ONE PARAMETER FAMILY OF SOLITONS FROM MINIMAL SURFACES

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Abstract. In this paper, we discuss a one parameter family of complex Born-Infeld solitons arising from a one parameter family of minimal surfaces. The process enables us to generate a new solution of the B-I equation from a given complex solution of a special type (which are abundant). We illustrate this with many examples. We find that the action or the energy of this family of solitons remains invariant in this family and find that the well-known Lorentz symmetry of the B-I equations is responsible for it.

1. Introduction

In a previous paper [1], using hodographic coordinates, we found the general solution to the minimal surface equation, namely a variant of the Weirstrass-Enneper representation of the minimal surface. This was done by wick rotating the general Born-Infeld soliton solution by Barbishov and Chernikov discussed in the last section of [5]. Underlying this, there was the observation that the minimal surface equation

$$(1+\phi_t^2)\phi_{xx} - 2\phi_x\phi_t\phi_{xt} + (1+\phi_x^2)\phi_{tt} = 0$$

and the Born-Infeld equation

$$(1-\phi_t^2)\phi_{xx} + 2\phi_x\phi_t\phi_{xt} - (1+\phi_x^2)\phi_{tt} = 0$$

can be obtained one from the other by wick rotation of the variable t.

We know that if $X(r,\bar{r}) = (x_1(r,\bar{r}), t_1(r,\bar{r}), \phi_1(r,\bar{r}))$

and $Y(r,\bar{r}) = (x_2(r,\bar{r}), t_2(r,\bar{r}), \phi_2(r,\bar{r}))$ are two minimal surfaces in isothermal coordinates (r_1, r_2), where $r = r_1 + ir_2$, which are harmonic conjugate to each other, then $\tilde{X}(r,\bar{r},\theta) = cos(\theta)X(r,\bar{r}) + sin(\theta)Y(r,\bar{r})$ is again a minimal surface for each θ, [2], page 213. Thus if we wick-rotate $t \to it$, we get a one parameter family of (complex) solitons, namely, $S(r,\bar{r},\theta) = cos(\theta)X^s + sin(\theta)Y^s$, where $X^s(r,\bar{r}) = (x_1(r,\bar{r}), it_1(r,\bar{r}), \phi_1(r,\bar{r}))$, $Y^s(r,\bar{r}) = (x_2(r,\bar{r}), it_2(r,\bar{r}), \phi_2(r,\bar{r}))$. We find the F and G functions corresponding to these complex solitons, (notation as in [5] page 617).

The process described here enables us to generate other solutions of the B-I, given one complex solution which can be wick rotated to get a real minimal surface (which can be then be written in isothermal coordinates using the Weierstrass-Enneper repesentation). Then one can easily write the harmonic conjugate of the minimal surface in the same form and then make the one-parameter combination of the two mentioned above and wick rotate back to get the soliton family which starts from a soliton solution which is the initial solution with $t \to -t$, (note that the B-I equation is invariant under $t \to -t$), and ends at a different soliton solution. We give many examples of this process.

The paper is organized as follows. We first give one example illustrating the case, namely that of the wick rotated helicoid and wick rotated the catenoid (since the catenoid is the harmonic conjugate of the helicoid).

Next we show that the first fundamental form, namely E^s, G^s and F^s are independent of θ and hence the action A^s is invariant under θ. This is due to a symmetry of the B-I equation which we explicitly show.

In the last section we give many examples illustrating the process described in the paper.

2. The one parameter family of solitons

Let $X(r,\bar{r}) = (x_1(r,\bar{r}), t_1(r,\bar{r}), \phi_1(r,\bar{r}))$
and $Y(r,\bar{r}) = (x_2(r,\bar{r}), t_2(r,\bar{r}), \phi_2(r,\bar{r}))$
be minimal surfaces which are harmonic conjugates of each other, given by the parameter r and its conjugate. They are isothermal in r_1 and r_2, where $r = r_1 + ir_2$. Then we know that $cos(\theta)X + sin(\theta)Y$ is a minimal surface for every θ, [2].

