

Orthogonal and Non-Orthogonal Signal Representations Using New Transformation Matrices Having NPM Structure

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Abstract—In this paper, we introduce two types of real-valued sums known as Complex Conjugate Pair Sums (CCPSs) denoted as CCPS⁽¹⁾ and CCPS⁽²⁾, and discuss a few of their properties. Using each type of CCPSs and their circular shifts, we construct two non-orthogonal Nested Periodic Matrices (NPMs). As NPMs are non-singular, this introduces two non-orthogonal transforms known as Complex Conjugate Periodic Transforms (CCPTs) denoted as CCPT⁽¹⁾ and CCPT⁽²⁾. We propose another NPM, which uses both types of CCPSs such that its columns are mutually orthogonal, this transform is known as Orthogonal CCPT (OCCPT). After a brief study of a few OCCPT properties like periodicity, circular shift, etc., we present two different interpretations of it. Further, we propose a Decimation-In-Time (DIT) based fast computation algorithm for OCCPT (termed as FOCCPT), whenever the length of the signal is equal to 2^v , $v \in \mathbb{N}$. The proposed sums and transforms are inspired by Ramanujan sums and Ramanujan Period Transform (RPT). Finally, we show that the period (both divisor and non-divisor) and frequency information of a signal can be estimated using the proposed transforms with a significant reduction in the computational complexity over Discrete Fourier Transform (DFT).

Index Terms—Complex exponential, Ramanujan sum, CCPS, DFT, RPT, NPM, OCCPT, FOCCPT, Frequency estimation.

I. INTRODUCTION

IN general, information of a finite length signal like period/frequency is not manifested in a recognizable fashion from the signal itself. Such information can be extracted by representing the signal using certain types of bases. This makes signal representation as one of the fundamental problems in signal processing.

In literature, different application specific representations are proposed like DFT, Discrete Cosine Transform, Discrete Sine Transform, etc., [1], [2]. In 1918, mathematician Srinivasa Ramanujan introduced an integer-valued summation known as *Ramanujan sum* [3]. It has some interesting properties like periodicity, orthogonality, etc., which attracted many researchers to use it for various applications [4]–[6]. In 2014, P. P. Vaidyanathan introduced a signal representation known as RPT by using Ramanujan sums and their circular shifts [7], [8]. It has certain periodicity properties, which are useful for period estimation [8]. Moreover, the presence of an integer basis makes it computationally efficient. Due to this, RPT has

been used in many period estimation applications [9]–[13]. Generalizing RPT, in [14], the authors introduced a family of full rank square matrices known as Nested Periodic Matrices (NPMs). Members of the NPM family include Natural Basis matrix, Hadamard matrix, DFT matrix, RPT matrix, etc.

Let p_1, p_2, \dots, p_m be all the divisors of N (where $N \in \mathbb{N}$) and s_{p_i} be a $\varphi(p_i)$ dimensional subspace consists of p_i periodic signals, where $\varphi(\cdot)$ is an Euler's totient function. As $\sum_{p_i|N} \varphi(p_i) = N$, an N^{th} order NPM is constructed by providing the basis for all $\{s_{p_i}\}_{i=1}^m$. The commonality between different NPMs is that they span the same subspaces ($\{s_{p_i}\}_{i=1}^m$) by providing alternate bases. Since $p_i|N$, s_{p_i} is known as divisor subspace. In the NPM family, the performance of RPT and DFT is good in period estimation [14].

Two discrete-time signals with different discrete frequencies may correspond to the same period. For example, two sinusoidal signals with discrete frequencies $\frac{2\pi}{5}$ and $\frac{2\pi(3)}{5}$ are periodic with period 5. Hence, extracting the frequency information is equally important as period extraction and it is crucial in many applications. For instance, brain signals exhibit periodic nature in response to external visual stimuli [10], [15]. Further, they are classified into five frequency sub-bands (alpha, beta, theta, delta, and gamma) over a periodic duration. Electrocardiogram (ECG) is another signal that exhibits periodic nature and we can segment it in QRS complex segment, ST segment, etc., based on the frequency range [16].

One of the limitations of RPT is that it does not provide the frequency information [17], whereas DFT gives period as well as frequency information with high computational complexity. In this paper, we address this problem of signal representation using certain new transformation matrices following NPM structure, such that both period and frequency information can be extracted with less computational complexity.

A. Contributions of This Paper

Ramanujan sums are generated by adding certain complex exponential sequences satisfying some periodicity property [3]. The RPT matrix is constructed by using Ramanujan sums and their circular shifts [7], [8]. Inspired by this, we have proposed two types of real-valued summations known as Complex Conjugate Pair Sum of type-1 (CCPS⁽¹⁾) and Complex Conjugate Pair Sum of type-2 (CCPS⁽²⁾). A few of their properties are also discussed. Two new NPMs are constructed by using each type of CCPSs and their circular shifts as a basis

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for $\{s_{p_i}\}_{i=1}^m$. Further, a finite length signal is represented by using these NPMs. The corresponding transforms are named as Complex Conjugate Periodic Transforms. Based on the type of summation used, it is denoted as either CCPT⁽¹⁾ or CCPT⁽²⁾, we have shown that both of these transforms are non-orthogonal. One of our previous works [17] contains the basic idea of generating CCPS⁽¹⁾, CCPT⁽¹⁾ and their application, which are discussed in brief in this paper.

Another NPM is proposed by using both types of complex conjugate pair sums as a basis for $\{s_{p_i}\}_{i=1}^m$, such that the columns of the matrix are mutually orthogonal. The corresponding transform is named as Orthogonal CCPT (OCCPT). The proposed transforms may have applications in communication, image processing, control applications [18]–[21]. Several important properties of OCCPT like periodicity, circular convolution, etc., are stated and proved, and its relation with DFT is also discussed. In addition, we have presented two different interpretations of OCCPT such that the period and frequency information is explicitly available in each interpretation. If $N = 2^v$, $v \in \mathbb{N}$, a DIT algorithm is proposed to reduce the computational complexity of OCCPT known as Fast OCCPT (FOCCPT). We have shown that this algorithm requires $N \log_2(N) - N + 1$ real multiplications and $2N \log_2(N) - 7 \left(\frac{N}{2}\right) + 5$ real additions for a given $x(n) \in \mathbb{R}^N$.

We have proved that we can estimate the divisor period and its corresponding frequency information of a signal using the proposed transforms. DFT is a standard transform that can serve for the same purpose, so, the computational complexity of proposed transforms is compared with DFT. If $N \neq 2^v$, the complexities of OCCPT and DFT are compared using the direct method, though there exist efficient algorithms for DFT [22], [23]. The following list of conclusions are drawn regarding the computational complexity between DFT and OCCPT:

- If $x(n) \in \mathbb{C}^N$ and $N = 2^v$, then their complexities are comparable with each other.
- If $x(n) \in \mathbb{R}^N$ and $N = 2^v$, then Fast Fourier Transform (FFT) requires $2N \log_2(N)$ real multiplications and $3N \log_2(N)$ real additions. This is approximately 50% higher than FOCCPT complexity.
- If $x(n) \in \mathbb{C}^N$ and $N \neq 2^v$, then the number of multiplications/additions required for OCCPT is approximately 50% lower than DFT.
- If $x(n) \in \mathbb{R}^N$ and $N \neq 2^v$, then DFT and OCCPT require the same complexity due to the complex conjugate symmetry of DFT coefficients.

So far we have dealt with NPMs, which are constructed using the basis of divisor subspaces. In general, there are scenarios where we deal with non-divisor subspaces. One such scenario is the estimation of non-divisor periods and their corresponding frequencies using a dictionary based approach, where the proposed CCPT dictionaries have approximately 75% less computational complexity over the DFT dictionary. Although, RPT dictionary has the computational advantage over CCPT dictionaries due to its integer-valued basis, but it does not provide the frequency information. As an example, we have evaluated the performance of proposed transforms

on an ECG signal by considering the problem of R peak delineation. The results are compared with DFT and RPT.

B. Outline and Notations

The structure of this paper is as follows: The NPM structure and its properties are briefly reviewed in Section II. The process of RPT matrix construction from the DFT matrix is explained in Section III. Section IV discusses two types of complex conjugate pair sums and their properties. Then, two non-orthogonal transforms are introduced by using each type of summation and its circular shifts in Section V. In Section VI, we introduce an orthogonal transform known as OCCPT and discuss a few of its properties. A fast computation algorithm for OCCPT is proposed in Section VII. Later, in Section VIII, the proposed transforms are compared with DFT and RPT. Conclusions are drawn in Section IX. The following notations are used throughout the paper:

\mathbb{N}, \mathbb{Z} : Set of natural numbers and integers respectively.

\mathbb{R}, \mathbb{C} : Set of real numbers and complex numbers respectively.

\mathbf{A}^T : The transpose of a matrix \mathbf{A} .

\mathbf{A}^H : The conjugate transpose of a matrix \mathbf{A} .

$r(\mathbf{A})$: The rank of a matrix \mathbf{A} .

(a, b) : Greatest common divisor of a and b .

lcm : Least common multiple.

$[a]$: The greatest integer less than or equal to a , where $a \in \mathbb{R}$.

$a|b$: a divides b and $a \nmid b$ denotes a does not divide b .

φ : Euler's totient function, defined as $\varphi(n) = \sum_{i=1}^n \left[\frac{1}{(i,n)} \right]$.

Since $(i, n) = (n - i, n)$, $\varphi(n)$ is even for $n \geq 3$.

$M_{m,n}(\mathbb{C})$: Set of all $m \times n$ matrices with entries from complex numbers. If $m = n$, $M_{m,n}(\mathbb{C}) = M_n(\mathbb{C})$.

D_N : The set of all positive divisors of N . We assume $D_N = \{p_1, p_2, \dots, p_m\}$, where $1 = p_1 < p_2 < \dots < p_m = N$.

$((n))_N$: Indicates $n \pmod{N}$.

U_n : $\{k \in \mathbb{N} | 1 \leq k \leq n, (k, n) = 1\}$. Hence the cardinality of U_n , i.e., $\#U_n$ is equal to $\varphi(n)$.

\hat{U}_n : $\{k \in \mathbb{N} | 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, (k, n) = 1, n > 2\}$. Hence $\#\hat{U}_n = \frac{\varphi(n)}{2}$. Let $\tilde{U}_n = U_n - \hat{U}_n$.

$\langle a, b \rangle$: Dot product between a and b .

II. NESTED PERIODIC MATRIX STRUCTURE

Definition 1:[14] A matrix $\mathbf{M} \in M_N(\mathbb{C})$ is said to have *NPM structure* (or simply NPM) if

$$\mathbf{M} = [\mathbf{M}_{p_1}, \mathbf{M}_{p_2}, \dots, \mathbf{M}_{p_i}, \dots, \mathbf{M}_{p_m}], \quad (1)$$

satisfies the following three properties:

- $\mathbf{M}_{p_i} \in M_{N, \varphi(p_i)}(\mathbb{C})$ and $r(\mathbf{M}_{p_i}) = \varphi(p_i)$. The size of \mathbf{M} is $N \times N$ by invoking $\sum_{p_i|N} \varphi(p_i) = N$ [7].
- \mathbf{M} is a full rank matrix.
- Each column in \mathbf{M}_{p_i} is a p_i periodic sequence.

Since \mathbf{M} is a non-singular matrix, the columns of \mathbf{M} form a basis for \mathbb{C}^N , known as *Nested Periodic Basis* (NPB). As a consequence, any finite N -length signal can be represented as a linear combination of columns of \mathbf{M} . Let s_{p_i} be the subspace spanned by the columns of \mathbf{M}_{p_i} , then

- 1) The period of every element in s_{p_i} is exactly p_i [14].

2) Let $x(n) = \sum_{i=1}^M x_{l_i}(n)$, where the period of $x_{l_i}(n)$ is equal to l_i . In general the period of $x(n)$ is a divisor of $lcm(l_1, \dots, l_M)$. But if $x_{l_i}(n) \in s_{l_i}$, then the period of $x(n)$ is exactly equal to $lcm(l_1, \dots, l_M)$ [14].

