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**PAIRWISE PARTITION GRAPHS
AND
STRATEGY-PROOF SOCIAL CHOICE
IN THE EXOGENOUS
INDIFFERENCE CLASS MODEL**

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PAIRWISE PARTITION GRAPHS AND STRATEGY-PROOF SOCIAL CHOICE IN THE EXOGENOUS INDIFFERENCE CLASS MODEL

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Abstract

In this paper we consider the exogenous indifference classes model of [Barberá and Ehlers \(2011\)](#) and [Sato \(2009\)](#) and analyze further the relationship between the structure of indifference classes across agents and dictatorship results. The key to our approach is the pairwise partition graph. We provide necessary conditions on these graphs for strategy-proofness and unanimity (or efficiency) to imply dictatorship. These conditions are not sufficient; we also provide separate stronger conditions that are sufficient. A full characterization is obtained in the case of two agents for domains where strategy-proofness and efficiency imply dictatorship.

KEYWORDS AND PHRASES: Social choice functions, Strategy-proofness, Dictatorial Domains, Exogenous Indifference Class, Gibbard- Satterthwaite theorem.

JEL CLASSIFICATION NUMBER: D71.

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1 INTRODUCTION

The seminal work of [Gibbard \(1977\)](#) and [Satterthwaite \(1975\)](#) showed that all deterministic strategy-proof SCFs with a range of at least three alternatives and defined over the complete domain, is dictatorial. A large body of literature has since focused on relaxing the underlying assumptions of this result. A natural way to do this is to impose domain restrictions. Indeed, many real life problems have inherent domain restrictions. This paper is a contribution to this stand of literature.

It is well known that the structure of strategy-proof SCFs becomes more complex when indifference is permitted in individual preferences.¹ In this paper, we investigate a model of domain restrictions involving indifference, *the exogenous indifference class model* first introduced in [Barberá and Ehlers \(2011\)](#). In this model, the indifference classes of agents' preferences is exogenously given. In particular, every individual has an exogenous partition of the set of alternatives. An individual is always indifferent between alternatives a and b iff both a and b belong to the same element of her partition set. But an individual's ranking of the different elements of her partition set, is complete.

This framework includes several well-studied models as special cases. For instance, the case of private goods and selfish preferences is one where an individual is indifferent between all alternatives that give her the same commodity bundle. It includes the one-sided matching model studied in [Svensson \(1999\)](#), [Pápai \(2000\)](#). It also includes the Gibbard-Satterthwaite framework where the elements of the partition are all singletons. Further examples are provided in [Sato \(2009\)](#).

The goal of [Barberá and Ehlers \(2011\)](#) was to study the Arrovain aggregation issue in the exogenous indifference class model. [Sato \(2009\)](#) examined the same model from the perspective of strategic voting. This is an interesting model in this respect as well because it covers both the complete domain over which strategy-proofness implies dictatorship (the Gibbard-Satterthwaite Theorem) as well as the private good allocation model for which it is well-known that a rich class of strategy-proof social choice functions exist ([Pápai \(2000\)](#)). Sato showed that the number of common indifference classes is critical to the existence of strategy-proof and non-dictatorial SCFs. He assumed that common indifference classes are singletons and obtained two results. First, an onto and strategy-proof social choice function is dictatorial whenever there are at least three common indifference classes. Second, the same result holds when the number of common indifference classes is two, provided that unanimity is strengthened to efficiency.

In this paper, we further examine the relationship between dictatorship results in this model and the structure of indifference classes across agents. Our results are formulated in terms of the *pairwise partition graph* induced by the indifference classes. Fix a pair of agents i and j and their indifference classes. The partition graph for this pair is a bipartite graph

¹See for instance, the literature of strategy-proofness in classical exchange economies originating from [Hurwicz \(1972\)](#) and [Satterthwaite and Sonnenschein \(1981\)](#).

whose vertices are i and j 's indifference classes. There are no edges between the vertices representing the indifference classes of a given agent; vertices for i and j have an edge if the indifference classes representing these vertices have *no common alternative*. We show that a necessary condition for strategy-proofness and unanimity to imply dictatorship in the domain induced by a partition is that each associated pairwise partition graph is connected with the degree of every vertex being at least two. This requirement can be weakened to the graphs being connected (with possibly isolated vertices), if unanimity is replaced by efficiency.

We are unfortunately, unable to show that these necessary conditions are sufficient for dictatorship. However we are able to identify a number of stronger conditions that are sufficient. The first of these is the existence of at least two common indifference classes with no restrictions on their size - a result which clearly generalizes that of Sato. In addition we have three sufficient conditions for the case of two voters. One applies to the case where there is exactly one common indifference class while another shows that strategy-proofness and unanimity imply dictatorship when the partition graph is a cycle. Finally, we show that with the stronger assumption of efficiency, strategy-proofness implies dictatorship when the partition graph is connected with possibly isolated vertices. The last condition implies that we have a necessary and sufficient condition for dictatorship for the case of efficiency when there are exactly two voters.

We now proceed to details.

2 THE MODEL

Let $A = \{a, b, c, \dots\}$ denote a finite set of alternatives with $|A| = m$. Let $I = \{1, \dots, n\}$, $n \geq 2$ be a finite set of agents. We impose restrictions on the domain of preferences following [Barberá and Ehlers \(2011\)](#). Each agent i has a partition S_i of A that is exogenously specified and independent of preferences. A typical element of S_i is s_i^j where $j = 1, \dots, J$. An ordering $R(S_i)$ over A *respects* S_i if (i) a pair of alternatives belonging to the same element of S_i are indifferent to each other and (ii) otherwise one is strictly preferred to the other. Let $I(S_i)$ and $P(S_i)$ denote the symmetric and asymmetric components of $R(S_i)$ respectively. Let $\mathbb{R}(S_i)$ be the set of all orderings respecting S_i . Let $S = (S_1, \dots, S_n)$ be an n -tuple of partitions, one for each agent. The admissible preference domain is $\mathbb{R}(S) = \prod_{i \in N} \mathbb{R}(S_i)$.

OBSERVATION 1 For a given partition S_i , $\mathbb{R}(S_i)$ consists of all possible strict orderings over the elements of S_i . If all agents have the same partition, the model reduces to the standard voting model where the elements of the partition can be thought of as an alternative. The essence of the problem is that partitions across agents can differ. Suitable choices of S_i 's yield the usual private goods allocation model with selfish preferences, the universal domain with strict orderings as well as several interesting intermediate cases as shown in [Sato \(2009\)](#).

EXAMPLE 1 Let $A = \{a, b, c, d, e\}$ and $I = \{1, 2\}$. Let $S_1 = \{ab, c, de\}$ and $S_2 = \{ab, cd, e\}$. Note that Table 1 and Table 2 below represent $\mathbb{R}(S_1)$ and $\mathbb{R}(S_2)$ respectively.

R_1^1	R_1^2	R_1^3	R_1^4	R_1^5	R_1^6
ab	ab	c	c	de	de
c	de	ab	de	ab	c
de	c	de	ab	c	ab

Table 1: $\mathbb{R}(S_1)$

R_2^1	R_2^2	R_2^3	R_2^4	R_2^5	R_2^6
ab	ab	cd	cd	e	e
cd	e	ab	e	ab	cd
e	cd	e	ab	cd	ab

Table 2: $\mathbb{R}(S_2)$

Whenever S or S_i is fixed, $\mathbb{R}(S)$, $R(S)$, $\mathbb{R}(S_i)$, and $R(S_i)$ will be simply written as \mathbb{R} , R , \mathbb{R}_i , and R_i respectively. Fix a partition S_i . An indifference class of agent i is an element of S_i . For any $R_i \in \mathbb{R}_i$ and $k \in \{1, \dots, J\}$, the k^{th} ranked indifference class in R_i is denoted by $r_k(R_i)$ (in other words, there are $k - 1$ elements of S_i ranked strictly higher than $r_k(R_i)$). For any $D \subseteq A$ and $R_i \in \mathbb{R}_i$, $M(R_i, B)$ is the set of maximal elements in B according to R_i i.e. $M(R_i, B) = \{a \in B : aR_i b \text{ for all } b \in B\}$.