Then $X^s(r,\bar{r}) = (x_1(r,\bar{r}), it_1(r,\bar{r}), \phi_1(r,\bar{r}))$
and $Y^s(r,\bar{r}) = (x_2(r,\bar{r}), it_2(r,\bar{r}), \phi_2(r,\bar{r}))$ are Born-Infeld solitons for *imaginary* time it_1 and it_2.

X^s and Y^s are complex solitons. The superscript s stands for soliton.

Proposition 2.1. *$S_\theta = cos(\theta)X^s + sin(\theta)Y^s$ are complex Born-Infeld solitons for every θ.*

Proof. We will put S_θ in the form in [5], last section. According to [1]
$X = (x_1(r,\bar{r}), t_1(r,\bar{r}), \phi_1(r,\bar{r}))$ is a minimal surface implies
$x_1 - it_1 = F_1(r) - \int \bar{r}^2 G_1'(\bar{r}) d\bar{r}$
$x_1 + it_1 = G_1(\bar{r}) - \int r^2 F_1'(r) dr$
$\phi_1 = \int r F_1'(r) + \int \bar{r} G_1'(\bar{r}) d\bar{r}$
where F_1 and G_1 are related by $F_1(r) = \overline{G_1(\bar{r})}$.
Similarly, $Y = (x_2(r,\bar{r}), t_2(r,\bar{r}), \phi_2(r,\bar{r}))$ is a minimal surface implies,
$x_2 - it_2 = F_2(r) - \int \bar{r}^2 G_2'(\bar{r}) d\bar{r}$
$x_2 + it_2 = G_2(\bar{r}) - \int r^2 F_2'(r) dr$
$\phi_2 = \int r F_2'(r) + \int \bar{r} G_2'(\bar{r}) d\bar{r}$
where F_2 and G_2 are related by $F_2(r) = \overline{G_2(\bar{r})}$.
Then

$$\begin{aligned} S_\theta &= (x^s_\theta, t^s_\theta, \phi^s_\theta) = cos(\theta)X^s + sin(\theta)Y^s \\ &= (cos(\theta)x_1 + sin(\theta)x_2, icos(\theta)t_1 + isin(\theta)t_2, cos(\theta)\phi_1 + sin(\theta)\phi_2) \end{aligned}$$

where recall superscript s stands for soliton.

$$\begin{aligned} x^s_\theta - t^s_\theta &= cos(\theta)F_1(r) + \sin(\theta)F_2(r) \\ &- \int (\bar{r}^2(cos(\theta)G_1'(\bar{r}) + \sin(\theta)G_2'(\bar{r}))d\bar{r} \\ &= F^s_\theta(r) - \int \bar{r}^2 G^{s\prime}_\theta(\bar{r}) d\bar{r} \end{aligned}$$

where $F^s_\theta(r) = cos(\theta)F_1(r) + \sin(\theta)F_2(r)$ and $G^s_\theta(\bar{r}) = cos(\theta)G_1(\bar{r}) + \sin(\theta)G_2(\bar{r})$.

One can easily check that
$x^s_\theta + t^s_\theta = G^s_\theta(\bar{r}) - \int r^2 F^{s\prime}_\theta(r) dr$
$\phi^s_\theta = \int r F^{s\prime}_\theta(r) + \int \bar{r} G^{s\prime}_\theta(\bar{r})$.

Renaming variables, $\bar{r} = s$, we get this is exactly in the form of solutions to the Born Infeld equation as in [5], page 617. Thus S_θ is a (complex) Born-Infeld soliton. □

Corollary 2.2. *The partial derivatives of S_θ with respective to θ are also soliton solutions.*

Proof. $\frac{\partial S_\theta}{\partial \theta} = cos(\theta + \frac{\pi}{2})X^s + sin(\theta + \frac{\pi}{2})Y^s$

$\frac{\partial^2 S_\theta}{\partial^2 \theta} = cos(\theta + \pi)X^s + sin(\theta + \pi)Y^s$

$\frac{\partial^3 S_\theta}{\partial^3 \theta} = cos(\theta - \frac{\pi}{2})X^s + sin(\theta - \frac{\pi}{2})Y^s$

$\frac{\partial^4 S_\theta}{\partial^4 \theta} = S_\theta$

These are again of the form $cos(\theta_0)X^s + sin(\theta_0)Y^s$ and thus are soliton solutions. □

3. An example:

Let us write the catenoid and the helicoid (two conjugate minimal surfaces) in a variant of their Weirstrass-Enneper representation, [1], [3], which is also isothermal.

