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# Mathematical model development of modified flow dispersion stress tensor in 2-D curvilinear flow domain

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**Abstract**. For 2-D simulation of curvilinear flow field, use of momentum equations involves flow dispersion stress terms. Dispersion Stress terms take into account the effect of secondary flow variation arisen due to integration of the product of discrepancy between depth averaged velocity and the true velocity distributions. The objective of this paper is to present empirical mathematical functions to evaluate these terms. These terms can be incorporated in the 2D depth averaged flow equations as an additional source/sink term. In this work, the derivation is done to get revised set of empirical relations are later used in development of enhanced 2D numerical model. When compared with earlier investigations, the proposed formulations are simplified and numerically compatible. It is expected that modified formulation for flow dispersion stress tensor will lead to more realistic and improved simulation of flow field in curved flow domain.

## 1. Introduction

The empirical relations for dispersion stress terms in 2D curvilinear flow field have been given by numerous researchers. Towards this, Engelund and Skovaard [1] and Shimizu and Itakura [2] predicted the transverse velocity, however it was valid only near the bed. Lien et al. (1) used orthogonal curvilinear coordinate system and incorporated the dispersion terms derived from stream-wise and transverse velocity profile [4]. Duan [5] employed the Cartesian Co-ordinate to facilitate model application in meandering and non-meandering channels. Duan [5] found that in meandering channel, mass diffusion coefficient is much larger than turbulent diffusion coefficient. Mathematical expressions for components of dispersion coefficient tensor have been deduced by integrating the product of discrepancy between the depth averaged and actual velocity. Duan [5] deduced the dispersion terms with the assumption that the stream-wise velocity satisfies the logarithmic law. It seems that integration by Duan [5] ignored the role of boundary sub-layer formation at the bed. However, this assumption may not always hold well in many situations. For example, for very mild bed gradient with highly sub-critical flow zones in alluvial river flow case, the boundary sub-layer is rationally assumed to be intact to satisfy logarithmic law of velocity distribution. This paper attempts to deal with the derivation of flow dispersion tensor in general curved channels which are common features in braided and dynamic alluvial streams. The objective of this paper is to derive the appropriate set of mathematical expressions for dispersion stress terms for depth averaged *2D* model to be used for complex non-orthogonal curvilinear flow domain with mild bed slope.

## 2. Governing Equations

The governing equations for flow simulation are (Reynold's Averaged Navier Stokes) *RANS* equations with depth averaged approximation of continuity and momentum equation [Eqs.(1), (2) and (3)] in Cartesian coordinate system.

$$\frac{\partial \rho h}{\partial t}+\frac{\partial}{\partial x}(\rho h U_x)+\frac{\partial}{\partial y}(\rho h U_y)=0 \tag{1}$$

$$\frac{\partial}{\partial t}(\rho h U)+\frac{\partial}{\partial x}(\rho h {U_x}^2)+\frac{\partial D_{xx}}{\partial x}+\frac{\partial}{\partial y}(\rho h U_x U_y)+\frac{\partial D_{xy}}{\partial y} \tag{2}$$
$$=-\rho g h\frac{\partial H}{\partial x}-C_d\rho U_x\sqrt{{U_x}^2+{U_y}^2}+\rho h\nu_t\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}\right)$$

$$\frac{\partial}{\partial t}(\rho h U_y)+\frac{\partial}{\partial x}(\rho h U_x U_y)+\frac{\partial D_{xy}}{\partial x}+\frac{\partial}{\partial y}(\rho h {U_y}^2)+\frac{\partial D_{yy}}{\partial y} \tag{3}$$
$$=-\rho g h\frac{\partial H}{\partial y}-C_d\rho U_y\sqrt{{U_x}^2+{U_y}^2}+h\rho\nu_t\left(\frac{\partial^2 U_y}{\partial x^2}+\frac{\partial^2 U_y}{\partial y^2}\right)$$

Where $U_x$ and $U_y$= depth-averaged velocity components in *x* and *y* directions; *t*=time; $\rho$=density of water ($kg/m^3$); *H*=water surface elevation; *h*=depth of the flow; *g*=acceleration of gravity; *Cd*=frictional stress coefficient (for friction shear stress at the bottom in x and y directions); and equals $n^2\rho g \Big/ h^{\frac{1}{3}}$ with *n*=Manning's roughness coefficient; $\nu_t$=eddy viscosity;

### 2.1. *Dispersion stress tensor*

Components of dispersion stress terms in Cartesian Coordinate which can be included in momentum transport equations are$D_{xx}$, $D_{xy}$ and $D_{yy}$ . These terms can be expressed as follows, [5];

$$D_{xx}=\int_{z_0}^{h+z_0}\rho(u_x-U_x)^2\,dz$$
$$D_{xy}=\int_{z_0}^{h+z_0}\rho(u_x-U_x)(u_y-U_y)\,dz \tag{4}$$
$$D_{yy}=\int_{z_0}^{h+z_0}\rho(u_y-U_y)^2\,dz$$

Where $z_0$=zero velocity level.

For open channel free surface gravity flow, cohesive terms are non-significant and can be neglected. The depth averaged parabolic eddy viscosity model (Zero equation model) is adopted for the turbulence term. The depth averaged eddy viscosity is computed as given in Eq.(5) [6][11].

$$\nu_t=\frac{1}{6}\kappa U_* h \tag{5}$$

Where $\kappa =$ *Von Karman' coefficient* and

$U*$= *Shear velocity* $= \left[C_d\left(U_x^2+U_y^2\right)\right]^{1/2}$.