As an NPM construction involves the basis of divisor subspaces $(\{s_{p_i}\}_{i=1}^m)$, it is useful to extract the divisor periods information of a signal [8].

Let $\mathbf{A}, \mathbf{B} \in M_N(\mathbb{C})$ denote DFT and RPT matrices respectively. Both \mathbf{A} and \mathbf{B} are NPMs [14]. In the following sections, we imitate the construction procedure of \mathbf{B} from \mathbf{A} for constructing three more matrices \mathbf{C}, \mathbf{D} and \mathbf{E} of order N . Further, we show that \mathbf{C}, \mathbf{D} and \mathbf{E} are NPMs. First, we recall \mathbf{A} and construction of \mathbf{B} from \mathbf{A} in the following section.

III. RPT MATRIX CONSTRUCTION FROM DFT MATRIX

A. DFT Matrix

Let $S_{N,k}(n) = e^{j\frac{2\pi kn}{N}}$, $0 \leq n \leq N-1$, then

$$[\mathbf{A}]_{N \times N} = [S_{N,0}(n), S_{N,1}(n), \dots, S_{N,N-1}(n)]. \quad (2)$$

The period of k^{th} column in \mathbf{A} is equal to $\frac{N}{(k,N)}$, i.e., divisor of N . For each $p_i \in D_N$, the number of columns in \mathbf{A} having period exactly equal to p_i is $\#U_{p_i} = \varphi(p_i)$ [17]. If $U_{p_i} = \{k_1, k_2, \dots, k_{\varphi(p_i)}\}$, then form a sub-matrix \mathbf{A}_{p_i} of \mathbf{A} as given below:

$$[\mathbf{A}_{p_i}]_{N \times \varphi(p_i)} = [\hat{\mathbf{A}}_{p_i}, \hat{\mathbf{A}}_{p_i} \dots, \hat{\mathbf{A}}_{p_i}]^T, \text{ where} \quad (3)$$

$$\hat{\mathbf{A}}_{p_i} = [S_{p_i, k_1}(n), S_{p_i, k_2}(n), \dots, S_{p_i, k_{\varphi(p_i)}}(n)]_{p_i \times \varphi(p_i)}.$$

So, \mathbf{A}_{p_i} is obtained by repeating $\hat{\mathbf{A}}_{p_i}$ periodically $\frac{N}{p_i}$ times. As $\sum_{p_i|N} \varphi(p_i) = N$, by constructing $\hat{\mathbf{A}}_{p_i}$ for every $p_i \in D_N$, we can build an $N \times N$ transformation matrix \mathbf{A} , whose columns are permutations of columns of \mathbf{A} given in (2). Using the orthogonality and periodicity properties of $S_{N,k}$, one can check that \mathbf{A} is an NPM [14]. In the following sections, we provide different alternative matrices to $\hat{\mathbf{A}}_{p_i}$ such as $\hat{\mathbf{B}}_{p_i}, \hat{\mathbf{C}}_{p_i}, \hat{\mathbf{D}}_{p_i}$ and $\hat{\mathbf{E}}_{p_i}$ to construct $\mathbf{B}_{p_i}, \mathbf{C}_{p_i}, \mathbf{D}_{p_i}$ and \mathbf{E}_{p_i} followed by the construction of $\mathbf{B}, \mathbf{C}, \mathbf{D}$ and \mathbf{E} respectively. Now before proceeding further, we prove the following:

Theorem 1. Let $x(n) = \sum_{i=1}^M S_{N, k_i}(n)$, where $M \leq N$, $0 \leq k_i, n \leq N-1$ and the values $\{k_i\}_{i=1}^M$ are unique. If $\mathbf{G}_N^M \in M_N(\mathbb{C})$ is a circulant matrix, whose first column is $x(n)$ and the remaining columns are the circular downshift of the previous columns, then $r(\mathbf{G}_N^M) = M$.

Proof. The given circulant matrix \mathbf{G}_N^M can be decomposed as $\mathbf{G}_N^M = \mathbf{B}_{N \times M} \mathbf{B}^H_{M \times N}$, where

$$\mathbf{B} = \begin{bmatrix} S_{N, k_1}(0) & S_{N, k_2}(0) & \dots & S_{N, k_M}(0) \\ S_{N, k_1}(1) & S_{N, k_2}(1) & \dots & S_{N, k_M}(1) \\ \vdots & \vdots & \ddots & \vdots \\ S_{N, k_1}(N-1) & S_{N, k_2}(N-1) & \dots & S_{N, k_M}(N-1) \end{bmatrix}.$$

Here, the columns of \mathbf{B} are orthogonal, so $r(\mathbf{B}) = M$. As $r(\mathbf{G}_N^M) = r(\mathbf{B})$ [24], this implies $r(\mathbf{G}_N^M) = M$. \square

B. RPT Matrix

If we add all the columns of $\hat{\mathbf{A}}_{p_i}$ given in (3), it generates an integer-valued, p_i periodic sequence $c_{p_i}(n)$, known as *Ramanujan sum* [3], [25]. From **Theorem 1**, if we construct a $p_i \times p_i$ circulant matrix \mathbf{G}_{p_i} using $c_{p_i}(n)$, then, $r(\mathbf{G}_{p_i}) = \varphi(p_i)$. So, using $c_{p_i}(n)$ we can build a matrix $\hat{\mathbf{B}}_{p_i}$ as an alternative to $\hat{\mathbf{A}}_{p_i}$ as follows:

$$\hat{\mathbf{B}}_{p_i} = [c_{p_i}^0(n), c_{p_i}^1(n), \dots, c_{p_i}^{\varphi(p_i)-1}(n)]_{p_i \times \varphi(p_i)}, \quad (4)$$

where $c_{p_i}^j(n)$ indicates the circular downshift of the sequence $c_{p_i}(n)$ by j times. Let $\mathbf{B} \in M_N(\mathbb{C})$ be the matrix constructed using $\hat{\mathbf{B}}_{p_i}$, by following the similar way of \mathbf{A} construction from $\hat{\mathbf{A}}_{p_i}$. Then by invoking the orthogonality and periodicity properties of $c_{p_i}(n)$, it is shown in [14] that \mathbf{B} satisfies all the NPM properties. Here \mathbf{B} is known as RPT matrix [8].

Remark 1: As $(k_j, p_i) = (p_i - k_j, p_i)$, for every complex sequence $S_{p_i, k_j}(n) \in \hat{\mathbf{A}}_{p_i}$, there exists a complex conjugate sequence $S_{p_i, p_i - k_j}(n) \in \hat{\mathbf{A}}_{p_i}$. Both together form a complex conjugate pair. So, there are $\frac{\varphi(p_i)}{2}$ complex conjugate pairs in $\hat{\mathbf{A}}_{p_i}$, i.e., $\#\hat{U}_{p_i}$ [17]. In [26], the authors introduced a two dimensional subspace spanned by $\{S_{p_i, k}(n), S_{p_i, p_i - k}(n)\}$ for each $k \in \hat{U}_{p_i}$, known as *Complex Conjugate Subspace (CCS)*, denoted as $v_{p_i, k}$. So, $v_{p_i, k}$ consists of signals having period exactly equal to p_i with discrete frequency $\frac{2\pi k}{p_i}$ (or) $\frac{2\pi(p_i - k)}{p_i}$. Let $\hat{U}_{p_i} = \{k_1, k_2, \dots, k_{\frac{\varphi(p_i)}{2}}\}$, then $\hat{\mathbf{A}}_{p_i}$ can be rewritten with permutation of its columns as follows:

$$\hat{\mathbf{A}}_{p_i} = \underbrace{[S_{p_i, k_1}, S_{p_i, p_i - k_1}, \dots, S_{p_i, k_{\frac{\varphi(p_i)}{2}}}, S_{p_i, p_i - k_{\frac{\varphi(p_i)}{2}}}]_{p_i \times \varphi(p_i)}}_{\text{Basis of } v_{p_i, k_1}} \quad \underbrace{\dots}_{\text{Basis of } v_{p_i, k_{\frac{\varphi(p_i)}{2}}}}. \quad (5)$$

The following section introduces two types of arithmetic sums and their properties, which are used to construct alternate bases for CCS.

IV. COMPLEX CONJUGATE PAIR SUMS AND THEIR PROPERTIES

In [17], we proposed a real-valued summation by adding each complex conjugate pair known as *Complex Conjugate Pair Sum of type-1 (CCPS⁽¹⁾)*. Given any $L \in \mathbb{N}$, the CCPS⁽¹⁾ ($c_{L,k}^{(1)}(n)$) is defined as follows:

$$c_{L,k}^{(1)}(n) = 2M \cos\left(\frac{2\pi kn}{L}\right), \quad (6)$$

where

$$M = \begin{cases} \frac{1}{2}, & \text{if } L = 1 \text{ (or) } 2 \\ 1, & \text{if } L \geq 3 \end{cases}, \quad (7)$$

and if $L \geq 3$ then $k \in \hat{U}_L$, otherwise $k = 1$.

Similar to addition, subtraction is another arithmetical operation, which can be performed on each complex conjugate pair without changing its periodicity. This defines another real-valued sum known as *Complex Conjugate Pair Sum of type-2 (CCPS⁽²⁾)*, denoted as $c_{L,k}^{(2)}(n)$ and defined as,

$$c_{L,k}^{(2)}(n) = \begin{cases} 1, & \forall n, \text{ if } L = 1 \\ (-1)^n, & \text{if } L = 2 \\ 2\sin\left(\frac{2\pi kn}{L}\right), & \text{if } L \geq 3, k \in \hat{U}_L \end{cases}. \quad (8)$$

Let $c_{L,k}^{(*)}(n)$ hereafter denotes either $c_{L,k}^{(1)}(n)$ (or) $c_{L,k}^{(2)}(n)$.

A. Properties

1) *Periodicity*: As $c_{L,k}^{(*)}(n+L) = c_{L,k}^{(*)}(n)$, and $c_{L,(k+L)}^{(*)}(n) = c_{L,k}^{(*)}(n)^1$, CCPs are periodic with respect to both n and k with period L . If L is even, then $c_{L,(k+\frac{L}{2})}^{(*)}(n) = (-1)^n c_{L,k}^{(*)}(n)$ and $c_{L,k}^{(*)}(n+\frac{L}{2}) = (-1)^k c_{L,k}^{(*)}(n)$.

2) *Symmetric*: $c_{L,k}^{(1)}(n)$ and $c_{L,k}^{(2)}(n)$ (for $L \geq 3$) are even and odd symmetric sequences respectively, with respect to both k and n , i.e., $c_{L,k}^{(1)}(L-n) = c_{L,(L-k)}^{(1)}(n) = c_{L,k}^{(1)}(n)$ and $c_{L,k}^{(2)}(L-n) = c_{L,(L-k)}^{(2)}(n) = -c_{L,k}^{(2)}(n)$.

3) *DFT of CCPs*: For a given $L \in \mathbb{N}$ and $l \in \hat{U}_L$,

$$\begin{aligned} \text{DFT}[c_{L,l}^{(1)}(n)] &= C_{L,l}^{(1)}(k) = \begin{cases} L, & \text{if } k = l \text{ (or) } L-l \\ 0, & \text{Otherwise} \end{cases}, \\ \text{DFT}[c_{L,l}^{(2)}(n)] &= C_{L,l}^{(2)}(k) = \begin{cases} -jL, & \text{if } k = l \\ jL, & \text{if } k = L-l \\ 0, & \text{Otherwise} \end{cases} \end{aligned}$$

where $0 \leq k \leq L-1$. The above results are obvious from the definition of Complex Conjugate Pair Sums (CCPSs).

4) *Sum and sum-of-squares*: For a given $L > 1$ and $l \in \hat{U}_L$ $\sum_{n=0}^{L-1} c_{L,l}^{(*)}(n) = 0$. Given $L \in \mathbb{N}$, using the Parseval's relation between $c_{L,l}^{(*)}(n)$ and $C_{L,l}^{(*)}(k)$, we can write:

$$\sum_{n=0}^{L-1} \left(c_{L,k}^{(*)}(n) \right)^2 = \frac{1}{L} \sum_{k=0}^{L-1} \left(|C_{L,l}^{(*)}(k)| \right)^2 = 2LM, \quad (9)$$

where M is defined in (7).