Let $V(S) = \bigcap_{i=1}^n S_i$ denote the set of common indifference classes for the partition n -tuple S . Also let $v(S) = |V(S)|$ denote the number of common indifference classes. Note that in Example 1, $V(S) = \{ab\}$ and $v(S) = 1$. Moreover, in the private goods allocation model with selfish preferences $v(S) = 0$ while in the standard voting model $v(S) = m$.

DEFINITION 1 A social choice function (SCF) is a mapping $f : \mathbb{R} \rightarrow A$.

Each agent's preference ordering is private information, i.e. known only to herself. These preferences must therefore be elicited by the mechanism designer. If a SCF is *strategy-proof*, then no agent can benefit by misrepresenting her preferences irrespective of her beliefs about the preference announcement of other agent.

DEFINITION 2 A SCF $f : \mathbb{R} \rightarrow A$ is manipulable by agent i at a profile $R \in \mathbb{R}$ via R'_i if

$$f(R'_i, R_{-i}) P_i f(R).$$

A SCF f is *strategy-proof* if it is not manipulable by any agent at any profile.

The following additional properties of SCFs are standard.

DEFINITION 3 A SCF f is unanimous if for all $R \in \mathbb{R}$

$$f(R) \in \bigcap_{i=1}^n r_1(R_i) \text{ whenever } \bigcap_{i=1}^n r_1(R_i) \neq \emptyset.$$

A unanimous SCF always respects consensus whenever it exists, i.e. if all agents agree on some set of alternatives as their best, then the SCF must pick an element from this set. A stronger condition than unanimity is efficiency.

DEFINITION 4 A SCF f is efficient if for all $R \in \mathbb{R}$, for all $a, b \in A$

$$\left[\begin{array}{l} aR_i b \text{ for all } i \in I \\ aP_i b \text{ for some } j \in I \end{array} \right] \Rightarrow [f(R) \neq b]$$

If agents preferences are strict but sufficiently “rich”, strategy-proofness and unanimity implies efficiency. In our setting where indifference is permitted, this proposition typically does not hold. We therefore consider the consequences of unanimity and efficiency separately.

DEFINITION 5 A SCF f is dictatorial if there exists an agent i such that $f(R) \in r_1(R_i)$ for all $R \in \mathbb{R}$.

An important feature of our model is that a SCF is not fully specified even when an agent is a dictator since this agent will typically have more than one best alternative at any profile.

An important subclass of dictatorial SCFs are *serial dictatorships* which we define below.

DEFINITION 6 A priority σ is a one-to-one map $\sigma : I \rightarrow I$. Let R be a profile and define the sets $T_{\sigma(1)}(R), T_{\sigma(2)}(R), \dots, T_{\sigma(n)}(R)$ inductively as follows:

$$\begin{aligned} T_{\sigma(1)}(R) &= M(R_{\sigma(1)}, A) \\ T_{\sigma(2)}(R) &= M(R_{\sigma(2)}, T_{\sigma(1)}(R)) \\ &\vdots \\ T_{\sigma(n)}(R) &= M(R_{\sigma(n)}, T_{\sigma(n-1)}(R)) \end{aligned}$$

A serial dictatorship with respect to σ satisfies $f^\sigma(R) \in T_{\sigma(n)}(R)$ for all R .

A priority is a linear arrangement of agents where $\sigma(1)$ is the first agent, $\sigma(2)$ the second and so on with $\sigma(n)$ being the last. A serial dictatorship (with respect to σ) works as follows: at any profile R , agent $\sigma(1)$ picks her maximal elements in A , $\sigma(2)$ picks her maximal elements in $\sigma(1)$'s maximal elements and so on. Note that a serial dictatorship is not fully specified because $T_{\sigma(n)}(R)$ may contain more than one alternative.

3 RESULTS

Our goal is to investigate the structure of indifference classes across agents for which unanimity (or efficiency) and strategy-proofness imply dictatorship. Our first step is to provide necessary conditions for dictatorship. Subsequently, we provide various sufficient conditions.

3.1 NECESSARY CONDITIONS FOR DICTATORSHIP

These condition will be formulated in terms of graphs arising from the set S . We briefly review some basic graph-theoretic concepts.

A Graph G is a pair of finite sets V and E where V is the set of vertices or nodes and E is the set of edges. An edge is a non-ordered pair of vertices. If $e = \{u, v\}$ is an edge, i.e. $e \in E$, then u and v are adjacent vertices and u and e are incident as are v and e . The *degree of a vertex v* , $deg_G(v)$ is the number of edges incident with v . An *isolated vertex* is a vertex with degree zero. A *path uv* is a finite sequence of distinct vertices (v_1, v_2, \dots, v_k) where $v_1 = u$, $v_k = v$ and $\{v_i, v_{i+1}\} \in E$ for all $i \in \{1, \dots, k-1\}$. If the path uv exist, then u and v are *connected*. A graph is connected if all pairs of distinct vertices are connected. A graph is *discrete* if $E = \{\emptyset\}$. A graph is a *connected graph with isolated vertices* if (i) it is not discrete and (ii) the sub-graph consisting of non-isolated vertices is connected. A graph is *bipartite* if the vertices of G can be partitioned in to two subsets V_1 and V_2 in such a way that no two vertices in the same subset have an edge.

Let i, j be agents and S_i and S_j be partitions for i and j respectively. The graph $G(S_i, S_j)$ is constructed as follows:

- (i) The set $V = V_1 \cup V_2$ where $V_1 = S_i$ and $V_2 = S_j$.
- (ii) There are no edges between vertices in V_1 or between vertices in V_2 .
- (iii) There is an edge between s_i^l and s_j^k iff $s_i^l \cap s_j^k \neq \emptyset$ for all $s_i^l \in S_i$ and $s_j^k \in S_j$.

The graph $G(S_i, S_j)$ is bipartite. We refer to it as the *Partition Graph for i and j* . We provide several examples of such graphs below.

EXAMPLE 2 Let $I = \{j, i\}$, $A = \{a, b, c, d, e, f, g, h\}$, $\bar{S}_j = \{ab, cd, ef, gh\}$ and $\bar{S}_i = \{ac, bd, eg, fh\}$. The induced partition Graph is shown in Figure 1 and it is not connected. There is no path between gh to fh . Also every vertex of it has degree 2.

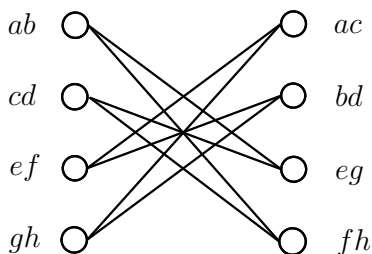


Figure 1: Disconnected graph

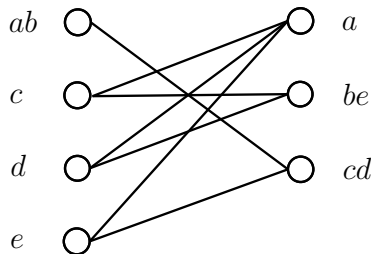


Figure 2: A graph with a degree one vertex

EXAMPLE 3 Let $I = \{j, i\}$, $A = \{a, b, c, d, e\}$, $\hat{S}_j = \{ab, c, d, e\}$ and $\hat{S}_i = \{a, be, cd\}$. Figure 2 shows the induced partition graph which is connected. Note that degree of the vertex ab is 1.

EXAMPLE 4 Let $I = \{j, i\}$, $A = \{a, b, c, d, e\}$, $\tilde{S}_j = \{ab, c, d, e\}$ and $\tilde{S}_i = \{a, be, cd\}$. The induced partition graph is shown in Figure 3 which is not connected. Note that degree of the vertex abc is 0.

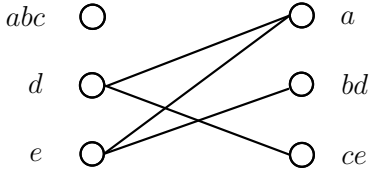


Figure 3: A graph with an isolated vertex

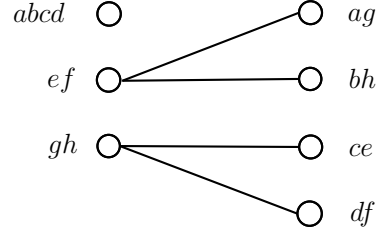


Figure 4: A non-connected graph with isolated vertices

EXAMPLE 5 Let $I = \{j, i\}$, $A = \{a, b, c, d, e, f, g, h\}$, $S_j^o = \{abcd, ef, gh\}$ and $S_i^o = \{ag, bh, ce, df\}$. The induced partition graph is shown in Figure 4 and it is not connected. Moreover it is not a connected graph with isolated vertices.