### 2.2. *Transformed Governing Equations with dispersion stress tensor*

The transformed depth averaged governing equations in generalized curvilinear coordinate system ($\xi$,$\eta$,$\tau$) from continuity and momentum equation [Eqs.(1),(2) and (3)] are derived as follows.

$$\frac{\partial}{\partial \tau}(\rho h J)+\frac{\partial}{\partial \xi}(\rho h J\hat{u}_\xi)+\frac{\partial}{\partial \eta}(\rho h J\hat{u}_\eta)=0 \tag{6}$$

$$\frac{\partial}{\partial\tau}(\rho hJU_x)+\frac{\partial}{\partial\xi}\left[\rho hJ\hat{u}_\xi U_x\right]+\frac{\partial}{\partial\eta}\left[\rho hJ\hat{u}_\eta U_x\right]$$
$$-\rho hJ\nu_t\left(\alpha_{11}\frac{\partial^2 U_x}{\partial\xi^2}+\alpha_{22}\frac{\partial^2 U_x}{\partial\eta^2}\right) \tag{7}$$
$$=-\rho ghJ\left(\xi_x\frac{\partial H}{\partial\xi}+\eta_x\frac{\partial H}{\partial\eta}\right)-C_d\rho J(U_x)\sqrt{(U_x)^2+(U_y)^2}$$
$$+\rho hJ\nu_t\alpha_{12}\frac{\partial^2 U_x}{\partial\xi\partial\eta}-\left(\xi_x\frac{\partial D_{xx}}{\partial\xi}+\eta_x\frac{\partial D_{xx}}{\partial\eta}+\xi_y\frac{\partial D_{xy}}{\partial\xi}+\eta_y\frac{\partial D_{xy}}{\partial\eta}\right)$$

$$\frac{\partial}{\partial\tau}(\rho hJU_y)+\frac{\partial}{\partial\xi}\left[\rho hJ\hat{u}_\xi U_y\right]+\frac{\partial}{\partial\eta}\left[\rho hJ\hat{u}_\eta U_y\right] \tag{8}$$
$$-\rho hJ\nu_t\left(\alpha_{11}\frac{\partial^2 U_y}{\partial\xi^2}+\alpha_{22}\frac{\partial^2 U_y}{\partial\eta^2}\right)$$
$$=-\rho ghJ\left(\xi_y\frac{\partial H}{\partial\xi}+\eta_y\frac{\partial H}{\partial\eta}\right)-C_d\rho J(U_y)\sqrt{(U_x)^2+(U_y)^2}$$
$$+\rho hJ\nu_t\left(\alpha_{12}\frac{\partial^2 U_y}{\partial\xi\partial\eta}\right)-\left(\xi_x\frac{\partial D_{xy}}{\partial\xi}+\eta_x\frac{\partial D_{xy}}{\partial\eta}+\xi_y\frac{\partial D_{yy}}{\partial\xi}+\eta_y\frac{\partial D_{yy}}{\partial\eta}\right)$$

Where $\xi_x = {\partial\xi}/{\partial x}$, $\eta_x = {\partial\eta}/{\partial x}$, $\xi_y = {\partial\xi}/{\partial y}$, $\eta_y = {\partial\eta}/{\partial y}$, $\alpha_{11}=\xi_x{}^2+\xi_y^2$, $\alpha_{22}=\eta_x{}^2+\eta_y^2$, $\alpha_{12}=2(\xi_x\eta_y+\xi_y\eta_x)$, $J=x_\xi y_\eta - x_\eta y_\xi$,

In Eqs.(6b) to (8b) , $\widehat{u_m}$ ($m$=$\xi$, $\eta$) are the velocity components in the curvilinear coordinate $(\xi,\eta,\tau)$ which relate to $U_x$ , $U_y$ as

$$\begin{pmatrix}\hat{u}_\xi\\ \hat{u}_\eta\end{pmatrix}=\begin{pmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{pmatrix}\begin{pmatrix}U_x\\ U_y\end{pmatrix} \tag{9}$$

## 3. Derivation of dispersion stress terms in Momentum Equation

The dispersion terms resulting from the integration of the product of the discrepancy between the *mean* velocity and actual vertical velocity distribution were included in the momentum equations to take into account the effect of secondary current. Free surface flow in natural rivers is generally classified as turbulent-subcritical within the ranges of corresponding values of *Reynolds numbers* and *Froude numbers*. One of the important aspects of the free surface flow is shear velocity parameter which causes variation in velocity in different layers of fluid flow. So from the literature, one can readily assume that the stream wise velocity profile satisfies the logarithmic distribution law, *i. e.*

$$\frac{u_s}{U^*}=\frac{1}{\kappa}\ln\left(\frac{z}{z_o}\right) \tag{10}$$

Where $z$=vertical coordinate level (See Fig. 1), $u_s$=velocity in stream-wise direction and $z_o$ is calculated according to flow Reynolds number as follows,

$$z_0 = 0.11\frac{\nu}{U^*}.........,\frac{U^*k_s}{\nu}\le 5;$$
$$z_0 = 0.033k_s..........,\frac{U^*k_s}{\nu}\ge 70; \tag{11}$$
$$z_0 = 0.11\frac{\nu}{U^*}+0.033k_s..,5\ge\frac{U^*k_s}{\nu}\ge 70$$

Where in Eq.(11), $\nu$ = Kinematic Viscosity;

$k_s$ = Roughness height(m); $U^*$ = Shear Velocity

Ideally, at the bed boundary, *u* (stream-wise actual velocity) is zero; but for developing numerical scheme, value of base velocity should judiciously be taken non-zero value to ensure feasible solutions. Hence, it is well justified to exclude boundary sub-layer thickness (depth up to which boundary sub-layer is formed) and assign non-zero base velocity to achieve numerical solution close to experimental results. In other words, solid physical boundary is replaced with fluvial boundary and corresponding fluvial boundary condition has to be taken into consideration when analyzing the velocity profile vertically.