5) *Orthogonality*: Given $L = \text{lcm}(L_1, L_2)$, $L_1 \in \mathbb{N}$, $L_2 \in \mathbb{N}$, $k_1 \in \hat{U}_{L_1}$, $k_2 \in \hat{U}_{L_2}$, $l_1 \in \mathbb{Z}$ and $l_2 \in \mathbb{Z}$, then we can prove the following theorem:

Theorem 2. Any two L length CCPs⁽¹⁾ (or) CCPs⁽²⁾ and their circular shifts are mutually orthogonal, i.e.,

$$\begin{aligned} \sum_{n=0}^{L-1} c_{L_1,k_1}^{(*)}(n-l_1) c_{L_2,k_2}^{(*)}(n-l_2) \\ = 2LM \cos\left(\frac{2\pi k_1(l_1-l_2)}{L_1}\right) \delta(L_1-L_2) \delta(k_1-k_2). \end{aligned} \quad (10)$$

Proof. Given in the appendix by assuming $c_{L_1,k_1}^{(*)} = c_{L_1,k_1}^{(1)}$. \square

Theorem 3. For a given $L_1 \geq 3$ and $L_2 \geq 3$, both CCPs⁽¹⁾ and CCPs⁽²⁾ are orthogonal to each other, i.e.,

$$\begin{aligned} \sum_{n=0}^{L-1} c_{L_1,k_1}^{(1)}(n-l_1) c_{L_2,k_2}^{(2)}(n-l_2) \\ = 2L \sin\left(\frac{2\pi k_1(l_1-l_2)}{L_1}\right) \delta(L_1-L_2) \delta(k_1-k_2). \end{aligned} \quad (11)$$

¹Use the property $(k, L) = (k+L, L)$, to verify $c_{L,(k+L)}^{(*)}(n) = c_{L,k}^{(*)}(n)$.

The above theorem can be proved using the same approach used to prove **Theorem 2**. If $L_1 < 3$ and $L_2 < 3$, then $c_{L_1,k_1}^{(1)}(n) = c_{L_2,k_2}^{(2)}(n)$, so **Theorem 3** satisfies **Theorem 2**.

Now we discuss, how these summations and their properties are used to construct the basis of CCS, followed by the construction of **C**, **D**, and **E** matrices.

V. NEW NESTED PERIODIC MATRICES

From **Theorem 1**, if we construct a $p_i \times p_i$ circulant matrix $\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(1)}$ using $c_{p_i, k}^{(1)}(n)$, then, $r(\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(1)}) = 2$. Let $c_{p_i, k}^{(*)j}$ indicate the circular downshift of the sequence $c_{p_i, k}^{(*)}$ by j times. As $c_{p_i, k}^{(1)}$ and $c_{p_i, k}^{(1)1}$ are linearly independent, the first two columns of $\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(1)}$ act as a basis for CCS [17]. So, the matrix $\hat{\mathbf{C}}_{\mathbf{p}_i}$ (an alternative to $\hat{\mathbf{A}}_{\mathbf{p}_i}$ given in (5)) built using this new basis is

$$\hat{\mathbf{C}}_{\mathbf{p}_i} = \left[\underbrace{c_{p_i, k_1}^{(1)}, c_{p_i, k_1}^{(1)1}}_{\text{Basis of } v_{p_i, k_1}}, \dots, \underbrace{c_{p_i, k_{\frac{\varphi(p_i)}{2}}}^{(1)}, c_{p_i, k_{\frac{\varphi(p_i)}{2}}}^{(1)1}}_{\text{Basis of } v_{p_i, k_{\frac{\varphi(p_i)}{2}}}} \right]_{p_i \times \varphi(p_i)}.$$

It is shown in [17] that the **C** built by using $\hat{\mathbf{C}}_{\mathbf{p}_i}$ is an NPM. Then any N length signal \mathbf{x} can be represented/synthesized as

$$\begin{aligned} \mathbf{x} = \mathbf{C}\boldsymbol{\beta}^{(1)} &= [\mathbf{C}_{\mathbf{p}_1}, \dots, \mathbf{C}_{\mathbf{p}_i}, \dots, \mathbf{C}_{\mathbf{p}_m}]_{N \times N} \boldsymbol{\beta}^{(1)}, \\ \text{where } [\mathbf{C}_{\mathbf{p}_i}]_{N \times \varphi(p_i)} &= [\hat{\mathbf{C}}_{\mathbf{p}_i}, \hat{\mathbf{C}}_{\mathbf{p}_i} \dots, \hat{\mathbf{C}}_{\mathbf{p}_i}]^T \end{aligned} \quad (12)$$

and $\boldsymbol{\beta}^{(1)}$ is the transform coefficient vector. Here **C** is a non-orthogonal matrix (refer **Theorem 2**). So,

$$\boldsymbol{\beta}^{(1)} = \mathbf{C}^{-1} \mathbf{x}. \quad (13)$$

This transformation from \mathbf{x} to $\boldsymbol{\beta}^{(1)}$ is known as *Complex Conjugate Periodic Transform (CCPT)* [17], denoted as $\text{CCPT}^{(1)}$. Both (12) and (13) together form a $\text{CCPT}^{(1)}$ pair.

In a similar way, if we construct a $p_i \times p_i$ circulant matrix $\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(2)}$ using $c_{p_i, k}^{(2)}(n)$, then $r(\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(2)}) = 2$. Further, $\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(2)}$ can be factorized as (refer **Theorem 1** proof for $M = 2$):

$$\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(2)} = -j(\mathbf{F}\hat{\mathbf{F}}^{\mathbf{H}}), \quad \text{where} \quad (14)$$

$$\mathbf{F}^{\mathbf{H}} = \begin{bmatrix} S_{p_i, p_i-k}(0) & S_{p_i, p_i-k}(1) & \dots & S_{p_i, p_i-k}(p_i-1) \\ S_{p_i, k}(0) & S_{p_i, k}(1) & \dots & S_{p_i, k}(p_i-1) \end{bmatrix}_{2 \times p_i}$$

and

$$\hat{\mathbf{F}}^{\mathbf{H}} = \begin{bmatrix} S_{p_i, p_i-k}(0) & S_{p_i, p_i-k}(1) & \dots & S_{p_i, p_i-k}(p_i-1) \\ -S_{p_i, k}(0) & -S_{p_i, k}(1) & \dots & -S_{p_i, k}(p_i-1) \end{bmatrix}_{2 \times p_i}.$$

From (14), the column space of $\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(2)}$ is same as the column space of \mathbf{F} [24], which is $v_{p_i, k}$. Moreover, one can check that the first two columns of $\mathbf{G}_{\mathbf{p}_i, \mathbf{k}}^{(2)}$ i.e., $c_{p_i, k}^{(2)}$ and $c_{p_i, k}^{(2)1}$ are linearly independent. So, they act as a basis for CCS. This provides another alternative for $\hat{\mathbf{A}}_{\mathbf{p}_i}$ as given below:

$$\hat{\mathbf{D}}_{\mathbf{p}_i} = \left[\underbrace{c_{p_i, k_1}^{(2)}, c_{p_i, k_1}^{(2)1}}_{\text{Basis of } v_{p_i, k_1}}, \dots, \underbrace{c_{p_i, k_{\frac{\varphi(p_i)}{2}}}^{(2)}, c_{p_i, k_{\frac{\varphi(p_i)}{2}}}^{(2)1}}_{\text{Basis of } v_{p_i, k_{\frac{\varphi(p_i)}{2}}}} \right]_{p_i \times \varphi(p_i)}. \quad (15)$$

From **Theorem 2** and (15), there are a few points worth noting:

- As $\langle c_{p_i, k_i}^{(2)l_1}, c_{p_i, k_j}^{(2)l_2} \rangle = 0$, $\forall k_i \neq k_j$, the columns of $\hat{\mathbf{D}}_{\mathbf{p}_i}$ are CCS wise orthogonal, this implies $r(\hat{\mathbf{D}}_{\mathbf{p}_i}) = \varphi(p_i)$.

- Since $c_{p_i, k_i}^{(2)}$ is a periodic sequence with period p_i , each column in $\hat{\mathbf{D}}_{p_i}$ is a p_i periodic sequence.
- As $\langle c_{p_i, k_i}^{(2)l_1}, c_{p_j, k_j}^{(2)l_2} \rangle = 0$, $\forall p_i \neq p_j$, $p_i | N$, and $p_j | N$, the rank of $\mathbf{D} \in M_N(\mathbb{C})$ built using $\hat{\mathbf{D}}_{p_i}$ is equal to N .

From the above three points \mathbf{D} results in an NPM. Hence, any finite N length signal \mathbf{x} is represented/synthesized as:

$$\mathbf{x} = \mathbf{D}\boldsymbol{\beta}^{(2)} = [\mathbf{D}_{p_1}, \dots, \mathbf{D}_{p_i}, \dots, \mathbf{D}_{p_m}]_{N \times N} \boldsymbol{\beta}^{(2)}, \quad (16)$$

where $\boldsymbol{\beta}^{(2)}$ is the transform coefficient vector. Along with this, the product $[\mathbf{D}_{p_i}]^T [\mathbf{D}_{p_j}] = [\mathbf{0}]_{\varphi(p_i) \times \varphi(p_j)}$, $\forall p_i \neq p_j$, i.e., s_{p_i} is orthogonal to s_{p_j} . If the subspaces are orthogonal, they can be uniquely determined as *Ramanujan Subspaces* [14]. From **Theorem 2**, $\langle c_{p_i, k}^{(2)1}(n), c_{p_i, k}^{(2)1}(n) \rangle \neq 0$, hence \mathbf{D} is a non-orthogonal matrix. So,

$$\boldsymbol{\beta}^{(2)} = \mathbf{D}^{-1} \mathbf{x}. \quad (17)$$

The above transform is known as *CCPT* and denoted as *CCPT*⁽²⁾. Both (16) and (17) together form a *CCPT*⁽²⁾ pair.

The non-orthogonality of the transformation matrices \mathbf{C} and \mathbf{D} results in both *CCPT*⁽¹⁾ and *CCPT*⁽²⁾ as non-orthogonal transforms. One can say that \mathbf{C} and \mathbf{D} are partially orthogonal as their columns are CCS wise orthogonal. In the following section, we propose an orthogonal basis for CCS, this leads to the construction of another NPM \mathbf{E} .

VI. ORTHOGONAL CCPT (OCCPT) AND ITS PROPERTIES

Here, both $c_{p_i, k}^{(1)}$ and $c_{p_i, k}^{(2)}$ together act as a basis for $v_{p_i, k}$.

Theorem 4. Given $v_{p_i, k}$ is the subspace spanned by $\{S_{p_i, k}(n), S_{p_i, p_i - k}(n)\}$ and $u_{p_i, k}$ is the subspace spanned by $\{c_{p_i, k}^{(1)}(n), c_{p_i, k}^{(2)}(n)\}$. Then $v_{p_i, k}$ is equal to $u_{p_i, k}$.