EXAMPLE 6 (The Allocation Problem with Selfish Preferences) There are three objects, say houses a , b and c which have to be allocated among three agents. An allocation is an ordered triple such as abc where the first, second and third components refer to the houses allocated to agents 1, 2 and 3 respectively. The set of allocations is the set $A = \{abc, acb, bac, bca, cab, cba\}$. Consider the standard model of selfish preferences. The induced partition $S^* = (S_1^*, S_2^*, S_3^*)$ is as follows:

- $S_1^* = \{\{abc, acb\}, \{bac, bca\}, \{cab, cba\}\}$.
- $S_2^* = \{\{abc, cba\}, \{bac, cab\}, \{bca, acb\}\}$.
- $S_3^* = \{\{abc, bac\}, \{cba, bca\}, \{cab, acb\}\}$.

The Partition Graph $G(S_1^*, S_2^*)$ is shown in Figure 5. Note that there are no common indifference classes. Clearly $G(S_1^*, S_2^*)$ is not connected, for instance there is no path between $\{cab, cba\}$ and $\{bca, acb\}$.

EXAMPLE 7 Let $I = \{j, i\}$, $A = \{a, b, c, d, e, f, g, h, i\}$, $S_j^i = \{abc, def, ghi\}$ and $S_i^i = \{adg, beh, cfi\}$. The induced partition graph in Figure 6, is a discrete graph.

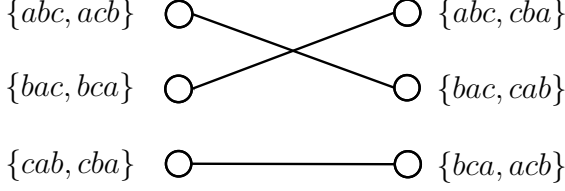


Figure 5: $G(S_1^*, S_2^*)$



Figure 6: Discrete graph

OBSERVATION 2 Suppose $I = \{i, j\}$ and S is such that $G(S_i, S_j)$ is discrete (Example 7). Then *every* unanimous SCF is dictatorial. Therefore the discrete partition graph case is trivial when there are two agents and can be excluded from consideration. In our necessary conditions (Theorem 1 below), we assume that the partition graph is not discrete for all pairs of agents. Note that if there are three or more agents, the pairwise discreteness of all partition graphs does not imply that every unanimous SCF is dictatorial. We are unable to provide an answer for the case where partition graphs are pairwise discrete (or even some of them are pairwise discrete) but unanimity does not imply dictatorship.

THEOREM 1 Let $S = (S_1, \dots, S_n)$ be such that $G(S_i, S_j)$ is not discrete for all $i, j \in I$.

- A. Suppose $[f : \mathbb{R} \rightarrow A \text{ is strategy-proof and unanimous} \Rightarrow f \text{ is dictatorial}]$. Then $G(S_i, S_j)$ is connected and the degree of every vertex is at least 2 for all $i, j \in I$.
- B. Suppose $[f : \mathbb{R} \rightarrow A \text{ is strategy-proof and efficient} \Rightarrow f \text{ is dictatorial}]$. Then $G(S_i, S_j)$ is a connected graph with isolated vertices for all $i, j \in I$.

Proof: We first prove Part A. Pick an arbitrary pair $i, j \in I$. We consider three cases regarding $G(S_i, S_j)$: (I) it is not connected but has no isolated vertices (such as Example 2) (II) it is connected and has a vertex with degree 1 (Example 3) and (III) it has an isolated vertex (Example 4). These cases cover all possible cases in the statement of A and we show that in each case it is possible to construct a unanimous, strategy-proof and non-dictatorial SCF $f : \mathbb{R} \rightarrow A$.

Case I: The vertices of $G(S_i, S_j)$ can be partitioned into subsets V' and V'' with $|V'|, |V''| \geq 2$ and $\{u, v\} \notin E$ for all $u \in V'$ and $v \in V''$ (In Figure 2, V' is the set $\{ab, cd, eg, fh\}$ and V'' is the set $\{ac, bd, ef, gh\}$). We fix two priority functions σ_1 and σ_2 as follows: $\sigma_1(1) = i, \sigma_1(2) = j, \sigma_2(1) = j, \sigma_2(2) = i$ and $\sigma_1(k) = \sigma_2(k)$ for all $k \neq 1, 2$. Let \mathbb{D} be the set of preference profiles such that $r_1(R_i), r_1(R_j) \in V'$. The SCF will be a serial dictatorship with priority σ_1 in the domain \mathbb{D} and σ_2 elsewhere, i.e.

$$f(R) = \begin{cases} \text{a serial dictatorship with respect to the priority function } \sigma_1 & \text{if } R \in \mathbb{D} \\ \text{a serial dictatorship with respect to the priority function } \sigma_2 & \text{if } R \notin \mathbb{D} \end{cases}$$

We first check the strategy-proofness of f . Observe that an agent $k \neq i, j$ cannot manipulate because of the nature of serial dictatorships and the fact that they cannot affect

their priority. Consider a profile $R \in \mathbb{D}$. The outcome is a R_j -maximal element outcome in the first-ranked indifference class of i in R_i , say s_i^1 . Since i is a dictator, she does not manipulate. Clearly j cannot manipulate via an ordering R'_j such that $(R_i, R'_j, \dots) \in \mathbb{D}$ because i will remain the dictator. Suppose j attempts to manipulate via an ordering R'_j such that $(R_i, R'_j, \dots) \notin \mathbb{D}$. He is now dictator and the outcome belongs to $r_1(R'_j)$. However, $r_1(R'_j) \cap s_i^1 \neq \emptyset$ by properties of the sets V' and V'' ; the outcome is therefore belongs to s_i^1 . Hence j cannot manipulate. The only remaining case to consider is a possible manipulation by i from a profile not belonging to \mathbb{D} to a profile in \mathbb{D} . The argument to show that i cannot gain is identical to the case where j attempts to manipulate from \mathbb{D} to a profile outside it.

Unanimity is satisfied because f is a serial dictatorship at every profile. There are profiles in \mathbb{D} and outside it where i and j 's best ranked alternatives have an empty intersection. Therefore, neither i nor j are dictators. If $l \neq i, j$ is a dictator, then $G(S_i, S_l)$ and $G(S_j, G_l)$ are discrete, contradicting our assumption- i.e. f is non-dictatorial.

Case II: Let u be a vertex with degree 1. Assume w.l.o.g that $u \in S_j$ and $u = s_j^k$. By assumption, there exists a (unique) vertex $s_i^m \in S_i$ such that $\{s_i^m, s_j^k\} \in E$ (For instance in Example 3, s_j^k and s_i^m are ab and cd respectively). Let \mathbb{D} be the set of preference profiles such that $r_1(R_i) = s_i^m$. We fix a priority function σ such that $\sigma(1) = j$ and $\sigma(2) = i$. Let $R \in \mathbb{D}$ and define the sets $T_{\sigma(1)}^*(R), T_{\sigma(2)}^*(R), \dots, T_{\sigma(n)}^*(R)$ inductively as follows:

$$\begin{aligned} T_{\sigma(1)}^*(R) &= M(R_{\sigma(1)}, s_j^k \cup s_i^m) \\ T_{\sigma(2)}^*(R) &= M(R_{\sigma(2)}, T_{\sigma(1)}^*(R)) \\ &\vdots \\ T_{\sigma(n)}^*(R) &= M(R_{\sigma(n)}, T_{\sigma(n-1)}^*(R)) \end{aligned}$$

We define a SCF $f_1^\sigma : \mathbb{D} \rightarrow A$ as follows: $f_1^\sigma(R) \in T_{\sigma(n)}^*(R)$ for all $R \in \mathbb{D}$.

