Pseudo-Hermiticity, weak pseudo-Hermiticity and η-orthogonality condition

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Abstract

We discuss certain features of pseudo-Hermiticity and weak pseudo-Hermiticity conditions and point out that, contrary to a recent claim, there is no inconsistency if the correct orthogonality condition is used for the class of pseudo-Hermitian, PT-symmetric Hamiltonians of the type $H_\beta = [p + \mathrm{i}\beta\nu(x)]^2/2m + V(x)$.

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In recent times it has been stressed that neither Hermiticity nor PT symmetry serves as a necessary condition for a quantum Hamiltonian to preserve the reality of its bound-state eigenvalues [1, 2, 3, 4, 5]. In fact, it has been realized [5] that the existence of real eigenvalues can be associated with a non-Hermitian Hamiltonian provided it is η-pseudo-Hermitian:

$$\eta H = H^\dagger \eta, \tag{1}$$

where η is a Hermitian linear automorphism and, assuming $\hbar = 2m = 1$,

$$H = p^2 + V(x) \tag{2}$$

for $V(x) \in \mathbb{C}$ and $p = -\mathrm{i}\partial_x$. Then, in such a case, the spectrum of a diagonalizable H is real if there exists a linear invertible operator O such that $\eta = (OO^\dagger)^{-1}$. Moreover, one can relax H to be only weak pseudo-Hermitian [6] by not restricting η to be Hermitian.

The purpose of this Letter is to establish the following results:

(i) The twin concepts of pseudo-Hermiticity and weak pseudo-Hermiticity are complementary to one another.

(ii) For a first-order differential realization, η may be anti-Hermitian but for the second-order case, η is necessarily Hermitian. For both cases, we make connections to the same PT-symmetric Scarf II Hamiltonian (having normalizable eigenfunctions) to show that the choice of η is not unique in ascertaining the character of the Hamiltonian.

(iii) For the class of η-pseudo-Hermitian, PT-symmetric Hamiltonians described by [7, 8] [1]

$$H_\beta = [p + \mathrm{i}\beta\nu(x)]^2 + V(x), \qquad \beta \in \mathbb{R}, \tag{3}$$

where the odd function $\nu(x) \in \mathbb{R}$, $V(x)$ is PT-symmetric, and

$$\eta = \exp\left[-2\beta \int^x \nu(y)dy\right], \tag{4}$$

our earlier derivation [9] of the generalized continuity equation for Hamiltonians of the form (2) [with $V(x)$ PT-symmetric] can be extended to H_β as well. The resulting η-orthogonality condition needs to be implemented judiciously.

[1] Note that in Ref. [8], it is assumed that $\hbar = m = 1$ instead of $\hbar = 2m = 1$.

We begin by addressing to the point (i) above. Consider some non-Hermitian η that is subject to the condition (1). Taking Hermitian conjugate, we obtain, on adding and subtracting, the following combinations

$$\eta_+ H = H^\dagger \eta_+, \qquad \eta_- H = H^\dagger \eta_-, \tag{5}$$

where $\eta_\pm = \eta \pm \eta^\dagger$. While the first one of (5) corresponds to strict pseudo-Hermiticity, the second one points to weak pseudo-Hermiticity with a new anti-Hermitian operator η_-. Note that η_+ is Hermitian. It is thus clear that weak pseudo-Hermiticity is not more general than pseudo-Hermiticity but works complementary to it.

We now turn to (ii). Decomposing $V(x)$ and η as

$$\begin{aligned} V(x) &= V_R(x) + \mathrm{i}V_I(x), \\ \eta &= \frac{d}{dx} + f(x) + \mathrm{i}g(x), \end{aligned} \tag{6}$$

where V_R, V_I, f, $g \in \mathbb{R}$, we get, on inserting (6) into (1), the relations

$$\begin{aligned} V_I &= \mathrm{i}(f' + \mathrm{i}g'), \\ V_R' + \mathrm{i}V_I' &= -(f'' + \mathrm{i}g'') - 2\mathrm{i}V_I(f + \mathrm{i}g). \end{aligned} \tag{7}$$

In (7) the primes denote the order of differentiations with respect to the variable x. We are then led to the conditions

$$V_R' = -2gg', \qquad cg' = 0, \qquad c \in \mathbb{R}, \tag{8}$$

which imply the existence of two solutions corresponding to $c = 0$ and $g' = 0$, respectively. In the following we concentrate on the case $c = 0$ because $g' = 0$ yields a trivial result that corresponds to a real constant potential with no normalizable eigenfunction.