Proof. Let $x_{p_i, k} \in v_{p_i, k}$, then there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ s.t. $x_{p_i, k}(n) = \alpha_1 S_{p_i, k}(n) + \alpha_2 S_{p_i, p_i - k}(n)$. Using Euler's identity, we can rewrite $x_{p_i, k}(n) = \gamma_1 c_{p_i, k}^{(1)}(n) + \gamma_2 c_{p_i, k}^{(2)}(n)$, $\gamma_1, \gamma_2 \in \mathbb{C}$, hence $x_{p_i, k} \in u_{p_i, k}$. Similarly, any $y_{p_i, k} \in u_{p_i, k}$ also belongs to $v_{p_i, k}$. Hence $v_{p_i, k}$ is equal to $u_{p_i, k}$. \square

So, another alternative to $\hat{\mathbf{A}}_{p_i}$ can be written as given below:

$$\hat{\mathbf{E}}_{p_i} = \underbrace{[c_{p_i, k_1}^{(1)}, c_{p_i, k_1}^{(2)}, \dots]}_{\text{Basis of } v_{p_i, k_1}} \underbrace{[c_{p_i, k}^{(1) \varphi(p_i/2)}, c_{p_i, k}^{(2) \varphi(p_i/2)}]}_{\text{Basis of } v_{p_i, k} \varphi(p_i/2)}]_{p_i \times \varphi(p_i)}.$$

If $p_i = 1$ (or) 2 , then $\hat{\mathbf{E}}_{p_i} \in M_{p_i, 1}$ and $c_{p_i, k}^{(1)} = c_{p_i, k}^{(2)}$. So, we can consider either $c_{p_i, k}^{(1)}$ or $c_{p_i, k}^{(2)}$. From **Theorem 3**, we can summarize two points:

- As $\langle c_{p_i, k_i}^{(1)}, c_{p_i, k_j}^{(2)} \rangle = 0$, $\forall p_i | N$, k_i and $k_j \in \hat{U}_{p_i}$, the columns of $\hat{\mathbf{E}}_{p_i}$ are orthogonal to each other.
- As $\langle c_{p_i, k_i}^{(1)}, c_{p_j, k_j}^{(2)} \rangle = 0$, $\forall p_i | N$, $p_j | N$, $k_i \in \hat{U}_{p_i}$ and $k_j \in \hat{U}_{p_j}$, the columns of the matrix \mathbf{E} constructed using $\hat{\mathbf{E}}_{p_i}$ are mutually orthogonal.

Furthermore, \mathbf{E} is an NPM as both CCPSs are periodic. So, an N length signal $x(n)$ is represented/synthesized as

$$\mathbf{x} = \mathbf{E}\boldsymbol{\beta} = [\mathbf{E}_{p_1}, \dots, \mathbf{E}_{p_i}, \dots, \mathbf{E}_{p_m}]_{N \times N} \boldsymbol{\beta}, \quad (18)$$

where $\boldsymbol{\beta}$ is the transform coefficient vector. This transform is known as *Orthogonal CCPT*. Though the columns of \mathbf{E} are mutually orthogonal, the product $\mathbf{E}^T \mathbf{E} = 2NM \mathbf{I}$, where $M = \begin{cases} \frac{1}{2}, & \text{if } p_i = 1 \text{ (or) } 2 \\ 1, & \text{if } p_i \geq 3 \end{cases}$, and $\mathbf{I} \in M_N(\mathbb{C})$ is an identity matrix. So the resultant analysis equation is

$$\boldsymbol{\beta} = \frac{1}{2NM} \mathbf{E}^T \mathbf{x}. \quad (19)$$

The representation in (18) can be written algebraically as

$$x(n) = \sum_{p_i | N} x_{p_i}(n) = \sum_{p_i | N} \sum_{\substack{k=1 \\ (k, p_i)=1}}^{\lfloor \frac{p_i}{2} \rfloor} \underbrace{\beta_{0ki} c_{p_i, k}^{(1)}(n) + \beta_{1ki} c_{p_i, k}^{(2)}(n)}_{x_{p_i, k} \in v_{p_i, k}}, \quad (20)$$

where $0 \leq n \leq N - 1$ and $x_{p_i} \in s_{p_i}$. Manipulating (20) algebraically with $c_{p_j, k_1}^{(1)}(n)$ leads to (where $p_j | N$ and $k_1 \in \hat{U}_{p_j}$),

$$\sum_{n=0}^{N-1} x(n) c_{p_j, k_1}^{(1)}(n) = \sum_{p_i | N} \sum_{\substack{k=1 \\ (k, p_i)=1}}^{\lfloor \frac{p_i}{2} \rfloor} (\mathbf{Q} + \mathbf{R}), \quad \text{here} \quad (21)$$

$$\mathbf{Q} = \sum_{n=0}^{N-1} \beta_{0ki} c_{p_i, k}^{(1)}(n) c_{p_j, k_1}^{(1)}(n) = \begin{cases} 2NM \beta_{0k_1 j}, & \text{if } p_i = p_j \\ & \text{and } k = k_1, \\ 0, & \text{Otherwise} \end{cases}$$

and $\mathbf{R} = \sum_{n=0}^{N-1} \beta_{1ki} c_{p_i, k}^{(2)}(n) c_{p_j, k_1}^{(1)}(n) = 0$. Similarly, by manipulating with $c_{p_j, k_1}^{(2)}(n)$, we can obtain the following

$$\begin{aligned} \beta_{0ki} &= \frac{1}{2NM} \sum_{n=0}^{N-1} x(n) c_{p_i, k}^{(1)}(n), \quad p_i | N \ \& \ k \in \hat{U}_{p_i}, \\ \beta_{1ki} &= \frac{1}{2NM} \sum_{n=0}^{N-1} x(n) c_{p_i, k}^{(2)}(n), \quad p_i | N \ \& \ k \in \hat{U}_{p_i}. \end{aligned} \quad (22)$$

Here (22) is an analysis equation and (20) is a synthesis equation, both together form an N -point *OCCPT pair*.

A. Properties

Let $x(n)$, $x_1(n)$ and $x_2(n)$ be the signals, whose OCCPT coefficients are $(\beta_{0ki}, \beta_{1ki})$, $(\hat{\beta}_{0ki}, \hat{\beta}_{1ki})$ and $(\tilde{\beta}_{0ki}, \tilde{\beta}_{1ki})$ respectively. These relationships are denoted as

$$\begin{aligned} x(n) &\xleftrightarrow{N\text{-OCCPT}} (\beta_{0ki}, \beta_{1ki}), \quad x_1(n) \xleftrightarrow{N\text{-OCCPT}} (\hat{\beta}_{0ki}, \hat{\beta}_{1ki}) \\ &\text{and } x_2(n) \xleftrightarrow{N\text{-OCCPT}} (\tilde{\beta}_{0ki}, \tilde{\beta}_{1ki}). \end{aligned}$$

Then we can derive the following properties:

1) *Circular Shift of a Sequence:* The N -point OCCPT of $x(((n-m))_N)$, for an arbitrary $m \in \mathbb{Z}$, is defined as

$$x_1(n) = x(((n-m))_N) \xleftrightarrow{N\text{-OCCPT}} (\hat{\beta}_{0ki}, \hat{\beta}_{1ki}). \quad (23)$$

$$\text{If } p_i = 1 \text{ (or) } 2: \hat{\beta}_{0ki} = \beta_{0ki} \cos(\theta). \quad (24)$$

$$\text{If } p_i \geq 3: \begin{bmatrix} \hat{\beta}_{0ki} \\ \hat{\beta}_{1ki} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \beta_{0ki} \\ \beta_{1ki} \end{bmatrix}, \quad (25)$$

where $\theta = \frac{2\pi k((-m))_N}{p_i}$. So, the circular delay in time results in proportionate rotation of transform coefficients.

2) *Circular Convolution*: The N -point OCCPT of $x(n) = x_1(n) \otimes x_2(n)$ is defined as

$$x(n) = \sum_{l=0}^{N-1} x_1(l)x_2(((n-l))_N) \xleftrightarrow{N\text{-OCCPT}} (\beta_{0ki}, \beta_{1ki}).$$

$$\text{If } p_i = 1 \text{ (or) } 2 : \beta_{0ki} = N \hat{\beta}_{0ki} \tilde{\beta}_{0ki}. \quad (26)$$

$$\text{If } p_i \geq 3 : \begin{bmatrix} \beta_{0ki} \\ \beta_{1ki} \end{bmatrix} = N \begin{bmatrix} \hat{\beta}_{0ki} & -\hat{\beta}_{1ki} \\ \hat{\beta}_{1ki} & \hat{\beta}_{0ki} \end{bmatrix} \begin{bmatrix} \tilde{\beta}_{0ki} \\ \tilde{\beta}_{1ki} \end{bmatrix}. \quad (27)$$

From above, the transform coefficients of circular convolution are satisfying the commutative property.

3) *Parseval's Relation*: The OCCPT conserves the energy of a given signal. In specific,

$$\sum_{n=0}^{N-1} |x(n)|^2 = N \left(|\beta_{011}|^2 + |\beta_{012}|^2 \right) + 2N \sum_{\substack{p_i | N \\ p_i \geq 3}} \sum_{\substack{k=1 \\ (k, p_i)=1}}^{\lfloor \frac{p_i}{2} \rfloor} |\beta_{0ki}|^2 + |\beta_{1ki}|^2. \quad (28)$$

Here the term $\beta_{012} = 0$, if 2 is not a divisor of N . Proofs for the above three properties are given in the appendix.

4) *Periodicity*: Using the periodicity property (with respect to k) of CCPSSs, it can be proved that $(\beta_{0ki}, \beta_{1ki})$ are periodic with period N , i.e.,

$$\beta_{0(k+N)i} = \beta_{0ki} \text{ and } \beta_{1(k+N)i} = \beta_{1ki}. \quad (29)$$

B. Relation Between Orthogonal CCPT and DFT Coefficients

Let $x(n) \in \mathbb{C}^N$, then the DFT coefficients $X(k)$, $0 \leq k \leq N-1$ are obtained by performing the dot product between $x(n)$ and $e^{-j\frac{2\pi kn}{N}} = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)$. For a given k , let $k_i = \frac{k}{d_i}$ and $p_i = \frac{N}{d_i}$ where $d_i = (k, N)$. Then the basis (cosine and sine terms) of v_{p_i, k_i} establish relationship between $X(k)$ and $(\beta_{0k_i i}, \beta_{1k_i i})$. Let

$$x(n) \xleftrightarrow{N\text{-DFT}} X(k) \text{ \& } x(n) \xleftrightarrow{N\text{-OCCPT}} (\beta_{0k_i i}, \beta_{1k_i i}), \quad (30)$$

where $\beta_{0k_i i} = \hat{\beta}_{0k_i i} + j\tilde{\beta}_{0k_i i}$ and $\beta_{1k_i i} = \hat{\beta}_{1k_i i} + j\tilde{\beta}_{1k_i i}$. Then, by exploiting the analysis equation of DFT, OCCPT and symmetry properties of CCPSSs, we can establish the following relation for every $p_i \in D_N$ and $k_i \in \hat{U}_{p_i}$:

$$\text{If } p_i = 1 \text{ (or) } 2 : X(k) = X\left(\frac{Nk_i}{p_i}\right) = N\beta_{0k_i i}. \quad (31)$$

$$\text{If } p_i \geq 3 : X(k) = X\left(\frac{Nk_i}{p_i}\right) = \begin{cases} N \left[\left(\hat{\beta}_{0k_i i} + \tilde{\beta}_{1k_i i} \right) + j \left(\tilde{\beta}_{0k_i i} - \hat{\beta}_{1k_i i} \right) \right], & \text{if } k_i \in \hat{U}_{p_i} \\ N \left[\left(\hat{\beta}_{0k_i i} - \tilde{\beta}_{1k_i i} \right) + j \left(\tilde{\beta}_{0k_i i} + \hat{\beta}_{1k_i i} \right) \right], & \text{if } k_i \in \tilde{U}_{p_i} \end{cases}. \quad (32)$$

If $x(n) \in \mathbb{R}^N$, then $\tilde{\beta}_{0k_i i} = \tilde{\beta}_{1k_i i} = 0$ in the above equation. Since the DFT coefficients are periodic with N , consider $k =$

N whenever $k = 0$, as it is an invalid case for $v_{N, k}$. From the relations given in (31) and (32), we can get both magnitude and phase information of a discrete frequency $\left(\frac{2\pi k}{N}\right)$ from OCCPT coefficients. The presence of circular downshift terms (in both **C** and **D**) allows us to extract magnitude, and not the phase information from $\beta^{(1)}$ and $\beta^{(2)}$.

The notion of the period is explicitly available from the N -point OCCPT pair given in (20) and (22), whereas the notion of frequency is represented in a succinct way. To get explicit frequency information, a reinterpretation of OCCPT followed by a DIT based fast computation algorithm for it are discussed in the following section.