Proposition 3.1. *a) The helicoid can be written in a parametrised form in the following way:*

$x_1 = -\frac{1}{2}\mathrm{Im}(\mathrm{r} + \frac{1}{\mathrm{r}})$

$t_1 = \frac{1}{2}\mathrm{Re}(\mathrm{r} - \frac{1}{\mathrm{r}})$

$\phi_1 = \mathrm{Im}(\ln\mathrm{r})$

b) The catenoid can be written in a parametrised form in the following way:

$x_2 = \frac{1}{2}\mathrm{Re}(\mathrm{r} + \frac{1}{\mathrm{r}})$

$t_2 = \frac{1}{2}\mathrm{Im}(\mathrm{r} - \frac{1}{\mathrm{r}})$

$\phi_2 = -\mathrm{Re}(\ln\mathrm{r})$

Proof. a) The non parametric form of helicoid is $\phi(x,t) = \tan^{-1}\frac{t}{x}$. As $\phi_x = \frac{-t}{x^2+t^2}$ and $\phi_t = \frac{x}{x^2+t^2}$, we have $u = \phi_{\bar{z}} = \phi_x x_{\bar{z}} + \phi_t t_{\bar{z}}$.

That is $u = \frac{-t+ix}{2(x^2+t^2)} = \frac{iz}{2|z|^2}$, where $z = x + it$.

$$u = \frac{i}{2\bar{z}} \tag{1}$$

Similarly we have

$$v = \frac{-i}{2z} \tag{2}$$

Let us make the following coordinate change, [1], [5]:

$$r = \frac{\sqrt{1+4uv} - 1}{2v}. \tag{3}$$

Then

$$u = \frac{r}{1-|r|^2} \text{ and } v = \frac{\bar{r}}{1-|r|^2} \tag{4}$$

Equation 1, 2 and 4 gives

$$z = \frac{i}{2}(r - \frac{1}{\bar{r}}) \tag{5}$$

which in turn gives

$$x = -\frac{1}{2}\mathrm{Im}(\mathrm{r} + \frac{1}{\mathrm{r}}) \text{ and } \mathrm{t} = \frac{1}{2}\mathrm{Re}(\mathrm{r} - \frac{1}{\mathrm{r}}) \tag{6}$$

also from equation 5, we have $F(r) = \frac{i}{2r}$ and hence $G(\bar{r}) = \frac{-i}{2\bar{r}}$, [1].

Then we have $\phi(r) = \int rF'(r)dr + \int \bar{r}G'(\bar{r})d\bar{r}$, [1], and thus $\phi(r) = \frac{-i}{2}[\ln r - \ln \bar{r}]$, that is we have:

$$\phi(r) = \mathrm{Im}(\ln \mathrm{r}) \tag{7}$$

(b) The nonparametric form of catenoid is
$\phi(x,t) = \cosh^{-1}\sqrt{x^2+t^2}$. As seen in helicoid case, for the catenoid we have:

As $\phi_x = \frac{x}{\sqrt{x^2+t^2-1}\sqrt{x^2+t^2}}$ and $\phi_t = \frac{t}{\sqrt{x^2+t^2-1}\sqrt{x^2+t^2}}$, and

$$u = \phi_{\bar{z}} = \phi_x x_{\bar{z}} + \phi_t t_{\bar{z}} = \frac{\phi_x + i\phi_t}{2}$$

That is $u = \frac{z}{2\sqrt{x^2+t^2-1}\sqrt{x^2+t^2}}$.

