(x) = \frac{1}{2} \left[W_+(x) + \frac{W'_+(x) - W'_+(x_0)}{W_+(x)} \right], \tag{38}$$

if $W_+(x_0) = \mathrm{i} g_+(x_0) = 0$ and $\epsilon = W'_+(x_0)$.

To summarize, in the non-Hermitian PT-symmetric case, we get two types of solutions: the one given by (38) is obtained as a straightforward consequence of the complexification of Tkachuk's result, while the other given by (37) is new.

3 Applications

3.1 A family of complexified non-polynomial oscillators

As the first application of our scheme we consider the case of a family of complexified non-polynomial oscillators. This corresponds to the choice $f_+ = ax$, $g_+ = bx^{2m}$ ($a > 0$, $b \neq 0$, $m \in \mathbb{N}$), which leads to

$$W_+(x) = ax + \mathrm{i}bx^{2m}, \qquad m \in \mathbb{N}. \tag{39}$$

Here $x_0 = 0$ and the first type of solutions, given in (37), applies to the case $m = 0$, while the second one, given in (38), has to be used for $m \in \mathbb{N}_0$.

The former case reduces to the well-studied exactly solvable PT-symmetric oscillator potential [4, 11, 12], thus partly accounting for the name of the family of potentials. Indeed, to get it in the standard form we have to set $a = 2$, $b = -2c$, and $\epsilon = 4\alpha$. Then $V^{(+)}$ assumes the form

$$V^{(+)}(x) = (x - \mathrm{i}c)^2 + 2(\alpha - 1) + \frac{\alpha^2 - \frac{1}{4}}{(x - \mathrm{i}c)^2}, \tag{40}$$

along with

$$W(x) = x - \mathrm{i}c + \frac{\alpha - \frac{1}{2}}{x - \mathrm{i}c}, \qquad W_1(x) = x - \mathrm{i}c - \frac{\alpha - \frac{1}{2}}{x - \mathrm{i}c}. \tag{41}$$

These agree with the two independent forms of the complex superpotentials proposed by us previously [11] in connection with para-SUSY and second-derivative SUSY of (40). The potential (40) can be looked upon as a transformed three-dimensional radial oscillator for the complex shift $x \to x - \mathrm{i}c$, $c > 0$, and replacing the angular momentum parameter l by $\alpha - \frac{1}{2}$. The presence of a centrifugal-like core notwithstanding, the shift of the singularity off the integration path makes (40) exactly solvable on the entire real line for any $\alpha > 0$ like the harmonic oscillator to which (40) reduces for $\alpha = \frac{1}{2}$ and $c = 0$.

In contrast, the case $m \in \mathbb{N}_0$ is entirely new. For such m values, the superpotentials are given by

$$W(x) = \frac{1}{2}\left[ax + \mathrm{i}bx^{2m} - \frac{2m\mathrm{i}bx^{2m-2}}{a + \mathrm{i}bx^{2m-1}}\right], \qquad W_1(x) = \frac{1}{2}\left[ax + \mathrm{i}bx^{2m} + \frac{2m\mathrm{i}bx^{2m-2}}{a + \mathrm{i}bx^{2m-1}}\right], \tag{42}$$

where we used $\epsilon = W_+'(0) = a$.

The corresponding potentials turn out to be

$$V^{(+)}(x) = \frac{1}{4}\Bigg[-b^2x^{4m} + 2\mathrm{i}abx^{2m+1} - 8m\mathrm{i}bx^{2m-1} + a^2x^2 - 2a + \frac{4m(m-1)\mathrm{i}bx^{2m-3}}{a+\mathrm{i}bx^{2m-1}} + \frac{4m(m-1)\mathrm{i}abx^{2m-3}}{(a+\mathrm{i}bx^{2m-1})^2}\Bigg], \qquad m = 1, 2, 3, \ldots, \tag{43}$$

which are QES and are seen to be PT-symmetric as well. The ground and first-excited state wave functions corresponding to the above QES potentials are easily determined to be

$$\psi_0^{(+)}(x) \propto (a+\mathrm{i}bx^{2m-1})^{m/(2m-1)} \exp\left[-\frac{1}{4}ax^2 - \frac{\mathrm{i}b}{2(2m+1)}x^{2m+1}\right], \tag{44}$$

$$\psi_1^{(+)}(x) \propto x(a+\mathrm{i}bx^{2m-1})^{(m-1)/(2m-1)} \exp\left[-\frac{1}{4}ax^2 - \frac{\mathrm{i}b}{2(2m+1)}x^{2m+1}\right]. \tag{45}$$

It is worth noting that the first member of the set (43) obtained for $m = 1$,

$$V^{(+)}(x) = \frac{1}{4}\left(-b^2x^4 + 2\mathrm{i}abx^3 + a^2x^2 - 8\mathrm{i}bx - 2a\right), \tag{46}$$

is a quartic potential differing from the known QES ones [2, 9]. All the remaining members of the set, starting with that associated with $m = 2$,

$$V^{(+)}(x) = \frac{1}{4}\left[-b^2x^8 + 2\mathrm{i}abx^5 - 16\mathrm{i}bx^3 + a^2x^2 - 2a + \frac{8\mathrm{i}bx}{a+\mathrm{i}bx^3} + \frac{8\mathrm{i}abx}{(a+\mathrm{i}bx^3)^2}\right], \tag{47}$$

are non-polynomial potentials. As for the PT-symmetric oscillator (40), the shift of the singularity off the integration path makes such potentials QES.