### *3.1. Modified approach with respect to Duan's [5] approach*

Depth averaged stream wise velocity can be expressed as,

$$U_s = \frac{\int_{z_o}^{z_o+h} u_s dz}{\int_{z_o}^{z_o+h} dz} \tag{12}$$

Assuming velocity profile to be logarithmic. For the simplifying the derivation to obtain the analytical solution, one can assume

For zo<<h

$$z_o + h \cong h \tag{13}$$

In the first step, Duan(2004) apparently integrated the denominator without any approximation in Eq.(12) as follows.

$$U_s = \frac{\int_{z_o}^{z_o+h} u_s dz}{\left(z_o + h - z_o\right)} = \frac{1}{h}\int_{z_o}^{z_o+h} u_s dz \tag{14}$$

Then she approximated $\boldsymbol{z_0 + h \cong h}$ for simplifying the derivation for arriving the analytical solution as follows,

$$U_s = \frac{1}{h}\int_{z_o}^{h} u_s dz \tag{15}$$

In this way Duan [5] approximated to $\boldsymbol{z_0 + h \cong h}$ in the numerator only, however, in denominator she included zo in the upper bound of the integral in Eq.(12).

Here modification can be proposed through computing the depth averaged stream wise velocity through approximating upper bound of the integral both in the numerator and the denominator of Eq (12) through considering $\boldsymbol{z_0 + h \cong h}$ to obtain a consistent solution with mathematically symmetric approximation. Using this, one can simplify considerably in mathematical representation for dispersion stress tensor to be used in momentum transport equations in Eqs. (7) and (8). The derivation of modified approach and comparison with Duan's approach is discussed in detail in subsequent sections. Putting the value of the expression in Eq.(13) in Eq.(12), one can have,

$$U_s = \int_{z_0}^{h} u_s dz \Big/ \int_{z_0}^{h} dz \tag{16}$$

Integrating the logarithmic velocity profile along the depth, from Eq.(10), one can finally deduce an expression as follows.

$$U_s = \frac{\int_{z_o}^{h} u_s dz}{h - z_o} = \frac{1}{h}\frac{\int_{z_o}^{h} u_s dz}{\left(1 - \frac{z_o}{h}\right)} \tag{17}$$

Let $\eta_0 = {}^{z_0}/_{h}$ , combining Eq.(10) and Eq.(17), one gets the following expression,

$$\frac{U_s}{U^*} = \frac{1}{\kappa h(1-\eta_0)}\int_{z_0}^{h} \ln\left(\frac{z}{z_0}\right)dz, \tag{18a}$$

or,

$$\frac{U_s}{U^*} = \frac{1}{\kappa}\frac{z_0}{h(1-\eta_0)}\left[\frac{z}{z_0}\ln\left(\frac{z}{z_0}\right) - \frac{z}{z_0}\right]_{z_0}^{h}, \tag{18b}$$

or,

$$\frac{U_s}{U^*} = \frac{1}{\kappa}\frac{z_0}{h(1-\eta_0)}\left[\frac{h}{z_0}\ln\left(\frac{h}{z_0}\right) - \frac{h}{z_0} - \frac{z_0}{z_0}\ln\left(\frac{z_0}{z_0}\right) + \frac{z_0}{z_0}\right], \tag{18c}$$

or,

$$\frac{U_s}{U^*} = \frac{1}{\kappa(1-\eta_0)}\left[\ln\left(\frac{h}{z_0}\right) - 1 + \frac{z_0}{h}\right] = \frac{1}{\kappa(1-\zeta_0)}\left[\frac{z_0}{h} - 1 + \ln\left(\frac{h}{z_0}\right)\right], \tag{18d}$$

or,

$$\frac{U_s}{U^*} = \frac{1}{\kappa(1-\eta_0)}\left[-\ln(\eta_0) - 1 + \zeta_0\right] = \frac{1}{\kappa(1-\eta_0)}\left[\eta_0 - 1 - \ln(\eta_0)\right] \tag{18e}$$

Dividing Eq. (18e) with Eq. (10), one obtains,

$$\frac{u_s}{U_s} = (1-\eta_0)\frac{\ln\left(\frac{z}{z_0}\right)}{\eta_0 - 1 - \ln(\eta_0)} \tag{19}$$

Rearranging the above expression, one has

$$\frac{u_s}{U_s} - 1 = \frac{(1-\eta_0)\ln\left(\frac{z}{\eta_0 h}\right) - \eta_0 + 1 + \ln\eta_0}{\eta_0 - 1 - \ln(\eta_0)} \tag{20a}$$

or,

$$\frac{u_s - U_s}{U_s} = \frac{(1-\eta_0)\ln\left(\frac{z}{h}\right) - (1-\eta_0)\ln\eta_0 - \eta_0 + 1 + \ln\eta_0}{\eta_0 - 1 - \ln(\eta_0)} \tag{20b}$$