Again with the same coordinate change as in equation 3, 4 and u as above we have $\frac{z}{\bar{z}} = \frac{r}{\bar{r}}$, that is:

$$z = \frac{r}{\bar{r}}\bar{z}. \tag{8}$$

Now as we have

$$u = \frac{z}{2\sqrt{x^2+t^2-1}\sqrt{x^2+t^2}} = \frac{z}{2\sqrt{|z|^2-1}\sqrt{|z|^2}}.$$

That is

$$\frac{r}{1-|r|^2} = \frac{z}{2\sqrt{|z|^2-1}\sqrt{|z|^2}}$$

Squaring it we have

$$\frac{z^2}{4(|z|^2-1).|z|^2} = \frac{r^2}{(1-|r|^2)^2}$$

Using equation 8, we have

$$4|r|^2(\frac{r}{\bar{r}}\bar{z}^2 - 1) = (1-|r|^2)^2$$

That is

$$\bar{z}^2 = \frac{\bar{r}}{r}\left(\frac{(1-|r|^2)^2}{4|r|^2} + 1\right)$$

$$\bar{z} = \pm\frac{1}{2}(\bar{r} + \frac{1}{r}).$$

We take the positive sign, because this gives us the non-parametric form. Hence in this case we have:

$$x = \frac{1}{2}\mathrm{Re}(\mathrm{r} + \frac{1}{\mathrm{r}}),\ \mathrm{t} = \frac{1}{2}\mathrm{Im}(\mathrm{r} - \frac{1}{\mathrm{r}}),\ \phi(\mathrm{r}) = -\mathrm{Re}(\ln \mathrm{r})$$

□

It is easy to check that the catenoid is conjugate harmonic to the helicoid because
$x_1 + ix_2 = i(r + \frac{1}{r})$
$t_1 + it_2 = r - \frac{1}{r}$
$\phi_1 + i\phi_2 = -ilnr$
so that the right hand sides of all the expressions are analytic functions of the complex variable r.

Proposition 3.2. *$F^s_\theta = \frac{i}{2}\frac{e^{-i\theta}}{r}$ and $G^s_\theta = \frac{-i}{2}\frac{e^{i\theta}}{\bar{r}}$ are the F and G functions for our family of soliton solutions.*

Proof. $x^s_\theta = cos(\theta)x_1 + sin(\theta)x_2$, $t^s_\theta = icos(\theta)t_1 + isin(\theta)t_2$, $\phi^s_\theta = cos(\theta)\phi_1 + sin(\theta)\phi_2$.
$x^s_\theta - t^s_\theta = cos(\theta)(x_1 - it_1) + sin(\theta)(x_2 - it_2)$
$x^s_\theta + t^s_\theta = cos(\theta)(x_1 + it_1) + sin(\theta)(x_2 + it_2)$
$x_1 - it_1 = -\frac{i}{2}(\bar{r} - \frac{1}{r})$
$x_2 - it_2 = \frac{1}{2}(\bar{r} + \frac{1}{r})$
$x^s_\theta - t^s_\theta = -\frac{i}{2}\bar{r}e^{i\theta} + \frac{i}{2}\frac{e^{-i\theta}}{r}$.
$x^s_\theta + t^s_\theta = \frac{i}{2}re^{-i\theta} - \frac{i}{2}\frac{e^{i\theta}}{\bar{r}}$
Thus $F^s_\theta(r) = \frac{i}{2}\frac{e^{-i\theta}}{r}$ and $G^s(\bar{r}) = -\frac{i}{2}\frac{e^{i\theta}}{\bar{r}}$.
We can check that $F^s_\theta(r) = \overline{G^s_\theta(\bar{r})}$.
Recall:
$\phi^s_\theta = \int rF^{s\prime}_\theta(r)dr + \int \bar{r}G^{s\prime}_\theta(\bar{r})d\bar{r}$.
Thus
$\phi^s_\theta = -\frac{i}{2}(lnr)e^{-i\theta} + \frac{i}{2}(ln\bar{r})e^{i\theta}$.
If $\theta = 0$ this corresponds to the wick rotated helicoid, namely $\phi^s_0 = \mathrm{Im(lnr)}$ and if $\theta = \frac{\pi}{2}$, this corresponds to the wick rotated catenoid, namely,
$\phi^s_{\frac{\pi}{2}} = -\mathrm{Re(lnr)}$ □

4. θ-invariants

Let $X^s_\theta = (x^s_\theta, t^s_\theta, \phi^s_\theta)$ be a soliton solution as before.