For the choice $c = 0$, it turns out that

$$f = 0, \qquad V_R = -g^2 + k, \qquad V_I = -g', \tag{9}$$

where $k \in \mathbb{R}$. In consequence, we have the results

$$\begin{aligned} V(x) &= -g^2(x) + k - \mathrm{i}g'(x), \\ \eta &= \frac{d}{dx} + \mathrm{i}g(x). \end{aligned} \tag{10}$$

The above form of $V(x)$ shows that, in the framework of supersymmetric quantum mechanics, we can associate to it an imaginary superpotential $W(x) = \mathrm{i}g(x)$, its partner being the complex conjugate potential. We also observe that, for even g functions, the representation of η makes it anti-Hermitian in character. Let us consider the following specific example for $g = d \operatorname{sech} x$, $d \in \mathbb{R}$. We get from (10)

$$\begin{aligned} V(x) &= -d^2 \operatorname{sech}^2 x + k + \mathrm{i}d \operatorname{sech} x \tanh x, \\ \eta &= \frac{d}{dx} + \mathrm{i}d \operatorname{sech} x. \end{aligned} \tag{11}$$

It is obvious that $V(x)$ is a particular case of the generalized PT-symmetric Scarf II potential investigated previously by us [10] in connection with the complex algebra sl(2, $\mathbb{C}$). A comparison with the results obtained there shows that, in the present case, we have a single series of real eigenvalues with normalizable eigenfunctions provided $d > \frac{1}{2}$. The corresponding Hamiltonian is both P-pseudo-Hermitian and η-weak-pseudo-Hermitian with η given by (11). Our example confirms the assertion [11] that, for a given non-Hermitian Hamiltonian, there could be infinitely many η satisfying the weak-pseudo-Hermiticity or the pseudo-Hermiticity condition.

We next attend to a second-order differential representation of η:

$$\eta = \frac{d^2}{dx^2} - 2p(x)\frac{d}{dx} + b(x), \tag{12}$$

where p, $b \in \mathbb{C}$. Substituting (12) into the condition (1), we obtain the constraints

$$\begin{aligned} b &= -p' + p^2 - \frac{p''}{2p} + \left(\frac{p'}{2p}\right)^2 + \frac{\gamma}{4p^2}, \\ V &= 2p' + p^2 + \frac{p''}{2p} - \left(\frac{p'}{2p}\right)^2 - \frac{\gamma}{4p^2} - \delta, \\ V^* &= -2p' + p^2 + \frac{p''}{2p} - \left(\frac{p'}{2p}\right)^2 - \frac{\gamma}{4p^2} - \delta, \end{aligned} \tag{13}$$

where γ, $\delta \in \mathbb{R}$. From the last two relations in (13), it is clear that $p(x)$ must be pure imaginary,

$$p(x) = \mathrm{i}a(x), \tag{14}$$

where $a(x) \in \mathbb{R}$. As such $V(x)$ and η acquire the forms

$$\begin{aligned} V(x) &= 2\mathrm{i}a' - a^2 + \frac{a''}{2a} - \left(\frac{a'}{2a}\right)^2 + \frac{\gamma}{4a^2} - \delta, \\ \eta &= \frac{d^2}{dx^2} - 2\mathrm{i}a(x)\frac{d}{dx} + b(x), \end{aligned} \tag{15}$$

with $b(x) = -V(x) + \mathrm{i}a' - 2a^2 - \delta$. In (15), η can be easily recognized to be a Hermitian operator since it can be written in the form $\eta = -\tilde{O}^\dagger \tilde{O}$, where $\tilde{O} = \frac{d}{dx} + r - \mathrm{i}a$, $\tilde{O}^\dagger = -\frac{d}{dx} + r + \mathrm{i}a$, and $r^2 - r' = \frac{a''}{2a} - \left(\frac{a'}{2a}\right)^2 + \frac{\gamma}{4a^2}$. In Ref. [12], such a decomposition of η was assumed, a priori, to arrive at some non-Hermitian Hamiltonians with real spectra.

Let us, however, confine ourselves to the following choice

$$a(x) = -\tfrac{1}{2}B(2A+1)\,\mathrm{sech}\,x, \qquad \gamma = 0, \qquad \delta = \tfrac{1}{4}, \tag{16}$$

where $A + \frac{1}{2} > 0$, $B > 0$, and $A - B + \frac{1}{2}$ is not an integer. We are again led to the PT-symmetric Scarf II potential having a more general form than obtained with the first-order differential realization of η:

$$V(x) = -V_1\,\mathrm{sech}^2\,x - \mathrm{i}V_2\,\mathrm{sech}\,x\tanh x, \tag{17}$$

where $V_1 = \frac{1}{4}[B^2(2A+1)^2 + 3] > 0$ and $V_2 = -B(2A+1) \neq 0$. According to Refs. [13, 14], the condition for real eigenvalues for the Hamiltonian corresponding to (17) is $|V_2| \leq V_1 + \frac{1}{4}$. Here it amounts to $[B(2A+1) - 2]^2 \geq 0$, which is always met.