As $1 - \eta_0 \cong 1$, one assumes this term as 1 in the second term of the numerator for the ease and simplification, *i.e.*,

$$u_s - U_s = \frac{U}{\eta_0 - 1 - \ln(\eta_0)}\left((1-\eta_0)\ln\left(\frac{z}{h}\right) - \eta_0 + 1\right) \tag{21a}$$

or,

$$u_s - U_s = \frac{U_s(1-\eta_0)}{\eta_0 - 1 - \ln(\eta_0)}\left(\ln\left(\frac{z}{h}\right) + 1\right) \tag{21b}$$

The transverse velocity profile is assumed to be linear. As proposed by Odgaard [7][8], following relation is adopted for this model.

$$u_n = U_n + 2v_s\left(\frac{z}{h} - \frac{1}{2}\right) \tag{22}$$

Where $u_n$ , $U_n$, and $v_s$ are transverse velocity, depth averaged transverse velocity and the transverse velocity at the water surface. Engelund and Skovgaard [1] derived the deviation angle of the bottom shear as follows

$$\left(\frac{\tau_n}{\tau_s}\right)_b \approx \left(\frac{u_n}{u_s}\right)_b = 7.0\frac{h}{r}, \tag{23}$$

Where $r$ = radius of channel curvature and the secondary flow at the surface and the bottom are equal. Therefore Eq. (20) is used to express transverse velocity at the surface. Thus,

$$(v_s)_b = 7.0(h/r)(u_s)_b \tag{24}$$

or,

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Substituting Eq.(21) in Eq.(19), one obtains (same as Duan's approach),

$$u_n = U_n + 7.0\frac{h}{r}U_s\left(\frac{z}{h} - \frac{1}{2}\right) \tag{25}$$

Let us define

$$\frac{(1-\eta_0)}{\eta_0 - 1 - \ln(\eta_0)} \equiv \gamma \tag{26}$$

Substituting Eq.(26), Eq. (21b) and Eq. (25) convert to ;

$$u_s - U_s = \gamma U_s\left(\ln\left(\frac{z}{h}\right) + 1\right) \tag{27a}$$

$$u_n - U_n = 7.0\frac{h}{r}U_s\left(\frac{z}{h} - \frac{1}{2}\right) \tag{27b}$$

### *3.2. Expressions for Dispersion stress Tensors*

The dispersion stress terms at the stream-wise and transverse directions can be expressed as.

$$D^c{}_{xx} = \int_{z_0}^{h+z_0} \rho(u_s - U_s)^2\, dz \tag{28a}$$

$$D^c{}_{xy} = \int_{z_0}^{h+z_0} \rho(u_s - U_s)(u_n - U_n)dz \tag{28b}$$

$$D^c{}_{yy} = \int_{z_0}^{h+z_0} \rho(u_n - U_n)^2\, dz \tag{28c}$$

In Eqs. (28a), (28b) and (28c) $D^c_{xx}$ , $D^c_{xy}$ and $D^c_{yy}$ are dispersion stress terms in curvilinear coordinate system. Substituting Eqs.(27a, b) in Eqs.(28a,b and c), one can deduce dispersion stress tensor presented in following steps.

*3.2.1. The First Dispersion Term* $D_{xx^c}$. Substituting Eq.(24a) in Eq.(25a), one can get,

$$D^c_{xx} = \gamma^2 U_s U_s \int_{z_0}^{h}\left(\ln\left(\frac{z}{h}\right) + 1\right)^2 dz \tag{29}$$

Now consider *z/h=m,* then, *dz=h.dm,* $m_1 = z_0/h = \eta_0$ lower integral bound, $m_2 = h/h = 1$ upper integral bound. With these substitution Eq.(29) becomes

$$D^c_{xx} = \gamma^2 U_s U_s h\left[\int_{\eta_0}^{1}(\ln(m))^2\, dm + 2m\int_{\eta_0}^{1}\ln(m)dz + (1-\eta_0)\right] \tag{30a}$$

or,

$$D^c_{xx} = \gamma^2 U_s U_s h\left[\begin{array}{l}\left(m(\ln m)^2 - 2m\ln m + 2m\right)_{\eta_0}^{1} \\ + 2(m\ln m - m)_{\eta_0}^{1} + (1-\eta_0)\end{array}\right] \tag{30b}$$

or,

$$D^c_{xx} = \gamma^2 U_s U_s h\left[\begin{array}{l}2 - \eta_0(\ln\eta_0)^2 + 2\eta_0\ln\eta_0 - 2\eta_0 \\ + 2(-1 - \eta_0\ln\eta_0 + \eta_0) + 1 - \eta_0\end{array}\right] \tag{30c}$$

One gets the final expression as,

$$D^c_{xx} = \rho\gamma^2 U_s U_s h\left[-\eta_0(\ln\eta_0)^2 - \eta_0 + 1\right] \tag{30d}$$