Let $\bar{\sigma}$ be a priority function such that $\bar{\sigma}(1) = i$, $\bar{\sigma}(2) = j$ and $\bar{\sigma}(k) = \sigma(k)$ for all $k \neq 1, 2$. We define the following SCF $f : \mathbb{R} \rightarrow A$:

$$f(R) = \begin{cases} f_1^\sigma(R) & \text{if } R \in \mathbb{D} \\ \text{a serial dictatorship with respect to the priority function } \bar{\sigma} & \text{if } R \notin \mathbb{D} \end{cases}$$

In order to check the strategy-proofness of f , it suffices to check that i and j cannot manipulate. Consider $R \in \mathbb{D}$ - the outcome is a R_j -maximal element in $s_j^k \cup s_i^m$. Agent j cannot manipulate since $(R_i, R'_j, \dots) \in \mathbb{D}$ for all R'_j . If the outcome belongs to s_i^m then clearly i cannot manipulate. Otherwise, the outcome belongs to s_j^k . Since s_j^k is a vertex with degree one, it has a non-empty intersection with *every* indifference class other than s_i^m . Hence the outcome belongs to $r_2(R_i)$. By construction, i cannot obtain an alternative in $r_1(R_i) = s_i^m$. If $R \notin \mathbb{D}$, agent i is dictator and cannot manipulate. Note that j cannot manipulate at $R \notin \mathbb{D}$ because $(R'_j, R_i, \dots) \notin \mathbb{D}$ for all R'_j .

Unanimity and non-dictatorship of f is easily verified.

Case III: Let u be a vertex with degree 0. Assume w.l.o.g that $u \in S_j$ and $u = s_j^k$. Since $G(S_i, S_j)$ is not discrete, there exists an edge $\{s_i^m, s_j^l\}$ where $s_j^k \neq s_j^l$ (In Figure 3, the node s_j^k and the edge $\{s_i^m, s_j^l\}$ are abc and $\{d, a\}$ respectively). Let \mathbb{D} be the set of preference profiles where $r_1(R_i) = s_i^m$, $r_1(R_j) = s_j^l$ and $r_2(R_j) = s_j^k$. We fix a priority function σ such that $\sigma(1) = j$ and $\sigma(2) = i$. Let $R \in \mathbb{D}$ and define the sets $T_{\sigma(1)}^*(R), T_{\sigma(2)}^*(R), \dots, T_{\sigma(n)}^*(R)$ inductively as follows:

$$\begin{aligned} T_{\sigma(1)}^*(R) &= s_j^k \\ T_{\sigma(2)}^*(R) &= r_2(R_{\sigma(2)}) \cap s_j^k \\ T_{\sigma(3)}^*(R) &= M(R_{\sigma(3)}, T_{\sigma(2)}^*(R)) \\ &\vdots \\ T_{\sigma(n)}^*(R) &= M(R_{\sigma(n)}, T_{\sigma(n-1)}^*(R)) \end{aligned}$$

We define a SCF $f_1^\sigma : \mathbb{D} \rightarrow A$ as follows: $f_1^\sigma(R) \in T_{\sigma(n)}^*(R)$ for all $R \in \mathbb{D}$.

Let $\bar{\sigma}$ be a priority function such that $\bar{\sigma}(1) = i$, $\bar{\sigma}(2) = j$ and $\bar{\sigma}(k) = \sigma(k)$ for all $k \neq 1, 2$. We define the following SCF $f : \mathbb{R} \rightarrow A$:

$$f(R) = \begin{cases} f_1^\sigma(R) & \text{if } R \in \mathbb{D} \\ \text{a serial dictatorship with respect to the priority function } \bar{\sigma} & \text{if } R \notin \mathbb{D} \end{cases}$$

We check the strategy-proofness of f . Again, it suffices to check that i and j cannot manipulate. If $R \in \mathbb{D}$, the outcome belongs to $r_2(R_j)$ i.e. s_j^k . Agent j cannot manipulate for the following reason: for any R'_j if (i) $(R_i, R'_j \dots) \in \mathbb{D}$, then the outcome belongs to s_j^k (ii) $(R_i, R'_j \dots) \notin \mathbb{D}$ then the outcome does not belong to $r_1(R_j)$ because j is the dictator “after” i . Consider a possible manipulation by i from R . Since s_j^k is an isolated vertex, $s_j^k \cap r_2(R_i) \neq \emptyset$. Hence i obtains a second-ranked alternative in R_i . For any R'_i (i) if $(R_j, R'_i \dots) \in \mathbb{D}$ the outcome belongs to $r_2(R'_i)$ and $r_1(R_i) \cap r_2(R'_i) = \emptyset$ and (ii) if $(R_j, R'_i \dots) \notin \mathbb{D}$ the outcome belongs to $r_1(R'_i)$ and $r_1(R_i) \cap r_1(R'_i) = \emptyset$. If $R \notin \mathbb{D}$, then i is a dictator and cannot manipulate. For j , the only case to consider is the one where the manipulation R'_j is such that $(R'_j, R_i, \dots) \in \mathbb{D}$. In this case, the outcome is in s_j^k . At R the outcome is a R_j - maximal element in s_i^m . Since s_j^k is an isolated vertex, it has elements in common with s_i^m . Hence the outcome at R is weakly preferred to alternatives in s_j^k according to R_j . Thus, j cannot manipulate.

It is again straightforward to show that f is unanimous and non-dictatorial.

We now prove Part B. Pick an arbitrary pair $i, j \in I$. Suppose $G(S_i, S_j)$ is not a connected graph with isolated vertices i.e. the sub-graph of $G(S_i, S_j)$ consisting of non-isolated vertices is not connected. The vertices of $G(S_i, S_j)$ can be partitioned into subsets V' and V'' with

- $|V'|, |V''| \geq 2$.
- $\{u, v\} \notin E$ for all $u \in V'$ and $v \in V''$.
- The sub-graphs with vertices V' and V'' are not discrete.

For instance in Figure 4, V' is the set $\{abcd, ef, ag, bh\}$ and V'' is the set $\{gh, ce, df\}$. We now construct the same SCF as in Case I of Part A. This SCF is efficient, strategy-proof and non-dictatorial as required. ■

OBSERVATION 3 The SCFs constructed in Cases II and III of Part A are not efficient. For instance in Example 3 of Case II, the outcome at (R_1, R_2) (shown in Table 3) is b and in Example 4 of Case III, the outcome at (\bar{R}_1, \bar{R}_2) (shown in Table 4) is c .

R_1	R_2
e	cd
ab	be
d	a
c	

Table 3: (R_1, R_2)

\bar{R}_1	\bar{R}_2
d	a
abc	ce
e	bd

Table 4: (\bar{R}_1, \bar{R}_2)

Unfortunately the necessary condition in Part A of Theorem 1 is not sufficient. This is shown by the Example 8. In the next section, we will show that the condition in Part B is sufficient when there are two agents (Part D of Theorem 2).

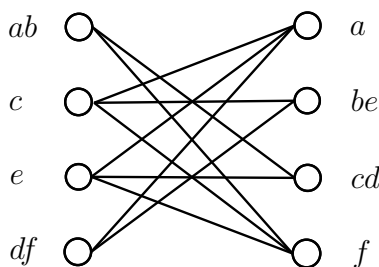


Figure 7: $G(S_1, S_2)$

EXAMPLE 8 Let $I = \{1, 2\}$ and $A = \{a, b, c, d, e, f\}$. Figure 7 shows the induced partition graph where $S_1 = \{ab, c, e, df\}$ and $S_2 = \{a, be, cd, f\}$. Observe that this graph is connected and each vertex has degree atleast 2.

The SCF is described as follows:

$$f(R) = \begin{cases} M(R_1, r_1(R_2)) & \text{if } r_1(R_2) \neq a \\ M(R_1, \{a, d\}) & \text{if } r_1(R_2) = a \text{ and } r_2(R_2) = cd \\ M(R_1, \{a, f\}) & \text{if } r_1(R_2) = a \text{ and } r_2(R_2) = f \\ a & \text{if } r_1(R_2) = a, r_2(R_2) = be, \\ & \text{and } ab \text{ is preferred to } df \text{ according to } R_1 \\ b & \text{if } r_1(R_2) = a, r_2(R_2) = be, \\ & \text{and } df \text{ is preferred to } ab \text{ according to } R_1 \end{cases}$$

The SCF is a serial dictatorship where 2 picks first and 1 second, for all profiles *except* those where the first-ranked alternative of 2 is a . In the latter case, the outcome specified depends on 2's second-ranked alternatives and 1's preferences. In all cases, 2 gets at least a second-ranked alternative. The only profiles that are candidates for manipulation by 2 are those where her first-ranked alternative is a but the outcome is not. However, in these cases, df is preferred to ab for 1 and 2 cannot obtain a by misrepresentation. The outcome at all profiles is determined by maximizing 1's preferences over a set determined by 2's ordering; hence 1 cannot manipulate.