VII. FAST OCCPT (FOCCPT)

A. Reinterpretation of OCCPT

Consider the following sets of irreducible rational numbers

$$H_1 = \left\{ \frac{k}{p_i} \mid 0 \leq k \leq \left\lfloor \frac{p_i}{2} \right\rfloor, (k, p_i) = 1, p_i | N \right\},$$

$$H_2 = \left\{ \frac{\hat{k}}{p_i} \mid \left\lfloor \frac{p_i}{2} \right\rfloor + 1 \leq \hat{k} \leq p_i - 1, (\hat{k}, p_i) = 1, p_i | N, p_i > 2 \right\}. \quad (33)$$

Both H_1 and H_2 can be rewritten as sets of all rational elements as given below:

$$H_1 = \left\{ \frac{K}{N} \mid 0 \leq K \leq \left\lfloor \frac{N}{2} \right\rfloor \right\} \text{ and}$$

$$H_2 = \left\{ \frac{K}{N} \mid \left\lfloor \frac{N}{2} \right\rfloor + 1 \leq K \leq N - 1 \right\}. \quad (34)$$

In an N -point OCCPT, the total columns of **E** can be divided into two sets as

$$H_3 = \left\{ 2M \cos\left(\frac{2\pi kn}{p_i}\right) \mid k \in \hat{U}_{p_i}, p_i | N \right\},$$

$$H_4 = \left\{ -2 \sin\left(\frac{2\pi \hat{k}n}{p_i}\right) \mid \hat{k} = (p_i - k) \in \tilde{U}_{p_i}, p_i | N, p_i > 2 \right\}.$$

Using (33) and (34), we can reinterpret H_3 and H_4 as follows:

$$H_3 = \left\{ 2\hat{M} \cos\left(\frac{2\pi Kn}{N}\right) \mid 0 \leq K \leq \left\lfloor \frac{N}{2} \right\rfloor \right\},$$

$$H_4 = \left\{ -2 \sin\left(\frac{2\pi Kn}{N}\right) \mid \left\lfloor \frac{N}{2} \right\rfloor + 1 \leq K \leq N - 1 \right\}, \quad (35)$$

where, if N is even then $\hat{M} = \begin{cases} \frac{1}{2}, & \text{if } K = 0 \text{ (or) } \frac{N}{2} \\ 1, & \text{otherwise} \end{cases}$ and

if N is odd then $\hat{M} = \begin{cases} \frac{1}{2}, & \text{if } K = 0 \\ 1, & \text{otherwise} \end{cases}$. That is, **E** can be rewritten with some permutations of its columns as given below:

$$\hat{\mathbf{E}} = \begin{bmatrix} 2\hat{M} \cos\left(\frac{2\pi(0)}{N}n\right), \dots, 2\hat{M} \cos\left(\frac{2\pi(\lfloor \frac{N}{2} \rfloor)}{N}n\right), \\ -2 \sin\left(\frac{2\pi(\lfloor \frac{N}{2} \rfloor + 1)}{N}n\right), \dots, -2 \sin\left(\frac{2\pi(N-1)}{N}n\right) \end{bmatrix}. \quad (36)$$

Now, we can rewrite the N -point OCCPT pair as

$$\beta(K) = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi Kn}{N}\right), & 0 \leq K \leq \lfloor \frac{N}{2} \rfloor \\ -\frac{1}{N} \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi Kn}{N}\right), & \lfloor \frac{N}{2} \rfloor + 1 \leq K \leq N-1. \end{cases} \quad (37)$$

$$x(n) = 2 \sum_{K=0}^{\lfloor \frac{N}{2} \rfloor} \hat{M}\beta(K) \cos\left(\frac{2\pi Kn}{N}\right) - 2 \sum_{K=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \beta(K) \sin\left(\frac{2\pi Kn}{N}\right). \quad (38)$$

The relation between the coefficients given in (22) and (37) is

$$\beta_{0ki} = \beta\left(\frac{Nk}{p_i}\right) \text{ and } \beta_{1ki} = \beta\left(\frac{N\hat{k}}{p_i}\right). \quad (39)$$

B. Decimation-In-Time FOCCPT (DIT-FOCCPT) Algorithm

Here, an N length sequence $x(n)$ is decomposed into successively smaller sub-sequences. The N -point OCCPT of $x(n)$ is computed by combining the OCCPT of these sub-sequences. The symmetry properties of CCPs are exploited in the combining process. As an initial step, we consider $N = 2^v$, $v \in \mathbb{N}$ (similar to radix-2 DIT-FFT [1]), this allows us to decompose $x(n)$ into two $\frac{N}{2}$ length sequences $h(n)$ and $g(n)$, where $h(n) = x(2n)$ and $g(n) = x(2n+1)$. By using the odd symmetry of $\sin(\cdot)$ function, the analysis equation of OCCPT given in (37) is modified as

$$N\beta(K) = X_x(K) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi Kn}{N}\right), \quad 0 \leq K \leq \frac{N}{2}$$

$$N\beta(N-K) = Y_x(K) = \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi Kn}{N}\right), \quad 1 \leq K \leq \frac{N}{2} - 1 \quad (40)$$

Here $X_x(K+N) = X_x(K)$ and $Y_x(K+N) = Y_x(K)$. Moreover, $X_x(N-K) = X_x(K)$ and $Y_x(N-K) = -Y_x(K)$ over the range of 0 to N . Now by decomposing $x(n)$ into $h(n)$ and $g(n)$, we obtain

$$X_x(K) = X_h(K) + \cos\left(\frac{2\pi K}{N}\right) X_g(K) - \sin\left(\frac{2\pi K}{N}\right) Y_g(K) = N\beta(K), \quad 0 \leq K \leq \frac{N}{4} \text{ and} \quad (41)$$

$$Y_x(K) = Y_h(K) + \cos\left(\frac{2\pi K}{N}\right) Y_g(K) + \sin\left(\frac{2\pi K}{N}\right) X_g(K) = N\beta(N-K), \quad 1 \leq K \leq \frac{N}{4}. \quad (42)$$

$$\text{Where, } X_f(K) = \sum_{n=0}^{\frac{N}{2}-1} f(n) \cos\left(\frac{2\pi Kn}{\frac{N}{2}}\right), \quad 0 \leq K \leq \frac{N}{4}$$

$$\text{and } Y_f(K) = \sum_{n=0}^{\frac{N}{2}-1} f(n) \sin\left(\frac{2\pi Kn}{\frac{N}{2}}\right), \quad 1 \leq K \leq \frac{N}{4} - 1 \quad (43)$$

represents the $\frac{N}{2}$ -point OCCPT of $f(n)$, here $f(n)$ can be either $h(n)$ or $g(n)$. Here, $X_x(K)$ is computed for $0 \leq K \leq \frac{N}{4}$, since the range of $X_h(K)$ and $X_g(K)$ is $0 \leq K \leq \frac{N}{4}$. The

remaining $\frac{N}{4}$ coefficients of $X_x(K)$ are computed using the symmetry property of $X_h(K)$, $X_g(K)$ and $Y_g(K)$, i.e.,

$$X_x\left(\frac{N}{2} - K\right) = X_h(K) - \cos\left(\frac{2\pi K}{N}\right) X_g(K) + \sin\left(\frac{2\pi K}{N}\right) Y_g(K) = N\beta\left(\frac{N}{2} - K\right), \quad 0 \leq K \leq \frac{N}{4} - 1 \quad (44)$$

Similarly, the remaining coefficients of $Y_x(K)$ are computed using the symmetry property of $Y_h(K)$, $Y_g(K)$ and $X_g(K)$, i.e.,

$$Y_x\left(\frac{N}{2} - K\right) = -Y_h(K) + \cos\left(\frac{2\pi K}{N}\right) Y_g(K) + \sin\left(\frac{2\pi K}{N}\right) X_g(K) = N\beta\left(\frac{N}{2} + K\right), \quad 1 \leq K \leq \frac{N}{4} - 1 \quad (45)$$

Note that in (41), (42) and (44) we have the terms $Y_h(K)$ and $Y_g(K)$, with possible K values to be 0 (or) $\frac{N}{4}$, but the actual range of $Y_h(K)$ and $Y_g(K)$ is $1 \leq K \leq \frac{N}{4} - 1$. Moreover, $Y_h(K) = Y_g(K) = 0$ whenever $K = 0$ (or) $\frac{N}{4}$. So, the equations (41), (42) and (44) can be further reduced as follows:

$$X_x(K) = \begin{cases} X_h(K) + \cos\left(\frac{2\pi K}{N}\right) X_g(K), & \text{if } K = 0 \text{ (or) } \frac{N}{4} \\ X_h(K) + \cos\left(\frac{2\pi K}{N}\right) X_g(K) - \sin\left(\frac{2\pi K}{N}\right) Y_g(K), & \text{if } 1 \leq K \leq \frac{N}{4} - 1 \end{cases} \quad (46)$$

$$X_x\left(\frac{N}{2} - K\right) = \begin{cases} X_h(K) - \cos\left(\frac{2\pi K}{N}\right) X_g(K), & \text{if } K = 0 \\ X_h(K) - \cos\left(\frac{2\pi K}{N}\right) X_g(K) + \sin\left(\frac{2\pi K}{N}\right) Y_g(K), & \text{if } 1 \leq K \leq \frac{N}{4} - 1 \end{cases} \quad (47)$$

and

$$Y_x(K) = \begin{cases} Y_h(K) + \cos\left(\frac{2\pi K}{N}\right) Y_g(K) + \sin\left(\frac{2\pi K}{N}\right) X_g(K), & \text{if } 1 \leq K \leq \frac{N}{4} - 1 \\ \sin\left(\frac{2\pi K}{N}\right) X_g(K), & \text{if } K = \frac{N}{4} \end{cases} \quad (48)$$

The equations (45)-(48) correspond to the decomposition of original N -point OCCPT into two $\frac{N}{2}$ -point OCCPT computations. As $N = 2^v$, we can further decompose each $\frac{N}{2}$ -point OCCPT into two $\frac{N}{4}$ -point OCCPTs. This process is repeated for $v = \log_2(N)$ times. Fig. 1 depicts the complete DIT decomposition flow graph of an 8-point OCCPT computation.

C. Computational Complexity of N -Point DIT-FOCCPT

Initially, we count the number of multiplications and additions required for v^{th} stage (final stage), i.e., computing the N -

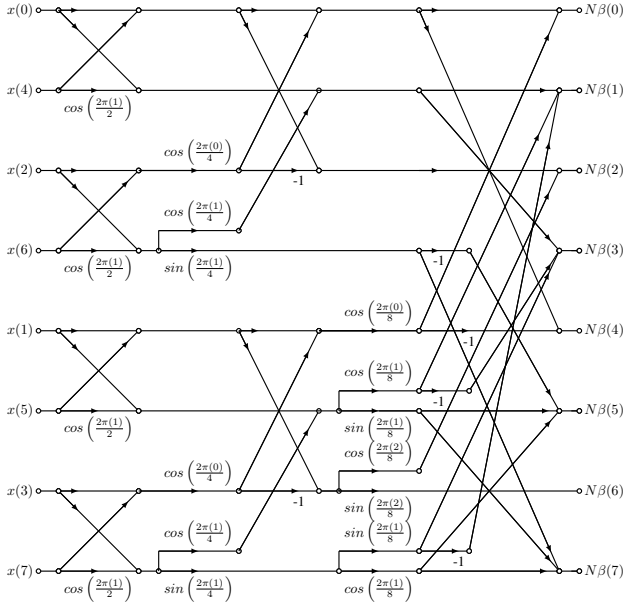


Fig. 1: Flow graph of complete DIT decomposition of an 8-point OCCPT computation.