We show that the coefficients of the first fundamental form, and hence the Born-Infeld action is independent of θ.

Proposition 4.1. *Let $r = r_1 + ir_2$. Then $E^s = x^{s2}_{\theta,r_1} - t^{s2}_{\theta,r_1} + \phi^{s2}_{\theta,r_1}$ remains invariant with respect to θ. Similarly, $G^s = x^{s2}_{\theta,r_2} - t^{s2}_{\theta,r_2} + \phi^{s2}_{\theta,r_2}$ remains invariant with respect to θ. Also, $F^s = x^s_{\theta,r_1}x^s_{\theta,r_2} - t^s_{\theta,r_1}t^s_{\theta,r_2} + \phi^s_{\theta,r_1}\phi^s_{\theta,r_2} = 0$ for all θ. Thus $A^s = \int \sqrt{E^sG^s - F^{s2}}dr_1dr_2 = \int \sqrt{1 + \phi^{s2}_{x^s} - \phi^{s2}_{t^s}}dx^sdt^s$ is θ invariant.*

Proof. We have
$$X^s_\theta = X^s_1 \cos\theta + X^s_2 \sin\theta$$
where corresponding X_1 and X_2 are harmonic conjugate minimal surafces in r_1 and r_2 variable, and
$$\frac{\partial X_1}{\partial r_1} = \frac{\partial X_2}{\partial r_2} \text{ and } \frac{\partial X_1}{\partial r_2} = -\frac{\partial X_2}{\partial r_1}.$$
If $X_i(r_1, r_2) = (x_i, t_i, \phi_i)$, we have
$$X^s_\theta = (x_1(r,s)\cos\theta + x_2(r,s)\sin\theta, i(t_1(r,s)\cos\theta$$
$$+t_2(r,s)\sin\theta), \phi_1(r,s)\cos\theta + \phi_2(r,s)\sin\theta)$$
As X_1 and X_2 are conjugate we have:

$$\frac{\partial X_1}{\partial r_1} = \frac{\partial X_2}{\partial r_2} \text{ and } \frac{\partial X_1}{\partial r_2} = -\frac{\partial X_2}{\partial r_1}.$$

Then

$$\begin{aligned} x^{s2}_{\theta,r_1} - t^{s2}_{\theta,r_1} + \phi^{s2}_{\theta,r_1} &= x^2_{\theta,r_1} + t^2_{\theta,r_1} + \phi^2_{\theta,r_1} \\ &= (X_{1r_1}\cos\theta + X_{2r_1}\sin\theta).(X_{1r_1}\cos\theta + X_{2r_1}\sin\theta) \\ &= (X_{1r_1}\cos\theta - X_{1r_2}\sin\theta).(X_{1r_1}\cos\theta - X_{1r_2}\sin\theta) \\ &= X_{1r_1}.X_{1r_1}\cos^2\theta + \sin^2\theta X_{1r_2}.X_{1r_2} \\ &\quad + \cos\theta\sin\theta X_{1r_1}.X_{1r_2} - \cos\theta\sin\theta X_{1r_1}.X_{1r_2} \end{aligned}$$

Now we have $X_{1r_1}.X_{1r_1} = X_{1r_2}.X_{1r_2}$, (since r_1 and r_2 are isothermal coordinates for X_1),

$$\begin{aligned} E^s &= x^{s2}_{\theta,r_1} - t^{s2}_{\theta,r_1} + \phi^{s2}_{\theta,r_1} \\ &= X_{1r_1}.X_{1r_1} \end{aligned}$$

Hence E^s is independent of θ.