It is straightforward to verify that the SCF is non-dictatorial and satisfies unanimity.

3.2 SUFFICIENT CONDITIONS FOR DICTATORSHIP

In this section we provide various sufficient conditions for dictatorship. We introduce some definitions.

DEFINITION 7 *The partition S satisfies Condition α , if (i) $v(S) \geq 2$ and (ii) there exists an agent i for whom S_i has at least three elements.*

DEFINITION 8 *Let $I = \{i, j\}$. The partition S satisfies Condition β if (i) $v(S) = 1$ and (ii) there exist $s_i^k, s_i^r \in S_i, s_i^{k'}, s_i^{r'} \in S_j$ such that $s_i^k \subset s_j^{k'}, s_j^{r'} \subset s_i^r$ and $s_j^{k'} \cap s_i^r = \emptyset$ (the subset relations are strict).*

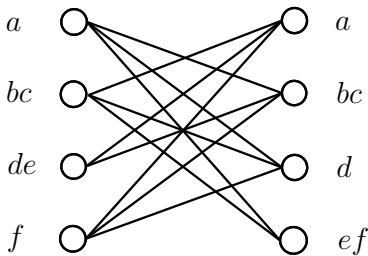


Figure 8: Partition graph for Condition α

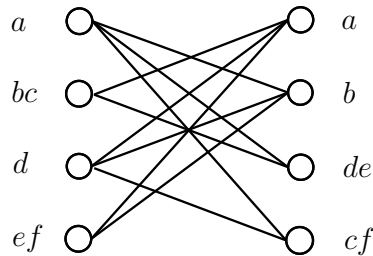


Figure 9: Partition graph for Condition β

OBSERVATION 4 Figure 8 shows an example of a partition graph for two players i and j where S satisfies Condition α (for instance, $S_i = \{a, bc, de, f\}$, $S_j = \{a, bc, d, ef\}$ and $V(S) = \{a, bc\}$). Observe that the vertices representing the elements of $V(S)$ have an edge with all vertices of the other agent (other than itself). This fact together with the assumption that $V(S) \geq 2$ ensures that the partition graph for any pair of agents is connected and the degree of every vertex is at least two.

Figure 9 illustrates Condition β (for instance, $S_i = \{a, bc, d, ef\}$, $S_j = \{a, b, de, cf\}$, $V(S) = a$ and $s_i^k, s_i^r, s_i^{k'}, s_i^{r'}$ are d, bc, de, b respectively). Note that the vertex representing the common element of the partition has an edge with all vertices of the other agent (other than itself). In addition, s_i^k has an edge with all vertices of other agent except $s_j^{k'}$. Similarly, $s_i^{r'}$ has an edge with all vertices of other agent except s_i^r . This ensures that the graph is connected and the degree of every vertex other than $s_j^{k'}$ and s_i^r is at least two. The assumption that $s_j^{k'} \cap s_i^r = \emptyset$ guarantees that the degree of these vertices is also at least two.

THEOREM 2 *Let $S = (S_1, \dots, S_n)$ be a partition, $n \geq 2$.*

- A. *If S satisfies Condition α , then a strategy-proof and unanimous SCF is dictatorial.*
- B. *If S satisfies Condition β , then a strategy-proof and unanimous SCF is dictatorial.*
- C. *If $I = \{i, j\}$ and $G(S_i, S_j)$ is a cycle graph², then a strategy-proof and unanimous SCF is dictatorial.*
- D. *If $I = \{i, j\}$ and $G(S_i, S_j)$ is a connected graph with isolated vertices, then a strategy-proof and efficient SCF is dictatorial.*

Proof of Part A: Let $V(S) = \{s^1, s^2, \dots, s^J\}$ where $J \geq 2$ and let $f : \mathbb{R} \rightarrow A$ be unanimous and strategy-proof SCF. We prove the Theorem via the following claims.

CLAIM 1 *Let $R, R' \in \mathbb{R}$ be such that*

- (i) $f(R) \in s^j$ where $s^j \in V(S)$ and
- (ii) $\{a \in A \mid f(R)R_i a\} \subseteq \{a \in A \mid f(R)R'_i a\}$ for all i .

Then $f(R') \in s^j$.

Proof: Pick an arbitrary agent i . We will show that $f(R'_i, R_{-i}) \in s^j$. Suppose not, i.e. $f(R'_i, R_{-i}) \notin s^j$. Clearly either $f(R)P_i f(R'_i, R_{-i})$ or $f(R'_i, R_{-i})P_i f(R)$ holds. The latter case immediately contradicts the strategy-proofness of f . If the former case holds, then condition (ii) above implies $f(R)P'_i f(R'_i, R_{-i})$. However, i will manipulate at (R'_i, R_{-i}) via R_i contradicting strategy-proofness again.

² A cycle graph is a connected graph with the degree of every vertex being 2.

The Claim is established by repeated application of the same argument for different agents. ■

Sato (2009) proves counterparts of Claim 1 when the common indifference classes are singletons.

CLAIM 2 *There exists an agent $k \in I$ for whom the following holds: for all $R \in \mathbb{R}$, $f(R) \in r_1(R_k)$ whenever $r_1(R_k) \in V(S)$.*

Proof: By assumption there exists $s^j, s^k \in V(S)$. Let R^1 denote the profile where s^j and s^k are the top and bottom indifference classes respectively for all agents. By unanimity, $f(R^1) \in s^j$.

Construct new profiles by progressively making s^k and s^j the best and second-best indifference classes respectively in each agent's preferences. After changing the preferences of all agents, the outcome must belong to s^k , by unanimity. Therefore, there must exist an agent i such that (i) before agent i changes his preference, the outcome belongs to s^j and (ii) when i lifts s^k to the top of his preference, the outcome is no longer in s^j . If the outcome does not belong to either s^j or s^k , agent i will manipulate by reverting to the ordering where s^j is first and s^k is last and thereby obtaining an outcome in s^j . Therefore the outcome when i changes her ordering, must belong to s^k . Let R^2 denote the profile where agents 1 through i have s^k first and s^j second. By the earlier argument $f(R^2) \in s^k$.

Next interchange s^j and s^k at R_i^2 . Let R^3 denote the resulting preference profile. By strategy-proofness, $f(R^3) \in s^j$.

At R^2 and R^3 , lower s^j to the bottom for $1, 2, \dots, i-1$ and to the second last position for $i+1, \dots, n$. Let $R^{2'}$ and $R^{3'}$ denote the resulting profiles respectively. By Claim 1, $f(R^{2'}) \in s^k$. In order for i not to manipulate at $R^{3'}$, we must have $f(R^{3'}) \in \{s^k \cup s^j\}$. But if $f(R^{3'}) \in s^k$, then Claim 1, implies $f(R^3) \in s^k$, which is a contradiction. Therefore $f(R^{3'}) \in s^j$.

Let $d \in A \setminus \{s^j \cup s^k\}$ and let s_i^d denote the indifference class of agent i to which d belongs. Let R^4 be the profile shown in Table 3.

Agent	1	.	.	.	i-1	i	i+1	.	.	.	n
Best	s^j
	s_i^d
	s^k
	s_1^d	.	.	.	s_{i-1}^d	.	s_{i+1}^d	.	.	.	s_n^d
	s^k	.	.	.	s^k	.	s^j	.	.	.	s^j
Worst	s^j	.	.	.	s^j	.	s^k	.	.	.	s^k

Table 5: R^4

Since $f(R^{3'}) \in s^j$, Claim 1 implies that $f(R^4) \in s^j$. Let R^5 be the profile obtained by switching s^j and s^k at R^4 for agents $i+1$ through n . By strategy-proofness $f(R^5) \in \{s^j \cup s^k\}$. Suppose $f(R^5) \in s^k$. Let R^6 be the profile in Table 4. Since $f(R^5) \in s^k$, Claim 1 implies $f(R^6) \in s^k$. But this contradicts unanimity at R^6 . Hence $f(R^5) \in s^j$. Since s^j is bottom ranked for all agents other than i and top ranked for i , it follows that the outcome belongs to s^j whenever agent i ranks s^j first.