On introducing the new variable $z = x(a+\mathrm{i}bx^{2m-1})^{-1/(2m-1)}$, the first-excited state wave function (45) can be rewritten in terms of the ground state one (44) as $\psi_1^{(+)}(z) \propto z\psi_0^{(+)}(z)$. By setting in general $\psi_n^{(+)}(z) = \psi_0^{(+)}(z)\phi_n^{(+)}(z)$, where $\phi_0^{(+)}(z) \propto 1$ and $\phi_1^{(+)}(z) \propto z$, the Schrödinger equation for the potentials (43) is transformed into the differential equation

$$T\phi_n^{(+)}(z) \equiv \left[-a^{-2/(2m-1)}(1-\mathrm{i}bz^{2m-1})^{4m/(2m-1)}\frac{d^2}{dz^2} + az\frac{d}{dz}\right]\phi_n^{(+)}(z) = E_n^{(+)}\phi_n^{(+)}(z). \tag{48}$$

For $m = 1$, the coefficient of the second-order differential operator in (48) becomes a quartic polynomial in z, thus showing that T can be expressed as a quadratic combination of the three sl(2) generators

$$J_+ = z^2\frac{d}{dz} - Nz, \qquad J_0 = z\frac{d}{dz} - \frac{N}{2}, \qquad J_- = \frac{d}{dz}, \tag{49}$$

corresponding to the two-dimensional irreducible representation (i.e., $N = 1$ in (49)) [16, 17]. The result reads

$$T = a^{-2}\left(-b^4 J_+^2 - 4\mathrm{i}b^3 J_+ J_0 + 6b^2 J_+ J_- + 4\mathrm{i}b J_0 J_- - J_-^2 - 2\mathrm{i}b^3 J_+ + 6b^2 J_0 + 2\mathrm{i}b J_- + 3b^2\right). \tag{50}$$

For higher m values, the differential operator T contains a non-vanishing element of the kernel [16]. It is worth stressing that in such a case, the sl(2) method becomes quite ineffective for constructing new QES potentials, whereas the SUSY one is not subject to such restrictions.

3.2 A complexified hyperbolic potential

Our next example is that of a complexified hyperbolic potential induced by the representations $f_+ = A\sinh\alpha x$, $g_+ = B$ (A, $\alpha > 0$, $B \neq 0$). We then get

$$W_+(x) = A\sinh\alpha x + \mathrm{i}B, \tag{51}$$

which gives for $x_0 = 0$

$$W(x) = \frac{1}{2}\left[A\sinh\alpha x + \mathrm{i}B - \frac{A\alpha\cosh\alpha x - \epsilon}{A\sinh\alpha x + \mathrm{i}B}\right], \tag{52}$$

$$W_1(x) = \frac{1}{2}\left[A\sinh\alpha x + \mathrm{i}B + \frac{A\alpha\cosh\alpha x - \epsilon}{A\sinh\alpha x + \mathrm{i}B}\right]. \tag{53}$$

The resulting expression for the complexified hyperbolic potential is

$$V^{(+)}(x) = \frac{1}{4}\left[A^2\sinh^2\alpha x - 4A\alpha\cosh\alpha x + 2\epsilon + \alpha^2 - B^2 + 2\mathrm{i}AB\sinh\alpha x + \frac{\epsilon^2 - \alpha^2(A^2 - B^2)}{(A\sinh\alpha x + \mathrm{i}B)^2}\right]. \tag{54}$$

Clearly $V^{(+)}(x)$ is PT-symmetric. The two known eigenstates of (54) correspond to the ground state and first excited state as outlined earlier. These are

$$\psi_0^{(+)}(x) \propto (A\cosh\alpha x - \nu)^{\frac{1}{4}\left(1-\frac{\epsilon}{\alpha\nu}\right)}(A\cosh\alpha x + \nu)^{\frac{1}{4}\left(1+\frac{\epsilon}{\alpha\nu}\right)} \times \exp\left[-\frac{A}{2\alpha}\cosh\alpha x - \frac{1}{2}\mathrm{i}Bx - \frac{\mathrm{i}}{2}\arctan\left(\frac{A}{B}\sinh\alpha x\right)\right.$$