$$\begin{aligned} & x^s_{\theta,r_1}x^s_{\theta,r_2} - t^s_{\theta,r_1}t^s_{\theta,r_2} + \phi^s_{\theta,r_1}\phi^s_{\theta,r_2} \\ = \; & x_{\theta,r_1}x_{\theta,r_2} + t_{\theta,r_1}t_{\theta,r_2} + \phi_{\theta,r_1}\phi_{\theta,r_2} \\ = \; & (X_{1r_1}\cos\theta + X_{2r_1}\sin\theta).(X_{1r_2}\cos\theta + X_{2r_2}\sin\theta) \\ = \; & (X_{1r_1}\cos\theta - X_{1r_2}\sin\theta).(X_{1r_2}\cos\theta + X_{1r_2}\sin\theta) \\ = \; & X_{1r_1}.X_{1r_2}\cos^2\theta - \sin^2\theta X_{1r_2}.X_{1r_1} \\ & + \cos\theta\sin\theta X_{1r_1}.X_{1r_1} - \cos\theta\sin\theta X_{1r_2}.X_{1r_2} \end{aligned}$$

Again $X_{1r_1}.X_{1r_1} = X_{1r_2}.X_{1r_2}$ and $X_{1r_1}.X_{1r_2} = 0$, we have $F^s = 0$.

Similiary we can prove for G^s. Hence we see that E^s, F^s, G^s all are independent of θ which in turn gives A^s is independent of θ.

□

Lorentz Invariance of the Born-Infeld equation

There is a well-known symmetry, namely, the Lorentz invariance of the Born-Infeld equation which is reponsible for these invariant quantities. We rederive it here.

Proposition 4.2. *There is a symmetry in the Born-Infeld equation, namely if* $\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} cosh(\theta) & sinh(\theta) \\ sinh(\theta) & cosh(\theta) \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$ *then* $\phi(x', t')$ *satisfies the same B-I equation with* x *and* t *replaced by* x' *and* t'.

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc \neq 0$, denote the symmetry to Born-Infield equation, then we have: $\phi_{x'} = a\phi_x + c\phi_t$, $\phi_{x'x'} = a^2\phi_{xx} + c^2\phi_{tt} + 2ac\phi_{xt}$, $\phi_{t'} = b\phi_x + d\phi_t$, $\phi_{t't'} = b^2\phi_{xx} + d^2\phi_{tt} + 2bd\phi_{xt}$, and $\phi_{x't'} = ab\phi_{xx} + cd\phi_{tt} + (bc + ad)\phi_{xt}$. Hence B-I equation for $\phi(x', t')$ changes to

$$\begin{aligned} & (1 - \phi^2_{t'})\phi_{x'x'} + 2\phi_{x'}\phi_{t'}\phi_{x't'} - (1 + \phi^2_{x'})\phi_{t't'} \qquad (9) \\ = \; & [1 - (b\phi_x + d\phi_t)^2](a^2\phi_{xx} + c^2\phi_{tt} + 2ac\phi_{xt}) + 2(a\phi_x + c\phi_t)(b\phi_x + d\phi_t) \\ & [ab\phi_{xx}cd\phi_{tt} + (ad + bc)\phi_{xt}] - [1 + (a\phi_x + c\phi_t)^2](b^2\phi_{xx} + d^2\phi_{tt} + 2bd\phi_{xt}) \end{aligned}$$

In above expression 9, coefficient of ϕ_{xx} is

$$\begin{aligned} & a^2-(b\phi_x+d\phi_t)^2a^2+2ab(a\phi_x+c\phi_t)(b\phi_x+d\phi_t)-b^2-(a\phi_x+c\phi_t)^2b^2 \\ = & (a^2-b^2)+\phi_x^2(a^2b^2+2a^2b^2-a^2b^2)+\phi_t^2(-a^2d^2+2abcd-b^2c^2) \\ & +\phi_x\phi_t(-bda^2+2abad+2abcb-2abb^2) \\ = & (a^2-b^2)-\phi_t^2(a^2d^2+b^2c^2-2abcd) \end{aligned}$$

Hence for invariance of B-I equation we must have $a^2-b^2=1$, and $a=d$, $b=c$. With these condition coefficient for ϕ_{xx} in equation 9 will be

$$(a^2-b^2)-\phi_t^2(a^2d^2+b^2c^2-2abcd)=(a^2-b^2)[1-\phi_t^2(a^2-b^2)]=1-\phi_t^2$$

When $a^2-b^2=1$ $a=d$ and $b=c$, we have coefficient of ϕ_{xt} in equation 9 as $2\phi_{xt}$.