Agent	1	.	.	.	i-1	i	i+1	.	.	.	n
Best	s_1^d	.	.	.	s_{i-1}^d	s_i^d	s_{i+1}^d	.	.	.	s_n^d
	s^j
	s^k

	s^k	.	.	.	s^k	.	s^k	.	.	.	s^k
Worst	s^j	.	.	.	s^j	.	s^j	.	.	.	s^j

Table 6: R^6

Consider $s^k \in V(S)$ with s^k distinct from s^j . The earlier arguments can be replicated to show that there exists an agent i' such that the outcome belongs to s^k whenever i' ranks s^k first. If i and i' are distinct, the single-valuedness of f is contradicted at any profile where s^j and s^k are ranked first by i and i' respectively. This establishes the Claim. ■

We now complete the proof by showing that agent i identified in Claim 2 is a dictator.

Suppose this is false, i.e. there exists profile \bar{R} and $f(R) \notin r_1(\bar{R}_i)$. In order not to contradict Claim 2 immediately, $r_1(\bar{R}_i) \notin V(S)$. Let s^j be the top-ranked indifference class in $V(S)$ in \bar{R}_i . By assumption there exists $s^k \in V(S)$ distinct from s^j . Construct profile R^7 as follows: (i) preferences of all agents other than i are the same as in \bar{R} and (ii) $r_1(R_i^7) = r_1(\bar{R}_i)$ and $r_2(R_i^7) = s^k$. By strategy-proofness $f(R^7) \notin r_1(R_i^7)$. However, i can obtain an alternative in s^k by raising s^k to the top of her ordering (Claim 2). Hence $f(R^7) \in s^k$. Consider profiles R^8 and R^9 described in Tables 5 and 6 below.

By Claim 1 $f(R^8) \in s^k$. In addition strategy-proofness and Claim 2 imply $f(R^9) \in s^j$. Pick an arbitrary alternative d in $r_1(\bar{R}_i)$ and let s^d denote the indifference class to which d belongs for all agents $l \neq i$. Observe that s^d is strictly preferred to s^j for all l in R^9 .

Agent	1	.	.	.	i-1	i	i+1	.	.	.	n
Best	s^k	.	.	.	s^k	$r_1(\bar{R}_i)$	s^k	.	.	.	s^k
	s^k
Worst	s^j	.	.	.	s^j	.	s^j	.	.	.	s^j

Table 7: R^8

Agent	1	.	.	.	i-1	i	i+1	.	.	.	n
Best	s^k	.	.	.	s^k	$r_1(\bar{R}_i)$	s^k	.	.	.	s^k
	s^j
Worst	s^j	.	.	.	s^j	.	s^j	.	.	.	s^j

Table 8: R^9

Construct profile R^{10} by raising s_l^d to the top of R_l^9 for all $l \neq i$ while keeping i 's preferences fixed at R_i^9 . By Claim 1, $f(R^{10}) \in s^j$. However unanimity requires $f(R^{10}) \in \bigcap_l r_1(R_l^{10})$ where $d \in \bigcap_l r_1(R_l^{10}) \neq \emptyset$. We have a contradiction. ■

Proof of Part B: Let S satisfy Condition β and suppose f is a strategy-proof and unanimous SCF defined for $I = \{i, j\}$. In order to simplify the notation, we will denote the common indifference class by s , the sets s_i^k and s_i^r by y and X respectively and $s_i^{k'}$, $s_i^{r'}$ by Y and x respectively. Condition β requires $y \subset Y$, $x \subset X$ and $X \cap Y = \emptyset$.

The proof uses the following claim.

CLAIM 3 For all profiles R such that $r_1(R_i) = s$ and $r_1(R_j) = Y$, $f(R) \in s \cup Y$.

Proof: Suppose not, i.e. there exists R' with $r_1(R'_i) = s$, $r_1(R'_j) = Y$ and $f(R') \notin \{s \cup Y\}$. Let R''_i and R''_j be such that $r_1(R''_i) = s$, $r_2(R''_i) = y$, $r_1(R''_j) = Y$ and $r_2(R''_j) = s$. Strategy-proofness and unanimity imply $f(R'_i, R''_j) \in s$ and $f(R''_i, R'_j) \in s$. However the same arguments also imply $f(R''_i, R'_j) \in y$ and $f(R'_i, R''_j) \in Y$. This leads to a contradiction since f is a function. ³ ■

Let \bar{R} be such that $r_1(\bar{R}_i) = s$, $r_1(\bar{R}_j) = Y$. By Claim 3, $f(\bar{R}_i, \bar{R}_j) \in s \cup Y$. We complete the proof by considering the following two cases.

Case 1: $f(\bar{R}) \in s$. Using the following sequence of claims we show that i is the dictator.

³These arguments closely follow counterparts in Sen (2001).

CLAIM 4 For all profiles R such that $r_1(R_i) = s$, $f(R) \in s$.

Proof: Let R^* be such that $R_i^* = \bar{R}_i$, $r_1(R_j^*) = Y$ and s is bottom-ranked according to R_j^* . If we can show $f(R^*) \in s$, then we are done. By Claim 3, $f(R^*) \in s \cup Y$. Note that $f(R^*) \notin Y$, otherwise j will manipulate at \bar{R} via R_j^* . Therefore, $f(R^*) \in s$. ■

CLAIM 5 For all profiles R such that $r_1(R_i) = y$, $f(R) \in y$.

Proof: Consider the profiles R^1 , R^2 , R^3 and R^4 shown in Tables 9, 10, 11 and 12 respectively. The ranking of indifference classes other than the top three is the same across the four profiles, for both agents.

Agent	i	j
Best	y	x
	s	Y
	X	s
	\cdot	\cdot
	\cdot	\cdot
Worst	\cdot	\cdot

Table 9: R^1

Agent	i	j
Best	y	x
	X	Y
	s	s
	\cdot	\cdot
	\cdot	\cdot
Worst	\cdot	\cdot

Table 10: R^2

Agent	i	j
Best	y	x
	X	s
	s	Y
	\cdot	\cdot
	\cdot	\cdot
Worst	\cdot	\cdot

Table 11: R^3

Agent	i	j
Best	y	x
	s	s
	X	Y
	\cdot	\cdot
	\cdot	\cdot
Worst	\cdot	\cdot

Table 12: R^4

We begin by showing $f(R^1) \in y$. If $f(R^1) \in A \setminus \{s \cup y\}$, i will manipulate via an ordering where s is top-ranked, thereby obtaining an alternative in s (Claim 4). If $f(R^1) \in s$, j will manipulate via an ordering where Y is at the top obtaining an alternative in Y by unanimity. Therefore $f(R^1) \in y$.

Strategy-proofness implies $f(R^2) \in y$. Our next step is to show $f(R^3) \in y$. If $f(R^3) \in A \setminus \{y \cup X\}$, i will manipulate via an ordering where X is top-ranked, obtaining an alternative in X by unanimity. If $f(R^3) \in X \setminus x$, the outcome is ranked below Y by j (since $X \cap Y = \emptyset$) and she will manipulate via an ordering where Y is top-ranked, obtaining an alternative in Y by unanimity. If $f(R^3) \in x$, j will manipulate at R^2 via R_j^3 . Therefore $f(R^3) \in y$. Again strategy-proofness implies $f(R^4) \in y$.

Finally, let R^5 be a profile such that $R_i^5 = R_i^4$ and Y is bottom-ranked according to R_j^5 . We complete the proof by showing that $f(R^5) \in y$. If $f(R^5) \in A \setminus \{s \cup y\}$, then i will manipulate via an ordering where s is top-ranked, obtaining an alternative in s by Claim 4. If $f(R^5) \in s$, j will manipulate at R^4 via R_j^5 . Therefore $f(R^5) \in y$. ■

CLAIM 6 For all profiles R such that $r_1(R_i) = X$, $f(R) \in X$.