*3.2.2. The Second Dispersion Term* $D_{xy^c}$. Similar to the above, one can get expression for the Second Dispersion Stress Term

$$D^c_{xy} = \int_{z_0}^{h}\gamma U_s\left(\ln\left(\frac{z}{h}\right) + 1\right) \times 7.0\frac{h}{r}U_s\left(\frac{z}{h} - \frac{1}{2}\right)dz \tag{31a}$$

or,

$$D_{xy}^{c} = -7.0\gamma \frac{h}{r} U_s U_s \int^{h} \left( \ln\left( \frac{z}{h} \right) + 1 \right)\left( \frac{1}{2} - \frac{z}{h} \right) dz \quad (31b)$$

or,

$$D_{xy}^{c} = -7.0\gamma \frac{h}{r} U_s U_s \left( \frac{1}{2}\int_{z_0}^{h} \ln\frac{z}{h}\, dz - \int_{z_0}^{h} \frac{z}{h}\ln\frac{z}{h}\, dz + \frac{1}{2}\int_{z_0}^{h} dz - \int_{z_0}^{h} \frac{z}{h}\, dz \right) \quad (31c)$$

Again taking $m=z/h$ and integrating and transforming the upper and lower bound as done earlier, and taking $h$ common, one has ,

$$D_{xy}^{c} = -7.0\gamma \frac{h^2}{r} U_s U_s \begin{pmatrix} \frac{1}{2}\left[ m\ln m - m \right]_{\eta_0}^{1} - \frac{1}{4}\left[ 2m^2 \ln m - m^2 \right]_{\eta_0}^{1} \\ + \frac{1}{2}\left( 1 - \eta_0 \right) - \frac{1}{2}\left[ m^2 \right]_{\eta_0}^{1} \end{pmatrix} \quad (32a)$$

$$D_{xy}^{c} = -7.0\gamma \frac{h^2}{r} U_s U_s \frac{1}{4} \begin{pmatrix} 2\left[ -1 - \eta_0 \ln\eta_0 + \eta_0 \right] \\ -\left[ -1 - 2\eta_0^2 \ln\eta_0 + \eta_0^2 \right] \\ + 2\left( 1 - \eta_0 \right) - 2\left[ 1 - \eta_0^2 \right] \end{pmatrix} \quad (32b)$$

or,

$$D_{xy}^{c} = -1.75\gamma \frac{h^2}{r} U_s U_s \left( \begin{bmatrix} -2 - 2\eta_0 \ln\eta_0 + 2\eta_0 \\ +1 + 2\eta_0^2 \ln\eta_0 - \eta_0^2 \\ +2 - 2\eta_0 - 2 + 2\eta_0^2 \end{bmatrix} \right) \quad (32c)$$

or,

$$D_{xy}^{c} = -1.75\gamma \frac{h^2}{r} U_s U_s \left( \left[ -2\eta_0 \ln\eta_0 + 2\eta_0^2 \ln\eta_0 + \eta_0^2 - 1 \right] \right) \quad (32d)$$

or,

or,

$$D_{xy}^{c} = -1.75\gamma \frac{h^2}{r} U_s U_s \left( \begin{bmatrix} -2\eta_0 \ln\eta_0 \left( 1 - \eta_0 \right) \\ -(1 + \eta_0)(1 - \eta_0) \end{bmatrix} \right) \quad (32e)$$

or,

$$D_{xy}^{c} = 1.75\rho\gamma \frac{h^2}{r} U_s U_s \left( 1 - \eta_0 \right)\left( 2\eta_0 \ln\eta_0 + \eta_0 + 1 \right) \quad (32f)$$

*3.2.3. The Third Dispersion Term $D_{yy}^{c}$.* Using the similar procedure as above, one can obtain the expression for $D_{yy}^{c}$

$$D_{yy}^{c} = (7)^2 \frac{h^2}{r^2} U_s U_s \int_{z_0}^{h} \left( \frac{z}{h} - \frac{1}{2} \right)^2 dz \quad (33a)$$

or,

$$D_{yy}^{c} = 49.0 \frac{h^2}{r^2} U_s U_s \left( \int_{z_0}^{h} \left( \frac{z}{h} \right)^2 dz - \int_{z_0}^{h} \left( \frac{z}{h} \right) dz + \frac{1}{4}\int_{z_0}^{h} dz \right) \quad (33b)$$

Again taking $m=z/h$ and integrating and transforming the upper and lower bound as done earlier, one can obtain,

$$D_{yy}^{c} = 49.0 \frac{h^3}{r^2} U_s U_s \left( \left[ \frac{m^3}{3} \right]_{\eta_0}^{1} - \left[ \frac{m^2}{2} \right]_{\eta_0}^{1} + \frac{1}{4}\left( 1 - \eta_0 \right) \right) \quad (34a)$$

or,

$$D_{yy}^{c} = 49.0 \frac{h^3}{r^2} U_s U_s \left( \frac{1}{3} - \frac{\eta_0^3}{3} - \left( \frac{1}{2} - \frac{\eta_0^2}{2} \right) + \frac{1}{4} - \frac{\eta_0}{4} \right) \quad (34b)$$

or,

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$$D_{yy}^{c} = 49.0\frac{h^3}{r^2}U_s U_s\left(\frac{1}{3}-\frac{\eta_0^3}{3}-\frac{1}{2}+\frac{\eta_0^2}{2}+\frac{1}{4}-\frac{\eta_0}{4}\right) \tag{34c}$$

or, finally one may obtain,

$$D_{yy}^{c} = 49.0\rho\frac{h^3}{r^2}U_s U_s\left(-\frac{\eta_0^3}{3}+\frac{\eta_0^2}{2}-\frac{\eta_0}{4}+\frac{1}{12}\right) \tag{34d}$$