In the same way the coefficient of ϕ_{tt} in equation 9 $=(1+\phi_x)^2$. Hence 9 changes to B-I equation in $\phi(x,t)$ that is we have $\phi(x',t')$ is a soliton if and only if $\phi(x,t)$ is a solition. Also we have $a^2-b^2=1$ if and only if $a=\cosh\theta$ and $b=\sinh\theta$.

That is under the coordinate change $\begin{bmatrix} cosh(\theta) & sinh(\theta) \\ sinh(\theta) & cosh(\theta) \end{bmatrix}$, solution to the Born-Infield equation remain invariant. □

It is easy to check that this symmetry keeps A^s invariant. This is expected since the B-I equation is obtained by minimizing this action.

5. Many more examples

Recall the Weierstrass-Enneper representation of minimal surfaces, namely, in the neighborhood of a nonumbilic interior point, any minimal surface can be represented in terms of w as follows, [3],

$$\begin{aligned} x(\zeta) &= x_0+\mathrm{Re}\int_{\zeta_0}^{\zeta}(1-\mathrm{w}^2)\mathrm{R(w)\,dw} \\ t(\zeta) &= t_0+\mathrm{Re}\int_{\zeta_0}^{\zeta}\mathrm{i}(1+\mathrm{w}^2)\mathrm{R(w)\,dw} \\ \phi(\zeta) &= \phi_0+\mathrm{Re}\int_{\zeta_0}^{\zeta}2\mathrm{wR(w)\,dw} \end{aligned}$$

This is an isothermal representation (w.r.t. ζ_1 and ζ_2 where $\zeta=\zeta_1+i\zeta_2$.)

Various examples of minimal surfaces are as follows, [3], page 148.

$R(w)=1$ leads to the Enneper minimal surface.

$R(w)=\frac{\kappa}{2w^2}$, κ real, leads to the catenoid, $\frac{z}{\kappa}=cosh^{-1}(\frac{\sqrt{x^2+t^2}}{|\kappa|})$.

$R(w)=\frac{i\kappa}{2w^2}$, κ real, leads to the right helicoid $\frac{z}{\kappa}=tan^{-1}(\frac{x}{t})$.

$R(w)=\frac{\kappa e^{i\alpha}}{2w^2}$ leads to the general helicoid.

$R(w)=\frac{2}{(1-w^4)}$ leads to the Scherk's minimal surface.

$R(w)=\frac{-2aisin(2\alpha)}{(1+2w^2cos(2\alpha)+w^4)}$, $0<\alpha<\pi/2$, $a>0$ leads to the general Scherk's minimal surface.

$R(w)=1-w^{-4}$ (and substituting $-t$ for t) leads to the Henneberg surface.

$R(w)=\frac{ia(w^2-1)}{w^3}-\frac{ib}{2w^2}$, a and b real, and setting $w=e^{-i\gamma/2}$, leads to the general Enneper surface and , in particular, for $a=1$ and $b=0$, to the Catalan's surface.

$R(w)=(1-14w^4+w^8)^{-1/2}$ leads to the Schwarz-Riemann minimal surface.

Description and pictures of these minimal surfaces can be found in [3].

These are in isothermal representation.

To find their *harmonic conjugate* minimal surfaces, we need to replace $R(w)$ by $-iR(w)$.

Because if

$$\begin{aligned} x_1(\zeta) &= x_{01} + \mathrm{Re}\int_{\zeta_0}^{\zeta} (1-\mathrm{w}^2)\mathrm{R(w)}\,\mathrm{dw} \\ t_1(\zeta) &= t_{01} + \mathrm{Re}\int_{\zeta_0}^{\zeta} \mathrm{i}(1+\mathrm{w}^2)\mathrm{R(w)}\,\mathrm{dw} \\ \phi_1(\zeta) &= \phi_{01} + \mathrm{Re}\int_{\zeta_0}^{\zeta} 2\mathrm{wR(w)}\,\mathrm{dw} \end{aligned}$$