Proof: Suppose the Claim is false, i.e. there exists a profile \tilde{R} with $r_1(\tilde{R}_i) = X$ but $f(\tilde{R}) \notin X$. Let R_i'' be such that $r_1(R_i'') = X$ and $r_2(R_i'') = y$. By strategy-proofness and Claim

5, $f(R'_i, \tilde{R}_j) \in y$. Let R''_j be such that top and bottom ranked indifference classes are Y and s respectively. Since $y \subset Y$, strategy-proofness implies that $f(R''_i, R''_j) \in Y$. Since $X \cap Y = \emptyset$, $f(R''_i, R''_j) \notin X$. Let R'''_i be such that $r_1(R'''_i) = X$ and $r_2(R'''_i) = s$. By strategy-proofness and Claim 4, $f(R'''_i, R''_j) \in s$. If $f(R'''_i, R''_j) \in s$, then j will manipulate via an ordering whose top-ranked indifference class has a non-empty intersection with X . This proves the Claim. ■

CLAIM 7 For all profiles R , $f(R) \in r_1(R_i)$.

Proof: Suppose the Claim is false, i.e. there exists a profile R but $f(R) \notin r_1(R_i)$ and $r_1(R_i) \neq X, y, s$. Let $r_1(R_i) = Z$. The following cases arise.

Case 1: $Z \cap Y = \emptyset$. The arguments in Claim 6 with Z substituted for X can be replicated to establish $f(R) \in Z$.

Case 2: $Z \cap Y \neq \emptyset$. At R_j , lift Y to the top keeping the relative ranking of other indifference classes same and let R'_j be the resulting ordering. By unanimity, $f(R_i, R'_j) \in Z \cap Y$. Let R'_i be such that $r_1(R'_i) = Z$ and $r_2(R'_i) = s$. By unanimity once again, $f(R'_i, R'_j) \in Z \cap Y$. We will now move Y downwards by progressive local switches in order to return to R_j while arguing that the outcome remains in Z , thereby contradicting our initial assumption.

Suppose such a local switch does not involve s and Y is above s in j 's ranking. The outcome at such a profile cannot be an alternative not in $Z \cup s$; in that case, i would manipulate by raising s to the top and getting s (Claim 4). Nor can the outcome belong to s because j would manipulate via R'_j obtaining an alternative in Y . Consequently, the outcome at such a profile must belong to Z .

Now suppose that we reach a profile where Y and s are contiguous in j 's ordering and Y is above s . Let this ordering for j be denoted by \hat{R}_j . By the arguments of the previous paragraph, $f(R'_i, \hat{R}_j) \in Z$. If $f(R'_i, \hat{R}_j) \notin Y$, then the outcome must in fact, be above Y in \hat{R}_j . By switching Y and s in \hat{R}_j , the outcome must remain in Z (by the earlier) arguments. Continuing with the switches as required, we can conclude that $f(R'_i, R_j)$ and $f(R)$ both belong to Z contradicting our assumption.

Suppose therefore that $f(R'_i, \hat{R}_j) \in Y$, i.e. $f(R'_i, \hat{R}_j) \in Y \cap Z$. Let \hat{R}_i be an ordering where Z and X are ranked first and second respectively. By strategy-proofness, $f(\hat{R}_i) \in Z$. Now switch Y and s in \hat{R}_j . The outcome at this profile must belong to $Y \cup s$. Since $X \cap (Y \cup s) = \emptyset$, the outcome does not belong to X . If the outcome is ranked below X in \hat{R}_i , i will manipulate by raising X to the top (Claim 6). Therefore the outcome belongs to Z . Now reverting back to R'_i , we observe that the outcome remains in Z . Continuing with the switches in j 's ordering, we conclude that $f(R'_i, R_j)$ and $f(R)$ both belong to Z , once again contradicting our assumption. ■

We have established that i is a dictator completing Case 1.

Case 2: Suppose $f(\bar{R}) \in Y$. We will show that j is the dictator.

CLAIM 8 For all profiles R , such that $r_1(R_j) = s$, $f(R) \in s$.

Proof: Consider the profiles R^6 , R^7 , R^8 and R^9 shown in Tables 13, 14, 15 and 16 respectively. The ranking of indifference classes other than the top three is the same across the four profiles, for both agents.

Agent	i	j
Best	s	Y
	X	s
	y	x
	\cdot	\cdot
Worst	\cdot	\cdot

Table 13: R^6

Agent	i	j
Best	X	Y
	s	s
	y	x
	\cdot	\cdot
Worst	\cdot	\cdot

Table 14: R^7

Agent	i	j
Best	X	s
	s	Y
	y	x
	\cdot	\cdot
Worst	\cdot	\cdot

Table 15: R^8

Agent	i	j
Best	X	s
	s	x
	y	Y
	\cdot	\cdot
Worst	\cdot	\cdot

Table 16: R^9

By Claim 3, $f(R_i^6, \bar{R}_j) \in s \cup Y$. If $f(R_i^6, \bar{R}_j) \in s$, then i will manipulate at \bar{R} via R_i^6 . Therefore $f(R_i^6, \bar{R}_j) \in Y$. Strategy-proofness implies $f(R^6) \in Y$. Strategy-proofness and $X \cap Y = \emptyset$ implies $f(R^7) \notin \{s \cup X\}$; otherwise i will manipulate at R^6 via R_i^7 . Unanimity implies that $f(R^7) \in y$. Since $y \subset Y$, $f(R^7) \in Y$. Note that strategy-proofness implies $f(R^8) \in s \cup Y$, because R^8 is obtained by switch s and Y for j at R^7 . However, $f(R^8) \in Y$ would lead to manipulation by i via an ordering where s is top-ranked. Hence $f(R^8) \in s$. By strategy-proofness, $f(R^9) \in s$. Let R^{10} be such that $R_j^{10} = R_j^9$, $r_1(R_i^{10}) = X$ and s is at the bottom according to R_i^{10} . Note that $f(R^{10}) \in x \cup s$, otherwise j will manipulate via an ordering where x is top-ranked, obtaining an alternative in x by unanimity. But $f(R^{10}) \notin x$, otherwise i would manipulate at R^9 via R_i^{10} . Therefore, $f(R^{10}) \in s$. This establishes the fact that $f(R) \in s$ for all R such that $r_1(R_j) = s$. ■

Claims 5, 6 and 7 can now be replicated with agent i replaced by j to show that j is the dictator. ■

Proof of Part C: Let $G(S_i, S_j)$ be a cycle graph and suppose f satisfies strategy-proofness and unanimity on this domain. In order to simplify the notation, we will denote the elements of S_i and S_j (i.e the vertices of $G(S_i, S_j)$) by X, Y, Z, P, Q, T, \dots etc.

We identify two properties of $G(S_i, S_j)$ that will be required for the argument.

CLAIM 9 1. $|S_i| = |S_j| \geq 3$. 2. If (X, P) is an edge where $X \in S_i$ and $P \in S_j$, there exist $Y \in S_i$, $Q \in S_j$ such that (i) $Y \cap Q = \emptyset$ (ii) $Y \cap P \neq \emptyset$ and (iii) $X \cap Q \neq \emptyset$.

Proof: 1. $|S_i| = |S_j|$ is a standard property of bipartite cycle graphs (see page 24 of West (2001)). The only case to rule out is $|S_i| = |S_j| = 2$. Suppose this is true. Let $S_i = \{X, Y\}$ and $S_j = \{P, Q\}$. Since the degree of each vertex is 2, it must be the case that

$(X, P), (X, Q), (Y, P)$ and (Y, Q) are all edges, i.e. $X \cap P, X \cap Q, Y \cap P$ and $Y \cap Q$ are all empty. This implies $(X \cup Y) \cap (P \cup Q) = \emptyset$ which is impossible because $X \cup Y = P \cup Q = A$.

2. Let (X, P) be an edge with $X \in S_i$ and $P \in S_j$. There must exist $Q \in S_j$ such that $X \cap Q \neq \emptyset$, i.e. (X, Q) is not an edge. Since Q has degree 2, there exist vertices $Y, Z \in S_i$ such that (Y, Q) and (Z, Q) are edges. Suppose $Y \cap P$ and $Z \cap P$ are both empty. Then P is degree at least three which is impossible. Hence either Y or Z (or both) has a non-empty intersection with P . ■

CLAIM 10 For all profiles R such that $r_1(R_i) \cap r_1(R_j) = \emptyset$, $f(R) \in r_1(R_i) \cup r_1(R_j)$.