The relation between depth averaged velocities in curvilinear coordinates and Cartesian coordinate can be given as [5]

$$U_x = U_s\cos\theta_s + U_n\cos\theta_n \tag{35a}$$

$$U_y = U_s\sin\theta_s + U_n\sin\theta_n \tag{35b}$$

Where $\theta_s$ and $\theta_n$ are angles between stream-wise, transverse directions pointing outward and positive $x$-axis respectively. Similarly the dispersion terms in Cartesian coordinates can be related to that in curvilinear coordinates as follows[5].

$$D_{xx} = D_{xx}^{c}\cos^2\theta_s + 2D_{xy}^{c}\cos\theta_s\cos\theta_n + D_{yy}^{c}\cos^2\theta_n \tag{36a}$$

$$D_{yy} = D_{xx}^{c}\sin^2\theta_s + 2D_{xy}^{c}\sin\theta_s\sin\theta_n + D_{yy}^{c}\sin^2\theta_n \tag{36b}$$

$$D_{xy} = D_{xx}^{c}\cos\theta_s\sin\theta_s + 2D_{xy}^{c}(\cos\theta_n\sin\theta_s + \sin\theta_n\cos\theta_s + D_{yy}^{c}\sin\theta_n\cos\theta_n \tag{36c}$$

The dispersion stress terms finally obtained in Eqs.(30d, 32f and 34d) can be transformed by Eqs.(32) and Eqs.(33) to get modified dispersion stress tensor in Cartesian coordinate system.

## 4. Results and Discussion

The correlations by Duan [5] for dispersion stress tensor in curvilinear coordinate system are as follows.

For the statistical comparison, theoretical data for a wide rectangular channel is analysed. A qualitative comparison of variations of dispersion stresses for varying sinuosity with the modified formulation (Eqs.30d, 32f and 34d) and Duan's formulations (Eqs. 37 a, 37b and 37c) have been compared. Four configurations (*Curvature* 0.34, 0.72, 1.00 and 1.05) were chosen to cover low, moderate, high and very high sinuosity curved channels[9]. The width ratio ($\beta=B/h$, where $h$=water depth, $B$=channel width) were chosen as 10, 15, and 20. Longitudinal slope is kept as 0.001 and 0.025 for creating sub-critical and supercritical condition, respectively. Average velocities are estimated using Manning's equation (Manning'*n* is kept 0.025). $z_0$ is taken as $D_{50}/30$ ( Roughness height ($k_s$) is kept equal to $D_{50}$ =0.44 mm).

It can be seen that expressions of dispersion stress terms as obtained in the present work are not in complete agreement of Duan's formulations as given in Eq. (37a). From Eq.(37b), it is apparent that any comparison between the two approaches, the value of '*C*' should have been available. However, in their paper [10] there is no *C* in Eq. (37b) .Thus, there is lack of enough insight into the adoption of any appropriate value of *C*. However, in view of Duan and Julien [10], the value of '*C*' is taken unity for comparative purposes only.

Assuming *C* as one, an attempt is made to relate Eq.(37b) with Eq. (32f), developed in the present work. Following empirical relation is obtained.

$D_{xy}^{c}=A_0 + A_1 \times D_{xy}^{c}$(Duan) (R-Square=0.9854; Adjusted R-Square=0.9847)

Where $A_0$=-3.147 and $A_1$=-103.676

To appreciate the difference between the two expressions for two approaches of dispersion terms (as given in Eqs.30d, 32f, 34d and 37), certain computations are done for a variety of conditions (Table 1).

The differences do appear in the formulations of the first term ($D_{xx}^{c}$) and second terms ($D_{xy}^{c}$). The formulation for $D_{xx}^{c}$ are different for the present approach and Duan's approach yet computed values of this term is similar and close valued, as shown in Table 1. However, the modified model (Eq. 30d) has much simpler mathematical representation than Duan's model (Eq. 37a).

Table 1 Computations of dispersion stress tensor by modified and Duan's (2004) expressions