$$
\begin{aligned}
&\quad + \frac{\mathrm{i}\epsilon}{2\alpha\nu} \arctan\left(\frac{\nu}{B}\tanh\alpha x\right)\Bigg] \qquad \text{if } 0 < B^2 < A^2, \\
&\propto \quad \cosh\frac{\alpha x}{2} \exp\left(-\frac{A}{2\alpha}\cosh\alpha x\right) \qquad \text{if } B = 0, \\
&\propto \quad \sqrt{\cosh\alpha x} \exp\left[-\frac{A}{2\alpha}\cosh\alpha x - \frac{1}{2}\mathrm{i}\delta A x + \frac{\epsilon}{2A\alpha}(\operatorname{sech}\alpha x + \mathrm{i}\delta\tanh\alpha x)\right] \\
&\qquad \text{if } B^2 = A^2, \\
&\propto \quad (B^2 + A^2\sinh^2\alpha x)^{1/4} \exp\Bigg[-\frac{A}{2\alpha}\cosh\alpha x - \frac{\epsilon}{2\alpha\mu}\arctan\left(\frac{A\cosh\alpha x}{\mu}\right) \\
&\quad - \frac{1}{2}\mathrm{i}Bx - \frac{\mathrm{i}}{2}\arctan\left(\frac{A}{B}\sinh\alpha x\right) + \frac{\mathrm{i}\epsilon}{2\alpha\mu}\operatorname{arctanh}\left(\frac{\mu}{B}\tanh\alpha x\right)\Bigg] \\
&\qquad \text{if } B^2 > A^2,
\end{aligned} \tag{55}
$$

and

$$
\begin{aligned}
\psi_1^{(+)}(x) \quad &\propto \quad (A\sinh\alpha x + \mathrm{i}B)(A\cosh\alpha x - \nu)^{-\frac{1}{4}\left(1-\frac{\epsilon}{\alpha\nu}\right)}(A\cosh\alpha x + \nu)^{-\frac{1}{4}\left(1+\frac{\epsilon}{\alpha\nu}\right)} \\
&\quad \times \exp\Bigg[-\frac{A}{2\alpha}\cosh\alpha x - \frac{1}{2}\mathrm{i}Bx + \frac{\mathrm{i}}{2}\arctan\left(\frac{A}{B}\sinh\alpha x\right) \\
&\quad + \frac{\mathrm{i}\epsilon}{2\alpha\nu}\arctan\left(\frac{\nu}{B}\tanh\alpha x\right)\Bigg] \qquad \text{if } 0 < B^2 < A^2, \\
&\propto \quad \sinh\frac{\alpha x}{2}\exp\left(-\frac{A}{2\alpha}\cosh\alpha x\right) \qquad \text{if } B = 0, \\
&\propto \quad (\sinh\alpha x + \mathrm{i}\delta)\sqrt{\operatorname{sech}\alpha x} \\
&\quad \times \exp\left[-\frac{A}{2\alpha}\cosh\alpha x - \frac{1}{2}\mathrm{i}\delta Ax - \frac{\epsilon}{2A\alpha}(\operatorname{sech}\alpha x + \mathrm{i}\delta\tanh\alpha x)\right] \qquad \text{if } B^2 = A^2, \\
&\propto \quad (A\sinh\alpha x + \mathrm{i}B)(B^2 + A^2\sinh^2\alpha x)^{-1/4} \\
&\quad \times \exp\Bigg[-\frac{A}{2\alpha}\cosh\alpha x + \frac{\epsilon}{2\alpha\mu}\arctan\left(\frac{A\cosh\alpha x}{\mu}\right) - \frac{1}{2}\mathrm{i}Bx \\
&\quad + \frac{\mathrm{i}}{2}\arctan\left(\frac{A}{B}\sinh\alpha x\right) - \frac{\mathrm{i}\epsilon}{2\alpha\mu}\operatorname{arctanh}\left(\frac{\mu}{B}\tanh\alpha x\right)\Bigg] \qquad \text{if } B^2 > A^2,
\end{aligned} \tag{56}
$$

where $\nu = \sqrt{A^2 - B^2}$, $\mu = \sqrt{B^2 - A^2}$, and $\delta = \operatorname{sgn}(B)$.

In (55) and (56), we have included the case $B = 0$ for which the potential $V^{(+)}(x)$ of Eq. (54) reduces to one of the potentials studied by Tkachuk [13], which itself is a special case of the Razavy potential [19].

4 Conclusion

To conclude, we have carried out in this paper a complexification of the SUSY method proposed recently by Tkachuk. This allows us to generate QES PT-symmetric potentials with two known real eigenvalues. We have also constructed suitable examples, namely those of a family of complexified non-polynomial oscillators and of a complexified hyperbolic potential, which serve to illustrate the viability of our scheme. For the former, we have also provided a connection with the sl(2) method, which illustrates the comparative advantages of the SUSY one.

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