and

$$\begin{aligned} x_2(\zeta) &= x_{02} + \mathrm{Re}(-\mathrm{i}\int_{\zeta_0}^{\zeta} (1-\mathrm{w}^2)\mathrm{R(w)}\,\mathrm{dw}) \\ &= x_{02} + \mathrm{Im}\int_{\zeta_0}^{\zeta} (1-\mathrm{w}^2)\mathrm{R(w)}\,\mathrm{dw} \\ t_2(\zeta) &= t_{02} + \mathrm{Re}(-\mathrm{i}\int_{\zeta_0}^{\zeta} \mathrm{i}(1+\mathrm{w}^2)\mathrm{R(w)}\,\mathrm{dw}) \\ &= t_{02} + \mathrm{Im}\int_{\zeta_0}^{\zeta} \mathrm{i}(1+\mathrm{w}^2)\mathrm{R(w)}\,\mathrm{dw} \\ \phi_2(\zeta) &= \phi_{02} + \mathrm{Re}(-\mathrm{i}\int_{\zeta_0}^{\zeta} 2\mathrm{wR(w)}\,\mathrm{dw}) \\ &= \phi_{02} + \mathrm{Im}\int_{\zeta_0}^{\zeta} 2\mathrm{wR(w)}\,\mathrm{dw} \end{aligned}$$

then,

$x_1 + ix_2 = x_{01} + ix_{02} + \int_{\zeta_0}^{\zeta} (1-w^2)R(w)\,dw$

$t_1 + it_2 = t_{01} + it_{02} + \int_{\zeta_0}^{\zeta} i(1+w^2)R(w)\,dw$

$\phi_1 + i\phi_2 = \phi_{01} + i\phi_{02} + \int_{\zeta_0}^{\zeta} 2wR(w)\,dw.$

Since the right-hand side are holomorphic functions of $\zeta = \zeta_1 + i\zeta_2$,

(x_2, t_2, ϕ_2) is harmonic conjugate of (x_2, t_2, ϕ_2) are and the representations above are isothermal (w.r.t. ζ_1 and ζ_2) .

Thus we can combine $cos\theta(x_1, t_1, \phi_1) + sin\theta(x_2, t_2, \phi_2)$ and get another minimal surface.

By "wick rotating", namely, $t \to it$, we get a one -paramter family of solitons, $cos\theta(x_1, it_1, \phi_1) + sin\theta(x_2, it_2, \phi_2)$

Each choice of $R(w)$ gives us an example. Thus we get many examples.

Remark: We re-emphasize that the process described here enables us to generate other solutions of the B-I, given one complex solution which can be wick rotated to get a real minimal surface (which can be then be written in isothermal coordinates using the Weierstrass-Enneper repesentation). Then one can easily write the harmonic conjugate of the minimal surface in the same form and then make the one-parameter combination of the two mentioned above and wick rotate back to get the soliton family which starts from a soliton solution which is the initial solution with $t \to -t$, (note that the B-I equation is invariant under $t \to -t$), and ends at a different soliton solution. We have given many examples of this process.

Remark: We are using the word solition for solutions of the B-I equations. But since these are complex solutions, they need not be actual solitons.

Remark: Given a minimal surface in isothermal coordinates, its harmonic conjugate in isothermal coordinates is also a minimal surface. This is because $\mathbf{X} = \mathbf{X}(u,v)$ is a minimal surface iff $\mathbf{X}$ is isothermal (w.r.t u and v) and harmonic, [4]. (Here $\mathbf{X}(u,v) = (x(u,v), t(u,v), \phi(u,v))$.)

Correction: There are corrections in [1]. Equation (14) should read $\bar{z} = \bar{z}_0 + F(\bar{\zeta}) - \int \bar{\zeta}^2 G'(\bar{\zeta})$.

Here $F(r) = \overline{G(\bar{r})}$.

Also, in [1] our representation is a little different from the Weierstrass-Enneper representation, though both are isothermal. The domain of validity of the W-E representation is away from the umbilical points, namely, $\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 0$, while our representation fails where $\phi_{zz}\phi_{\bar{z}\bar{z}} - \phi_{\bar{z}z}^2 = 0$.

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