| S No. | Curve Type | Curvature | Width Ratio ($\beta$) | Longitudinal Slope | Velocity (m/s) | Froude Number | Modified Terms | | | Duan(2004)-Terms | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $D^c_{xx}$ | $D^c_{xy}$ | $D^c_{yy}$ | $D^c_{xx}$ | $D^c_{xy}$ | $D^c_{yy}$ |
| 1 | Mild | 0.34 | 10 | 0.001 | 0.57 | 0.3304 | 1.204 | 1.90 | 4.00 | 1.204 | -0.0182 | 4.00 |
| 2 | | 0.34 | 15 | 0.001 | 0.43 | 0.2522 | 0.512 | 0.52 | 0.69 | 0.512 | -0.0068 | 0.69 |
| 3 | | 0.34 | 20 | 0.001 | 0.36 | 0.2082 | 0.280 | 0.20 | 0.20 | 0.280 | -0.0034 | 0.20 |
| 4 | | 0.34 | 10 | 0.025 | 2.83 | 1.6521 | 30.098 | 47.56 | 99.88 | 30.101 | -0.4540 | 99.88 |
| 5 | | 0.34 | 15 | 0.025 | 2.16 | 1.2608 | 12.800 | 12.89 | 17.23 | 12.802 | -0.1697 | 17.23 |
| 6 | | 0.34 | 20 | 0.025 | 1.79 | 1.0408 | 6.994 | 5.11 | 4.95 | 6.996 | -0.0844 | 4.95 |
| 7 | Medium | 0.72 | 10 | 0.001 | 0.57 | 0.3304 | 1.204 | 4.06 | 18.24 | 1.204 | -0.0388 | 18.24 |
| 8 | | 0.72 | 15 | 0.001 | 0.43 | 0.2522 | 0.512 | 1.10 | 3.15 | 0.512 | -0.0145 | 3.15 |
| 9 | | 0.72 | 20 | 0.001 | 0.36 | 0.2082 | 0.280 | 0.44 | 0.90 | 0.280 | -0.0072 | 0.90 |
| 10 | | 0.72 | 10 | 0.025 | 2.83 | 1.6521 | 30.098 | 101.61 | 455.88 | 30.101 | -0.9699 | 455.88 |
| 11 | | 0.72 | 15 | 0.025 | 2.16 | 1.2608 | 12.800 | 27.54 | 78.66 | 12.802 | -0.3626 | 78.66 |
| 12 | | 0.72 | 20 | 0.025 | 1.79 | 1.0408 | 6.994 | 10.92 | 22.61 | 6.996 | -0.1802 | 22.61 |
| 13 | High | 1.00 | 10 | 0.001 | 0.57 | 0.3304 | 1.204 | 5.69 | 35.70 | 1.204 | -0.0543 | 35.70 |
| 14 | | 1.00 | 15 | 0.001 | 0.43 | 0.2522 | 0.512 | 1.54 | 6.16 | 0.512 | -0.0203 | 6.16 |
| 15 | | 1.00 | 20 | 0.001 | 0.36 | 0.2082 | 0.280 | 0.61 | 1.77 | 0.280 | -0.0101 | 1.77 |
| 16 | | 1.00 | 10 | 0.025 | 2.83 | 1.6521 | 30.098 | 142.16 | 892.42 | 30.101 | -1.3570 | 892.42 |
| 17 | | 1.00 | 15 | 0.025 | 2.16 | 1.2608 | 12.800 | 38.53 | 153.98 | 12.802 | -0.5073 | 153.98 |
| 18 | | 1.00 | 20 | 0.025 | 1.79 | 1.0408 | 6.994 | 15.28 | 44.26 | 6.996 | -0.2522 | 44.26 |
| 19 | very high | 1.05 | 10 | 0.001 | 0.57 | 0.3304 | 1.204 | 5.95 | 39.09 | 1.204 | -0.0568 | 39.09 |
| 20 | | 1.05 | 15 | 0.001 | 0.43 | 0.2522 | 0.512 | 1.61 | 6.74 | 0.512 | -0.0212 | 6.74 |
| 21 | | 1.05 | 20 | 0.001 | 0.36 | 0.2082 | 0.280 | 0.64 | 1.94 | 0.280 | -0.0106 | 1.94 |
| 22 | | 1.05 | 10 | 0.025 | 2.83 | 1.6521 | 30.098 | 148.76 | 977.22 | 30.101 | -1.4200 | 977.22 |
| 23 | | 1.05 | 15 | 0.025 | 2.16 | 1.2608 | 12.800 | 40.32 | 168.62 | 12.802 | -0.5309 | 168.62 |
| 24 | | 1.05 | 20 | 0.025 | 1.79 | 1.0408 | 6.994 | 15.99 | 48.47 | 6.996 | -0.2639 | 48.47 |

The variability of $D^c_{xy}$ for both approaches are shown in Fig. 1. The trend of the variation is closely related (R-square=0.985), but values of $D^c_{xy}$ differs considerably. Plot of $D^c_{xy}$ against width ratio ($\beta$) for both approaches for variety of conditions are shown in Fig. 1.

The third term $D^c_{yy}$ is identical in both cases as in present formulation and Duan's work. The trend of second terms is statistically similar with very low difference in mean and standard deviation (With using statistically determined '$C$' value). R-square (0.99 for $D^c_{xx}$ and $D^c_{yy}$, 0.98 for $D^c_{xy}$) and Standard Error suggests high degree of goodness of fit for both models (For Eqs. 32f, and 37b). The inconsistency in the values of dispersion stress terms from Duan's model for different hydraulic conditions is evident in the Table 1. For example, in modified model's $D^c_{xx}$, $D^c_{xy}$, and $D^c_{yy}$ is are varying consistently for different width ratio ($\beta$). Whereas, same terms show inconsistent variations with different $\beta$ for Duan's Model. These models are developed for sub- critical flow condition; however, their trends are also analyzed for super critical flow condition. Plots of variation of different terms with width ratio for sub critical and super critical conditions are shown in Fig. s 2 and 3.

Fig. 3 shows that $D^c_{xx}$ remains nearly constant with varying sinuosity and β. $D^c_{yy}$ increases sharply with increasing sinuosity and reducing width ratio. For wider channels, variations are low. But for narrow bends transverse deviations in velocity is quite high. The trend remains same in case of supercritical flow condition (Fig. 3) except higher magnitude of three components of flow dispersion tensor. This is caused due to enhanced transverse mixing of the flow at high turbulence.