point OCCPT from two $\frac{N}{2}$ -point OCCPTs using (45)-(48). Notice that in both (46) and (47), we have the same multiplication terms with a difference in addition/subtraction operations. So, to find the number of multiplications, it is sufficient to consider one equation with a maximum variation range of K . Hence we consider (46) and it requires $2\left(\frac{N}{4} - 1\right) + 2 = \frac{N}{2}$ multiplications. Whereas for additions we have to consider both the equations. They require $4\left(\frac{N}{4} - 1\right) + 3 = N - 1$ additions. Similarly, both (45) and (48) require $2\left(\frac{N}{4} - 1\right) + 1 = \frac{N}{2} - 1$ multiplications and $4\left(\frac{N}{4} - 1\right) = N - 4$ additions. As a result, we have the following for v^{th} stage:

$$M_v = N - 1, \quad A_v = 2N - 5,$$

where M_i and A_i denote the number of multiplications and additions required in i^{th} stage respectively. Likewise, there are two $\frac{N}{2}$ -point OCCPTs computation from four $\frac{N}{4}$ -point OCCPTs in $(v - 1)^{\text{th}}$ stage. Hence

$$M_{v-1} = 2\left(\frac{N}{2} - 1\right), \quad A_{v-1} = 2\left(2\left(\frac{N}{2}\right) - 5\right).$$

Proceeding further, the first stage requires computation of 2-point OCCPT for $\frac{N}{2}$ times. Here each 2-point OCCPT requires one multiplication and two additions. Therefore

$$M_2 = \frac{N}{2}(2 - 1) = \frac{N}{2}, \quad A_2 = \frac{N}{2}(2) = N.$$

Now, by combining each stage complexity we can count the total number of multiplications (M_{total}) and additions (A_{total}) required for N -point OCCPT, i.e.,

$$\begin{aligned} M_{\text{total}} &= 1(N - 1) + 2\left(\frac{N}{2} - 1\right) + \dots + \frac{N}{4}(4 - 1) + \frac{N}{2}(2 - 1) \\ &= vN - (N - 1) = N \log_2(N) - N + 1, \\ A_{\text{total}} &= 1(2N - 5) + 2\left(2\left(\frac{N}{2}\right) - 5\right) + \dots + \frac{N}{4}(2(4) - 5) + N \\ &= 2vN - 5\left(\frac{N}{2}\right) + 5 - N = 2N \log_2(N) - 7\left(\frac{N}{2}\right) + 5. \end{aligned}$$

Therefore, a given $x(n) \in \mathbb{R}^N$ requires $N \log_2(N) - N + 1$ real multiplications and $2N \log_2(N) - 7\left(\frac{N}{2}\right) + 5$ real additions for computing N -point OCCPT using DIT-FOCCPT.

If $x(n) \in \mathbb{C}^N$, then it requires $2N \log_2(N) - 2N + 2$ real multiplications and $4N \log_2(N) - 7N + 10$ real additions, as OCCPT is a linear transform.

D. Comparison of Computational Complexity Between Different Transforms

For a given $x(n) \in \mathbb{C}^N$, the N -point OCCPT and DFT are computed using FOCCPT and FFT respectively, when $N = 2^v$, $v \in \mathbb{N}$. If $N \neq 2^v$, both are computed using the direct method, even though there exist fast computation algorithms for DFT [22], [23]. Whereas CCPT⁽¹⁾ and CCPT⁽²⁾ are computed using direct method for both $N = 2^v$ and $N \neq 2^v$ cases due to the non-orthogonality of transformation matrices **C** and **D**. Even for RPT, we use the direct method for both cases. But if $N = 2^v$, the RPT matrix is a sparse orthogonal matrix. To the best of our knowledge, there is no fast computation algorithm for computing RPT in the literature.

If $x \in \mathbb{C}^N$ and $y \in \mathbb{C}^N$, then the number of real multiplications and additions required for computing $\langle x, y \rangle$ is $4N$ and $4N - 2$ respectively. Similarly, if $x \in \mathbb{C}^N$ and $y \in \mathbb{R}^N$, then it requires $2N$ and $2(N - 1)$ real multiplications and additions respectively. Using this, the computational complexity (in terms of the number of real multiplications and additions) required for different transforms for a given $x(n) \in \mathbb{C}^N$ is tabulated in Table I. There are a few points we can summarize from Table I:

- If $N = 2^v$, then FOCCPT results in a reduction of $2N - 2$ real multiplications with an increase of $N \log_2(N) - 7N + 10$ real additions over FFT. Here the addition complexity becomes significant for a large value of N ($N > 2^8$).
- If $N = 2^v$ and $x(n) \in \mathbb{R}^N$, then FFT requires $2N \log_2(N)$ real multiplications and $3N \log_2(N)$ real additions. This implies FOCCPT has approximately 50% reduction in computational complexity over FFT.
- The number of multiplications/additions required for the proposed transforms and RPT is approximately 50% less when compared with DFT using the direct method.

Note that, both CCPT⁽¹⁾ and CCPT⁽²⁾ require some additional complexity to find **C**⁻¹ and **D**⁻¹ respectively, along with the complexity given in Table I. This additional complexity is required even for RPT when $N \neq 2^v$. In this paper, both CCPT⁽¹⁾ and CCPT⁽²⁾ are studied in brief. A complete study of these transforms, especially the importance of circular shift operation in the matrices is one of our future works. The proposed orthogonal and non-orthogonal transforms (CCPTs) may find applications in communication, image processing, control applications, etc., [18]–[21], [27].

TABLE I: COMPARISON OF COMPUTATIONAL COMPLEXITY BETWEEN DIFFERENT TRANSFORMATION TECHNIQUES

	If $N \neq 2^v$		If $N = 2^v$	
	Number of Real Multiplications	Number of Real Additions	Number of Real Multiplications	Number of Real Additions
CCPT ⁽¹⁾	$2N^2$	$2N^2 - 2N$	$2N^2$	$2N^2 - 2N$
CCPT ⁽²⁾	$2N^2$	$2N^2 - 2N$	$2N^2$	$2N^2 - 2N$
Orthogonal CCPT	$2N^2$	$2N^2 - 2N$	$2N \log_2(N) - 2N + 2$	$4N \log_2(N) - 7N + 10$
DFT	$4N^2$	$4N^2 - 2N$	$2N \log_2(N)$	$3N \log_2(N)$
RPT	$2N^2$	$2N^2 - 2N$	$2N^2$	$2N^2 - 2N$

If $x(n) \in \mathbb{R}^N$, then DFT coefficients follow the symmetry property, i.e., for each discrete frequency, we have four real coefficients, whereas CCPT has only two coefficients. This results in a *non-redundant representation*.

Remark 2: In literature, there exist methods based on a DIT algorithm for fast N -point DFT computation, whenever $N \neq 2^v$ [1], [22], [23]. In a similar way, extending the proposed DIT-FOCCPT for $N \neq 2^v$ is one of our future works.

Remark 3: There exists a rich literature [28]–[31] for fast computation of DFT, with complexity less than DIT-FFT complexity. So, further reducing the complexity of OCCPT to less than the DIT-FOCCPT complexity is another open problem, which needs to be addressed.

Notation: Now onwards, a general CCPT can be either CCPT⁽¹⁾ (or) CCPT⁽²⁾ (or) OCCPT.

VIII. PERIOD AND FREQUENCY ESTIMATION

For a given N length signal, the possible period can be a divisor (or) non-divisor of N . In the following subsections, we explore how CCPTs are used to estimate these periods.

A. Divisor Period and Frequency Estimation

As the CCPT matrix is an NPM, it estimates the period and hidden periods of a signal which are divisors of the signal length. We illustrate the same by considering two 54 length periodic sequences $\hat{x}_1(n)$ and $\hat{x}_2(n)$. The period of $\hat{x}_1(n)$ is 18 and the period of $\hat{x}_2(n)$ is 40. In this example, both $\hat{x}_1(n)$ and $\hat{x}_2(n)$ can be periodically decomposed as follows:

$$\hat{x}_1(n) = x_{11}(n) + x_{12}(n) \text{ and } \hat{x}_2(n) = x_{21}(n) + x_{22}(n), \quad (49)$$

where $x_{11}(n)$, $x_{12}(n)$, $x_{21}(n)$ and $x_{22}(n)$ are periodic signals with periods 9, 18, 5 and 8 respectively. These periods are known as *hidden periods* [32]. Here $x_{11}(n)$, $x_{21}(n)$ are the two periodic random signals, whose one period data is generated by following $\mathcal{N}(0, 1)$, $x_{12}(n) = 0.6 \cos(2\pi 100(\frac{n}{360}) + \frac{\pi}{3})$ and $x_{22}(n) = 0.3 \cos(2\pi 45(\frac{n}{360}) + \frac{\pi}{4})$. Moreover a zero mean Gaussian noise with $SNR = 6$ dB is added to both $\hat{x}_1(n)$ and $\hat{x}_2(n)$ to generate $x_1(n)$ and $x_2(n)$ respectively.

The three CCPT coefficients are computed for both $x_1(n)$ and $x_2(n)$. Fig. 2(a)–(c) and Fig. 2(d)–(f) depict the strength of each divisor period present in $x_1(n)$ and $x_2(n)$ respectively. The strength of a period p_i is computed by taking the square sum of $\varphi(p_i)$ transform coefficients that belong to s_{p_i} . From Fig. 2(a)–(c), the significant periods present in $x_1(n)$ are 3, 9 and 18. Hence, the period of $x_1(n)$ is equal to $lcm(3, 9, 18)$. While from Fig. 2(d)–(f), the period of $x_2(n)$ is equal to 54, as s_{54} has significant period strength. So, the proposed transforms failed to estimate the period of $x_2(n)$, as $40 \nmid 54$.

Moreover, CCPT estimates the frequency information of a signal. Fig. 2(g)–(i) depicts the absolute values of transform coefficients that belong to s_{18} , computed for $x_1(n)$. Note that there are only two significant non-zero coefficients in s_{18} . Because, in CCPT each $\varphi(p_i)$ dimensional s_{p_i} is further decomposed as $\frac{\varphi(p_i)}{2}$ orthogonal CCSs, where each CCS v_{p_i, k_i} consists of signals with frequency $\frac{2\pi k_i}{p_i}$. Now, in this example s_{18} is decomposed as $s_{18} = v_{18,1} \oplus v_{18,5} \oplus v_{18,7}$. Since the sampling frequency is 360 Hz, the CCSs $v_{18,1}$, $v_{18,5}$ and

$v_{18,7}$ consist of signals with frequencies 20 Hz, 100 Hz and 140 Hz accordingly. As the given $x_{12}(n)$ is a 100 Hz signal, the coefficients 15 and 16, which belong to $v_{18,5}$ are having significant strength. Along with this, we can find out the phase information using OCCPT coefficients as given in (32). From Fig. 2(i), the two significant coefficient values are 0.149 and -0.261 , then $-\tan^{-1}(\frac{-0.261}{0.149}) = 1.05 \approx \frac{\pi}{3} = 1.04$ rad.

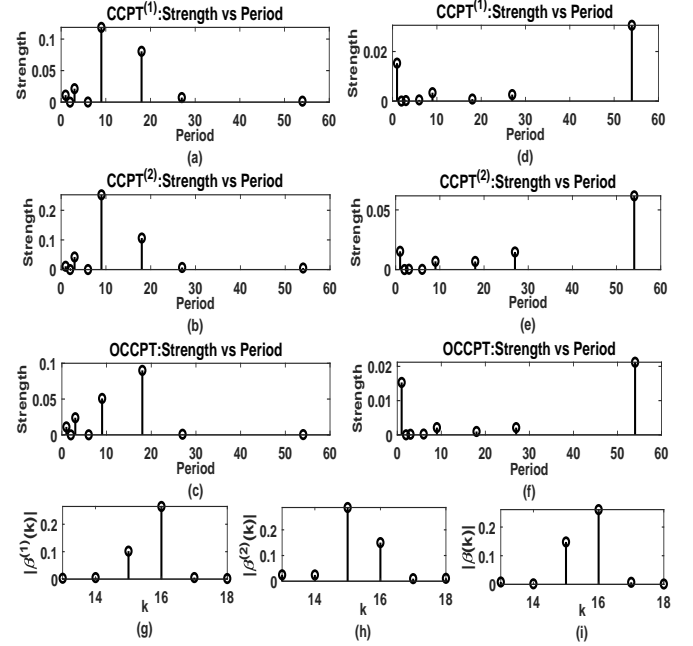


Fig. 2: (a)–(c) Strength vs Period plots obtained from the three CCPT coefficients of $x_1(n)$. (d)–(f) Strength vs Period plots obtained from the three CCPT coefficients of $x_2(n)$. (g)–(i) Absolute values of CCPT⁽¹⁾, CCPT⁽²⁾ and OCCPT coefficients that belong to s_{18} , computed for $x_1(n)$.