Of particular interest is the special case $B = 1$:

$$V(x) = -(A^2 + A + 1)\,\mathrm{sech}^2\,x + \mathrm{i}(2A+1)\,\mathrm{sech}\,x\tanh x. \tag{18}$$

On setting $A + \frac{1}{2} = -\lambda$ $(\lambda < 0)$, Eq. (18) can be seen to reduce to the potential $V^{(1)} - \frac{1}{4}$ of Ref. [15] for $\mu = 1$. The associated energy levels of (18) are [10]: $E_n^{(-\lambda)} = -(\lambda + n + \frac{1}{2})^2$ and coincide with $E_n^{(2)} - \frac{1}{4}$ of [15]. Note that there is, in general, a doubling of energy levels in transiting from the real to the PT-symmetric Scarf II potential. In fact, the second algebra of sl(2, $\mathbb{C}$) leads to an additional energy level $E_0^{(1)} = -\frac{1}{4}$ that is consistent with the zero-energy state of [15].

Finally, we take up a general derivation of the continuity equation for the class of Hamiltonians H_β given by (3). The associated Schrödinger equation reads

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} = -\left(-\frac{\partial}{\partial x} + \beta\nu(x)\right)^2 \psi(x,t) + V(x)\psi(x,t). \tag{19}$$

From this it follows that the function $\psi^*(-x,t)$ satisfies

$$-\mathrm{i}\frac{\partial\psi^*(-x,t)}{\partial t} = -\left(-\frac{\partial}{\partial x} + \beta\nu(x)\right)^2 \psi^*(-x,t) + V(x)\psi^*(-x,t). \tag{20}$$

On considering Eq. (19) for some solution $\psi_1(x,t)$ and Eq. (20) for some other solution $\psi_2(x,t)$ and then multiplying (19) and (20) by $\exp\left[-2\beta\int^x \nu(y)dy\right]\psi_2^*(-x,t)$ and $\exp\left[-2\beta\int^x \nu(y)dy\right]\psi_1(x,t)$, respectively, we obtain, on subtracting, a natural generalization of the continuity equation for PT-symmetric quantum mechanics to its η-pseudo-Hermitian extension, namely

$$\frac{\partial P_\eta(x,t)}{\partial t} + \frac{\partial J_\eta(x,t)}{\partial x} = 0, \tag{21}$$

where

$$\begin{aligned} P_\eta(x,t) &= \eta\psi_2^*(-x,t)\psi_1(x,t), \\ J_\eta(x,t) &= \frac{\eta}{\mathrm{i}}\left[\psi_2^*(-x,t)\frac{\partial\psi_1(x,t)}{dx} - \psi_1(x,t)\frac{\partial\psi_2^*(-x,t)}{dx}\right], \end{aligned} \tag{22}$$

and η is defined in (4). If $\psi_1(x,t) \to 0$ and $\psi_2(x,t) \to 0$ as $x \to \pm\infty$, as is normally expected for bound-state wave functions, then integration of (21) over the entire real line gives the conservation law

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\, \eta\, \psi_2^*(-x,t)\psi_1(x,t) = 0. \tag{23}$$

In the case of energy eigenfunctions

$$\psi_1(x,t) = u_1(x)e^{-\mathrm{i}E_1 t}, \qquad \psi_2(x,t) = u_2(x)e^{-\mathrm{i}E_2 t}, \tag{24}$$

corresponding to the eigenvalues E_1 and E_2 respectively, Eq. (23) reduces to

$$(E_1 - E_2^*)\int_{-\infty}^{\infty} dx\, \eta\, u_2^*(-x)u_1(x) = 0. \tag{25}$$

Equation (25) represents the η-orthogonality condition [5]. Obviously it transforms to the PT-orthogonality [9, 16]

$$(E_1 - E_2^*) \int_{-\infty}^{\infty} dx\, u_2^*(-x) u_1(x) = 0 \tag{26}$$

for $\nu(x) = 0$. Indeed Eq. (25) can be derived from (26) by effecting a gauge transformation on the wave functions u of H in a manner $u \to \exp[-\int^x \beta\nu(y)dy]u$, $V(x)$ being PT-symmetric. As such H_β may be looked upon as a gauge-transformed version of H. However, it needs to be emphasized that care should be taken to correctly implement the normalization conditions deriving from (25) and (26) and which are appropriate to the Hamiltonians H_β and H, respectively. Thus although PT-symmetric, the form of the η-pseudo-Hermitian Hamiltonian H_β at once suggests that the normalization condition related to (25) is to be used rather than that connected with (26), a point overlooked in Ref. [8].

In summary, we have shown that η-pseudo-Hermiticity and weak pseudo-Hermiticity are essentially complementary concepts. We have provided an explicit example of PT-symmetric Scarf II model to demonstrate that η does not necessarily have a unique representation to determine the character of the associated non-Hermitian Hamiltonian. We have also pointed out the correct use of the η-orthogonality condition when dealing with a pseudo-Hermitian gauge-transformed Hamiltonian.

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