Proof: Pick R such that $r_1(R_i) = X$ and $r_1(R_j) = P$ and $X \cap P = \emptyset$. Assume for contradiction that $f(R) \notin X \cup P$. From Claim 9 (Part 2), there exists $Y \in S_i, Q \in S_j$ such that $X \cap Q \neq \emptyset, Y \cap P \neq \emptyset$ and $Y \cap Q = \emptyset$.

Raise Y to the second-ranked position in R_i keeping the ranking of other indifference classes the same and let R'_i be the resulting ordering. Similarly, raise Q to the second-ranked position keeping the relative ranking of other indifference classes the same and let R'_j be the resulting ordering. Standard strategy-proofness and unanimity arguments can be applied to show that either $f(R'_i, R'_j) \in X \cap Q$ or $f(R'_i, R'_j) \in Y \cap P$. We complete the proof of the claim by showing that $f(R_i, R_j) \in X$ if $f(R'_i, R'_j) \in X \cap Q$ (an analogous argument establishes $f(R_i, R_j) \in P$ if $f(R'_i, R'_j) \in Y \cap P$).

Strategy-proofness and unanimity imply $f(\bar{R}_1, \bar{R}_2) \in X$ where $r_1(\bar{R}_1) = X, r_1(\bar{R}_2) = P$ and $r_2(\bar{R}_2) = Q$. If $r_2(R_2) = r_2(\bar{R}_2) = Q$, we are done. So, let $r_2(R_2) = T \neq Q$. We consider two cases.

Case 1: $T \cap X = \emptyset$. Let R'' be a profile such that $r_1(R''_i) = X, r_1(R''_j) = P, r_2(R''_j) = Q$ and $r_3(R''_j) = T$. Note that $f(R'') \in X$. Since degree of T is 2, there exists $Z \in S_i$ such that $Z \cap T = \emptyset$. Applying Claim 9 (Part 1), $Z \cap P \neq \emptyset$.

Suppose $Z \cap Q = \emptyset$. Raise Z to the second position in R''_i keeping the ranking of other indifference classes the same and let R'''_i be the resulting ordering. By strategy-proofness, $f(R'''_i, R''_j) \in X$. At R''_j switch Q and T and let R'''_j be the resulting ordering. By strategy-proofness and unanimity, $f(R'''_i, R'''_j) \in X$. If $r_3(R_j) = Q$, strategy-proofness and unanimity imply $f(R'''_i, R_j) \in X$; moreover $f(R) \in X$. Let $r_3(R_j) = P' \neq Q$. By the fact that the degree of P is 2, there exists $X' \in S_i$ such that $X' \cap P' = \emptyset$ and $X' \cap P \neq \emptyset$. Let \tilde{R}_i be an ordering such that $r_1(\tilde{R}_i) = X$ and $r_2(\tilde{R}_i) = X'$. By strategy-proofness, $f(\tilde{R}_i, R'''_j) \in X$. Let \tilde{R}_j be such that $r_1(\tilde{R}_j) = P, r_2(\tilde{R}_j) = T$ and $r_3(\tilde{R}_j) = P'$. By strategy-proofness, $f(\tilde{R}_i, \tilde{R}_j) \in X$. Again by strategy-proofness, $f(\tilde{R}_i, R_j) \in X$ and $f(R_i, R_j) \in X$.

Suppose $Z \cap Q \neq \emptyset$. Since the degree of Z is two, there exists $Q' \in S_j$ such that $Z \cap Q' = \emptyset$. Let \hat{R}_j be such that $r_1(\hat{R}_j) = P, r_2(\hat{R}_j) = Q, r_3(\hat{R}_j) = T$ and $r_4(\hat{R}_j) = Q'$. Note that $f(R'''_i, \hat{R}_j) \in X$. Let R^*_j be such that $r_1(R^*_j) = X, r_2(R^*_j) = T, r_3(R^*_j) = Q'$ and $r_4(R^*_j) = Q$. By strategy-proofness and unanimity, $f(R'''_i, R^*_j) \in X$. Applying these arguments repeatedly, we conclude that $f(R) \in X$.

Case 2: $T \cap X \neq \emptyset$. Since the degree of P is 2, there exists $Y' \in S_1$ such that $Y' \cap T = \emptyset$ and $Y' \cap P \neq \emptyset$. At R_i'' lift Y' to the second ranked position keeping the relative ranking of other indifference classes same and let R_i^* be the resulting ordering. By strategy-proofness and unanimity, we can infer that $f(R_i^*, R_j'') \in X$ and $f(R_i^*, R_j''') \in X$. Applying these arguments repeatedly, $f(R) \in X$ follows. ■

The proof is completed by showing that the agent from whose first-ranked indifference class the outcome is picked in Claim 10 is in fact, a dictator. This can be done by replicating the arguments in Claim B in Sen (2001). The only requirement for the argument is for the graph $G(S_i, S_j)$ to be connected. ■

Proof of Part D: Let $G(S_i, S_j)$ be a connected graph with isolated vertices and suppose f satisfies strategy-proofness and efficiency on this domain.

CLAIM 11 For all R such that $r(R_i) \cap r(R_j) = \emptyset$, either $f(R) \in r_1(R_i)$ or $f(R) \in r_1(R_j)$.

Proof: Pick R such that $r(R_1) \cap r(R_2) = \emptyset$ and denote $r_1(R_i)$ and $r_1(R_j)$ by X and P respectively. Suppose $f(R) \notin X \cup P$. There exists $Q \in S_j$ such that $X \cap Q \neq \emptyset$ and $Y \in S_i$ such that $Y \cap P \neq \emptyset$.

Let R'_i and R'_j be such that $r_1(R'_i) = X$, $r_2(R'_i) = Y$, $r_1(R'_j) = P$ and $r_2(R'_j) = Q$. By strategy-proofness and efficiency, $f(R'_i, R'_j) \notin Y \cap P$. By strategy-proofness, $f(R'_i, R'_j) \in P$. Similarly, strategy-proofness and efficiency imply that $f(R_i, R'_j) \in X \cap Q$. Once again applying strategy-proofness we get $f(R'_i, R'_j) \in X$ - a contradiction. ■

Suppose $X \in S_i$ is an isolated vertex in $G(S_i, S_j)$. If $r_1(R_i) = X$ at R , we have $r_1(R_i) \cap r_1(R_j) \neq \emptyset$. The same conclusion holds if X is isolated and belongs to S_j .

Finally, since non-isolated vertices are connected in $G(S_i, S_j)$, the arguments in Claim B in Sen (2001) can be used in conjunction with Claim 11 to demonstrate the existence of a dictator. ■

We make several observations about our result.

OBSERVATION 5 Part A of Theorem 2 generalizes Sato (2009) in two ways. It requires only two rather than three common indifference classes. Moreover, these common indifference classes need not be singletons. Our argument follows the general structure of Sato's arguments (with appropriate refinements) which in turn follows that of Reny (2001) and several other papers.

OBSERVATION 6 Parts B, C and D are two-person results. The generalization to an arbitrary number of agents is not straightforward. The main difficulty is that the standard "cloning" arguments for the situation where all agents have a common domain, cannot be used. It is not clear whether the pairwise conditions used in Parts B, C and D are sufficient for dictatorship generally.

OBSERVATION 7 Part B of Theorem 1 together with Part D of Theorem 2 provide a necessary and sufficient condition for strategy-proofness and efficiency to imply dictatorship, in the case of two agents.

4 CONCLUSION

The paper investigates the effect of the partition structure on dictatorship results in the exogenous indifference classes model. The focus of the analysis is the pairwise partition function graph. We provide necessary conditions and stronger sufficient conditions on these graphs for strategy-proofness and unanimity (or efficiency) to imply dictatorship.

Several natural questions remain open. Some of our results apply only to the two-voter case and a gap exists between our necessary and sufficient conditions. These questions appear to be difficult but we hope to make progress in answering them in the future.

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