Therefore, using the proposed transforms we can estimate the divisor period, frequency and phase information of a signal. Even DFT works for the same purpose, while RPT gives only the divisor period information [17].

Now, we evaluate the period estimation performance of different transforms (DFT, RPT, and CCPTs) in the presence of noise. By varying SNRs of $x_1(n)$, we compute the strength of each divisor period using different transforms. Consider the periods with significant strength, by keeping 20% of maximum strength as a threshold. The obtained results are tabulated in Table II. From the table, we claim that the performance of OCCPT is quite good in the presence of noise and it is comparable with DFT and RPT, whereas both CCPT⁽¹⁾ and CCPT⁽²⁾ are sensitive to noise.

TABLE II: EVALUATING THE PERIOD ESTIMATION PERFORMANCE OF DIFFERENT TRANSFORMATION TECHNIQUES IN THE PRESENCE OF NOISE

SNR (dB)	6	3	0	-3	-6	-9	-12
DFT	3,9,18	3,9,18	3,9,18	3,9,18	3,9,18	3,9,18,27,54	3,9,18,27,54
RPT	3,9,18	3,9,18	3,9,18	3,9,18	3,9,18	1,3,9,18	3,6,9,18,27
OCCPT	3,9,18	3,9,18	3,9,18	3,9,18	3,9,18	3,9,18,27,54	3,9,18,27,54
CCPT ⁽¹⁾	3,9,18	9,18	9,18,27,54	3,9,18,54	3,9,18,27,54	9,27,54	9,18,27,54
CCPT ⁽²⁾	9,18	9,18	9,18	3,9,18,54	9,18,27,54	9,18,54	9,27,54

In general, the required period may not be a divisor of the signal length. In the following subsection, this issue is addressed by considering a dictionary based approach.

B. Non-divisor Period and Frequency Estimation

In this scenario, the signal is projected onto each and every subspace s_1 to $s_{P_{max}}$, instead of projecting only onto the divisor subspaces (as in NPM). Here P_{max} is the maximum possible period that exists in the signal. This generates a fat matrix \mathbf{F} known as the CCPT dictionary. Since, \mathbf{F} is fat, there exist multiple solutions (b) for the given signal (\mathbf{x}) representation:

$$[\mathbf{x}]_{N \times 1} = [\mathbf{F}]_{N \times \hat{N}} [\mathbf{b}]_{\hat{N} \times 1}, \text{ where } \hat{N} = \sum_{i=1}^{P_{max}} \varphi(i). \quad (50)$$

In [14] and [32], the authors proposed a similar kind of approach for DFT (Farey dictionary) and RPT (RPT dictionary). Here the non-divisor period estimation is treated as a data fitting problem to reduce the computational complexity. To get the best fit of the given signal with the signals having smaller periods an optimization problem is formulated as follows:

$$\min \|\mathbf{T}\mathbf{b}\|_2 \text{ s.t. } \mathbf{x} = \mathbf{F}\mathbf{b}. \quad (51)$$

Here \mathbf{T} is a diagonal matrix consisting of $f(p_i)$ as elements and p_i is the period of i^{th} column in \mathbf{F} . This has a closed-form expression for the optimal solution ($\hat{\mathbf{b}}$) as given below:

$$\hat{\mathbf{b}} = \mathbf{T}^{-2} \mathbf{F}^H (\mathbf{F} \mathbf{T}^{-2} \mathbf{F}^H)^{-1} \mathbf{x}. \quad (52)$$

Fig. 3(a)-(c) show the strength of each period present in $x_2(n)$ using CCPT dictionaries with $f(p_i) = p_i^2$ and $P_{max} = 50$. For a detailed dictionary approach and for the results of Farey and RPT dictionaries refer [14] and [32]. From Fig. 3(a)-(c), the period of $x_2(n)$ is equal to $lcm(5, 8)$.

In addition to this, we can estimate the frequency and phase information from CCPT dictionary coefficients by following the same procedure used for CCPT coefficients earlier. Fig. 3(d) shows the absolute values of $\hat{\mathbf{b}}$ obtained using OCCPT dictionary. Here only 70 coefficients are displayed in the figure, as the rest of the coefficient values are almost equal to zero. From Fig. 3(d), the significant non-zero coefficient indices 19 and 20 are belong to $v_{8,1}$. It indicates the presence of $45Hz/315Hz$ frequency component in the signal. These values are 0.0927 and -0.0905 , then $-\tan^{-1} \left(\frac{-0.0905}{0.0927} \right) = 0.773 \approx \frac{\pi}{4} = 0.785 \text{ rad}$. So, the proposed transforms can be generalized to estimate the non-divisor period, frequency and phase information as well.

1) *Analysis of Computational Complexity:* In Farey dictionary $\mathbf{F} \in M_{N, \hat{N}}$, the columns of \mathbf{F} follow subspace wise complex conjugate symmetry. As a consequence, the computation of $(\mathbf{F} \mathbf{T}^{-2} \mathbf{F}^H)$ results in a real matrix involving complex multiplications. While for CCPT and RPT dictionaries this computation involves real multiplications. Table III tabulates the number of real multiplications and additions required for computing $(\mathbf{F} \mathbf{T}^{-2} \mathbf{F}^H)$.

If $x(n) \in \mathbb{R}^N$, then $\hat{\mathbf{b}}$ exhibits subspace wise complex conjugate symmetry for Farey dictionary. Due to this, the remaining complexity (apart from $(\mathbf{F} \mathbf{T}^{-2} \mathbf{F}^H)$ complexity) in computing

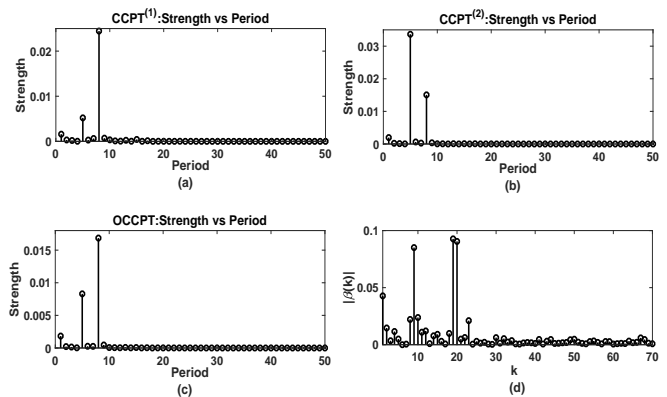


Fig. 3: (a), (b) and (c)- The strength vs period plots of $x_2(n)$ obtained from the optimal solution ($\hat{\mathbf{b}}$) using CCPT⁽¹⁾, CCPT⁽²⁾ and OCCPT dictionaries respectively. (d) Absolute values of optimal solution $\hat{\mathbf{b}}$ computed for $x_2(n)$ using OCCPT dictionary.

TABLE III: COMPARISON OF COMPUTATIONAL COMPLEXITY BETWEEN DIFFERENT DICTIONARIES

	CCPT Dictionary	Farey Dictionary	RPT Dictionary
Number of Multiplications	$N^2 \hat{N} + N \hat{N}$	$4N^2 \hat{N} + 2N \hat{N}$	$N^2 \hat{N} + N \hat{N}$
Number of Additions	$N^2 \hat{N} + N \hat{N} - N^2 - N$	$4N^2 \hat{N} + 2N \hat{N} - 2N^2 - 2N$	$N^2 \hat{N} + N \hat{N} - N^2 - N$

$\hat{\mathbf{b}}$ is same for both CCPT and Farey dictionaries. If $x(n) \in \mathbb{C}^N$, this symmetry fails, then CCPT and RPT dictionaries have a computational advantage over the Farey dictionary. From Table III, the complexity of CCPT and RPT dictionaries is approximately 75% less in comparison with the Farey dictionary. This computational benefit is also evident from the table given in [14], where the complexity of the Farey dictionary is compared with different other dictionaries, which involves real multiplications in computing $(\mathbf{F} \mathbf{T}^{-2} \mathbf{F}^H)$ and $\hat{\mathbf{b}}$. The overall comparison of the proposed transforms with RPT and DFT is tabulated in TABLE IV.

TABLE IV: COMPARISON OF DIFFERENT TRANSFORMS

	Basis Type	Orthogonality b/w Transformation Matrix columns	Period Information	Frequency Information	
				Magnitude	Phase
DFT	Complex	✓	✓	✓	✓
CCPT ⁽¹⁾	Real	×	✓	✓	×
CCPT ⁽²⁾	Real	×	✓	✓	×
OCCPT	Real	✓	✓	✓	✓
RPT	Integer	✓ (If matrix size is in power of 2)	✓	×	×

C. Usage of The Proposed Basis in Non-Divisor Subspaces

The usage of non-divisor subspaces in a signal representation do not hold the orthogonality between the basis elements. The proposed bases have computational benefit in such scenarios. Dictionary based approach discussed above is one such example. Now we mention another example: Given a periodic signal $x(n)$, the minimum data length (N_{min}) required to estimate its integer period from a list of candidate integer periods $P = \{P_1, P_2, \dots, P_K\}$ is $N_{min} = P_i, P_j \in P, P_i +$

$P_j - (P_i, P_j)$ [33], [34]. Construct a matrix $\mathbf{H} \in M_{N_{min}}(\mathbb{C})$ such that $\mathbf{x} = \mathbf{H}\mathbf{z}$, where \mathbf{H} includes non-divisor subspaces of N_{min} . For example, let $P = \{6, 8\}$, this implies $N_{min} = 12$. Then \mathbf{H} is constructed by using the basis of s_1, s_2, s_3, s_4, s_6 and s_8 . Notice that s_8 is not a divisor subspace of 12. So \mathbf{H} is a non-orthogonal matrix, then $\mathbf{z} = \mathbf{H}^{-1}\mathbf{x}$. One can verify that the \mathbf{H} constructed using complex exponential sequences (or) Ramanujan sums (or) CCPs is a full rank matrix. In such scenarios using the real-valued CCPs as basis is computationally efficient for the period and its corresponding frequency estimation over complex exponential sequences.

D. Real-World Example: ECG Signal Analysis

Here the problem of R peak (QRS complex) delineation in an ECG signal is considered, which is important in many ECG based applications [35], [36]. We discuss, how to address this problem using DFT, RPT, and CCPTs. A $10sec$ ECG data with a sampling frequency of $500Hz$ is considered for the analysis (record number 19 of person 1 from ECG-ID database [37]). For easy computation, we further down-sampled this data by a factor of 8, the resultant 625 length signal is depicted in Fig 4(a). Here the period of ECG, i.e., the average RR interval

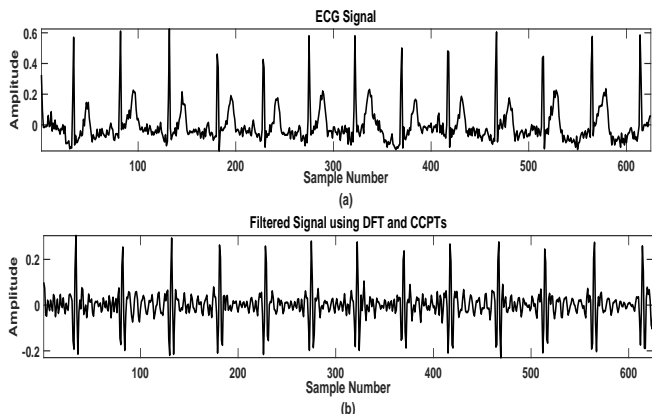


Fig. 4: (a) Raw ECG signal. (b) Filtered ECG signal using DFT and CCPTs (CCPT⁽¹⁾, CCPT⁽²⁾ and OCCPT).

is $0.7733sec$ (≈ 48 samples), and $48 \nmid 625$. As RPT gives only the divisor period information [17], it fails to estimate the R peak locations, whereas CCPTs and DFT give frequency information as well. Moreover, the frequency range of the QRS complex is $8 - 20Hz$ [16]. So we have reconstructed a signal (filtered) as shown in Fig. 4(b), by selecting the transform coefficients (of DFT and CCPTs) corresponding to $8 - 20Hz$ band. The reconstructed signal from both DFT and CCPTs is the same, since the basis of DFT is orthogonal and in CCPTs the basis is CCS wise orthogonal. Now, a better estimation of R peak locations can be achieved from this filtered signal, using a standard adaptive threshold algorithm [35].

Addressing the given problem using the dictionary based approach gives the results as shown in Fig. 5. Here we considered $P_{max} = 250$ and $f(p_i) = \varphi(p_i)$. Hence, the period of the ECG signal is $lcm(12, 16, 48) = 48$, $lcm(12, 16) = 48$, $lcm(8, 12, 16, 48) = 48$, $lcm(4, 8, 12, 16) = 48$ and $lcm(8, 12, 16, 48) = 48$ using CCPT⁽¹⁾, CCPT⁽²⁾, OCCPT,

RPT and DFT dictionaries respectively. Notice that CCPT⁽¹⁾ and CCPT⁽²⁾ are giving some spurious periods, as they are more sensitive to noise. Now, it is easy to estimate the R peak location from the period of the ECG signal.

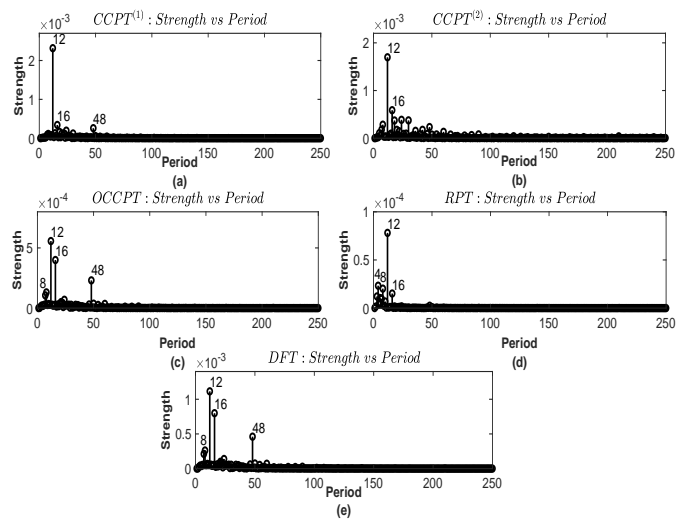


Fig. 5: (a)-(e) The strength vs period plots of ECG signal obtained from the optimal solution (b) using CCPT⁽¹⁾, CCPT⁽²⁾, OCCPT, RPT and DFT dictionaries respectively.

By considering the problem of R peak delineation we proved that the period and frequency of an ECG signal can be extracted using the proposed nested periodic matrices/dictionaries. The DFT matrix/dictionary also gives the same result, but with high computational complexity, whereas the RPT matrix/dictionary gives only the period information of an ECG signal.

IX. CONCLUSION

In this paper, we addressed the problem of finite length signal representation by introducing three NPMs. These NPMs are constructed by providing alternate bases for CCS using CCPs. Out of three NPMs, one has mutually orthogonal columns, this results in an orthogonal transform named OCCPT and the remaining two are non-orthogonal transforms. We proposed two different interpretations for OCCPT such that the information about the period and frequency is explicitly available in each interpretation. In addition, a DIT based fast computational algorithm is proposed for OCCPT, whenever the length of the signal is equal to 2^v , $v \in \mathbb{N}$. Further, we evaluated the performance and computational complexity of the proposed transforms in period and frequency estimation. The results are compared with DFT and RPT. The proposed theory is justified with some simulated and real-world examples.

APPENDIX

Proof of Theorem 2: Let $L_1 \geq 3$, $L_2 \geq 3$,

$$E = \sum_{n=0}^{L-1} c_{L_1, k_1}^{(1)}(n - l_1) c_{L_2, k_2}^{(1)}(n - l_2), \quad (53)$$

$x_1 = \frac{2\pi k_1}{L_1}$ and $x_2 = \frac{2\pi k_2}{L_2}$. Using Euler's identity and definition of CCPSSs, E can be decomposed as

$$E = \left[e^{-j(x_1 l_1 + x_2 l_2)} \sum_{n=0}^{L-1} e^{j(x_1 + x_2)n} + e^{-j(x_1 l_1 - x_2 l_2)} \sum_{n=0}^{L-1} e^{j(x_1 - x_2)n} + e^{j(x_1 l_1 - x_2 l_2)} \sum_{n=0}^{L-1} e^{-j(x_1 - x_2)n} + e^{j(x_1 l_1 + x_2 l_2)} \sum_{n=0}^{L-1} e^{-j(x_1 + x_2)n} \right]. \quad (54)$$

Since $L = lcm(L_1, L_2) \exists d_1, d_2 \in \mathbb{Z}$ s.t. $L = L_1 d_1$ and $L = L_2 d_2$. From this

$$\sum_{n=0}^{L-1} e^{\pm j(x_1 \pm x_2)n} = \frac{1 - e^{\pm j 2\pi(k_1 d_1 \pm k_2 d_2)}}{1 - e^{\pm j 2\pi(k_1 d_1 \pm k_2 d_2)/L}} = 0. \quad (55)$$

By substituting (55) in (54), we get, $E = 0$. The above condition is valid even if $L_1 = L_2 = L$ and $k_1 \neq k_2$. If $L_1 = L_2 = L$, $k_1 = k_2 = k$ and $l_1 \neq l_2$, then

$$\sum_{n=0}^{L-1} e^{\pm j(x_1 + x_2)n} = 0 \text{ and } \sum_{n=0}^{L-1} e^{\pm j(x_1 - x_2)n} = L. \quad (56)$$

In this case $E = 2L \cos\left(\frac{2\pi k_1(l_1 - l_2)}{L_1}\right)$. Combining the above cases with $E = \begin{cases} L, & \text{if } L_1 = L_2 = 1 \text{ (or) } 2 \\ 0, & \text{if } L_1 = 1 \text{ \& } L_2 = 2 \\ & \text{(or) } L_1 = 2 \text{ \& } L_2 = 1 \end{cases}$ leads to

$$E = 2LM \cos\left(\frac{2\pi k_1(l_1 - l_2)}{L_1}\right) \delta(L_1 - L_2) \delta(k_1 - k_2). \quad (57)$$

Proof of Circular Shift of a Sequence Property: Using $c_{p_i, k}^{(1)}\left(\left(\left(n - m\right)\right)_N\right) = \left[\frac{1}{2M} c_{p_i, k}^{(1)}(n) c_{p_i, k}^{(1)}\left(\left(\left(-m\right)\right)_N\right) - \frac{M}{2} c_{p_i, k}^{(2)}(n) c_{p_i, k}^{(2)}\left(\left(\left(-m\right)\right)_N\right)\right]$, $c_{p_i, k}^{(2)}\left(\left(\left(n - m\right)\right)_N\right) = \frac{1}{2M} \left[c_{p_i, k}^{(2)}(n) c_{p_i, k}^{(1)}\left(\left(\left(-m\right)\right)_N\right) + c_{p_i, k}^{(1)}(n) c_{p_i, k}^{(2)}\left(\left(-m\right)\right)_N \right]$ and the synthesis equation given in (20), we can write

$$x\left(\left(\left(n - m\right)\right)_N\right) = \sum_{p_i | N} \sum_{\substack{k=1 \\ (k, p_i)=1}}^{\lfloor \frac{p_i}{2} \rfloor} \hat{\beta}_{0ki} c_{p_i, k}^{(1)}(n) + \hat{\beta}_{1ki} c_{p_i, k}^{(2)}(n),$$

where $m \in \mathbb{Z}$ and

$$\hat{\beta}_{0ki} = \frac{1}{2M} \left[\beta_{0ki} c_{p_i, k}^{(1)}\left(\left(\left(-m\right)\right)_N\right) + \beta_{1ki} c_{p_i, k}^{(2)}\left(\left(\left(-m\right)\right)_N\right) \right], \\ \hat{\beta}_{1ki} = \left[\frac{1}{2M} \beta_{1ki} c_{p_i, k}^{(1)}\left(\left(\left(-m\right)\right)_N\right) - \frac{M}{2} \beta_{0ki} c_{p_i, k}^{(2)}\left(\left(\left(-m\right)\right)_N\right) \right]. \quad (58)$$

Now using $M = \begin{cases} \frac{1}{2}, & \text{if } p_i = 1 \text{ (or) } 2 \\ 1, & \text{if } p_i \geq 3 \end{cases}$, and the definitions of CCPSSs, we can simplify (58) as given in (25).

Proof of Circular Convolution Property: Given $x(n) = x_1(n) \otimes x_2(n)$, then using (22) we can write

$$\beta_{0ki} = \frac{1}{2NM} \sum_{n=0}^{N-1} \left[\sum_{l=0}^{N-1} x_1(l) x_2\left(\left(\left(n - l\right)\right)_N\right) \right] c_{p_i, k}^{(1)}(n).$$

Let $n - l = r$, then $\beta_{0ki} = \frac{\hat{\beta}_{0ki}}{2M} \mathbf{P} - \frac{M \hat{\beta}_{1ki}}{2} \mathbf{Q}$, where $\mathbf{P} = \sum_{r=-l}^{N-1-l} x_2\left(\left(\left(r\right)\right)_N\right) c_{p_i, k}^{(1)}(r) = 2NM \tilde{\beta}_{0ki}$ and $\mathbf{Q} = \sum_{r=-l}^{N-1-l} x_2\left(\left(\left(r\right)\right)_N\right) c_{p_i, k}^{(2)}(r) = 2NM \tilde{\beta}_{1ki}$. Hence

$$\beta_{0ki} = N \left[\hat{\beta}_{0ki} \tilde{\beta}_{0ki} - M^2 \hat{\beta}_{1ki} \tilde{\beta}_{1ki} \right]. \quad (59)$$

Similarly, we can derive

$$\beta_{1ki} = N \left[\hat{\beta}_{0ki} \tilde{\beta}_{1ki} + \hat{\beta}_{1ki} \tilde{\beta}_{0ki} \right]. \quad (60)$$

Equation (27) is an immediate consequence of these equations (obtained by substituting M value).

Proof of Parseval's Relation: Using the orthogonal CCPT synthesis equation, we can write

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} x(n) x^*(n) = \sum_{p_i | N} \sum_{\substack{k=1 \\ (k, p_i)=1}}^{\lfloor \frac{p_i}{2} \rfloor} \sum_{p_j | N} \sum_{\substack{k_1=1 \\ (k_1, p_j)=1}}^{\lfloor \frac{p_j}{2} \rfloor} \sum_{n=0}^{N-1} \left(\underbrace{\left(\beta_{0ki} \left(\beta_{0k_1j} \right)^* c_{p_i, k}^{(1)}(n) c_{p_j, k_1}^{(1)}(n) \right)}_{T_1} + \underbrace{\left(\beta_{0ki} \left(\beta_{1k_1j} \right)^* c_{p_i, k}^{(1)}(n) c_{p_j, k_1}^{(2)}(n) \right)}_{T_2} \right. \\ \left. + \underbrace{\left(\beta_{1ki} \left(\beta_{0k_1j} \right)^* c_{p_i, k}^{(2)}(n) c_{p_j, k_1}^{(1)}(n) \right)}_{T_3} + \underbrace{\left(\beta_{1ki} \left(\beta_{1k_1j} \right)^* c_{p_i, k}^{(2)}(n) c_{p_j, k_1}^{(2)}(n) \right)}_{T_4} \right).$$

Now using **Theorem 3**, the terms $T_2 = T_3 = 0$ over $0 \leq n \leq N - 1$. Similarly, using **Theorem 2**, the terms $T_1 = T_4 = 2NM$, whenever $p_i = p_j$ and $k = k_1$. This implies

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{p_i | N} \sum_{\substack{k=1 \\ (k, p_i)=1}}^{\lfloor \frac{p_i}{2} \rfloor} 2NM \left[|\beta_{0ki}|^2 + |\beta_{1ki}|^2 \right]. \quad (61)$$

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