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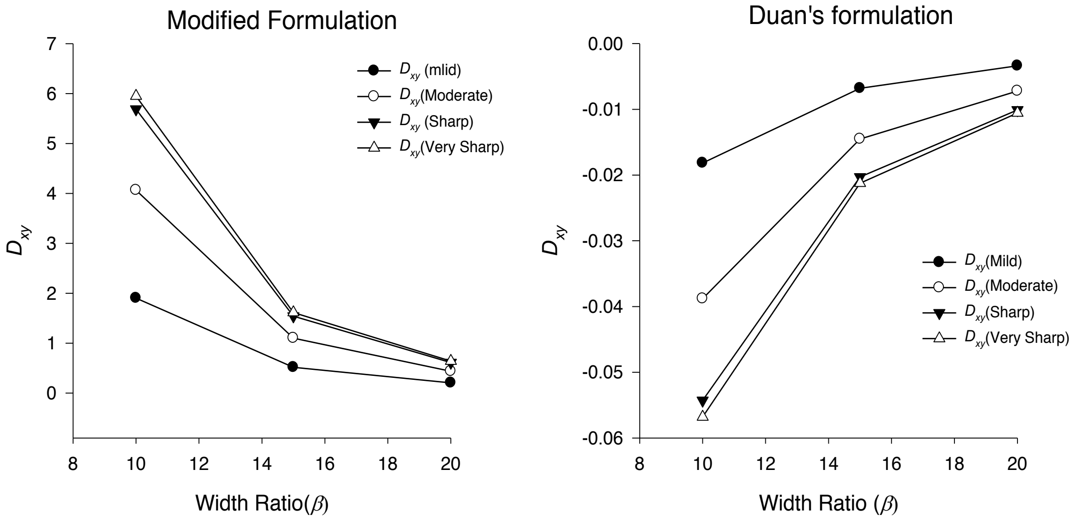


Fig. 1. Variation of $D_{xy}$ with width ratio ($\beta$) for two approaches.

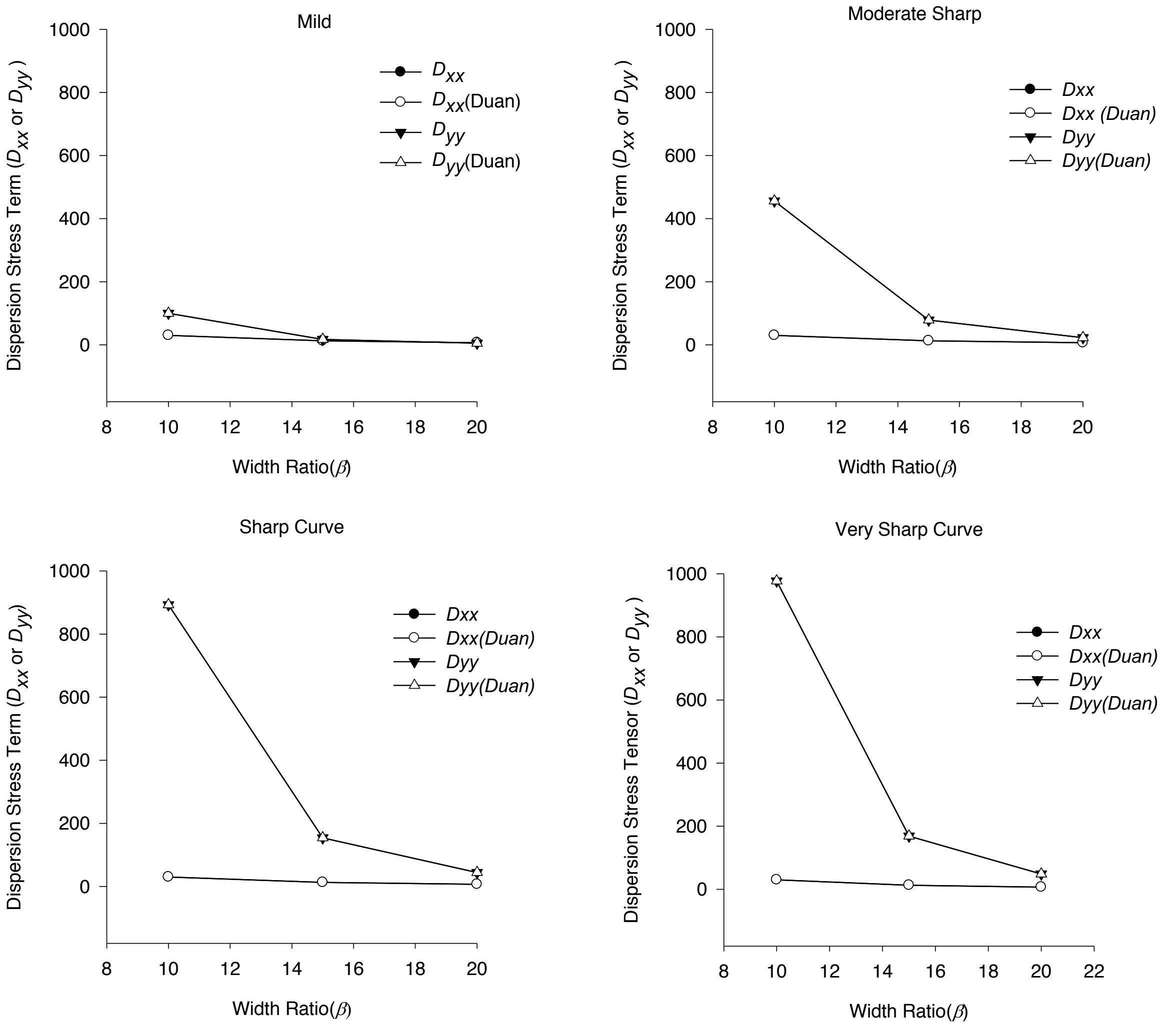


Fig. 2 Variation of $D_{xx}$, and $D_{yy}$ with width ratio ($\beta$) for *sub- critical flow condition*

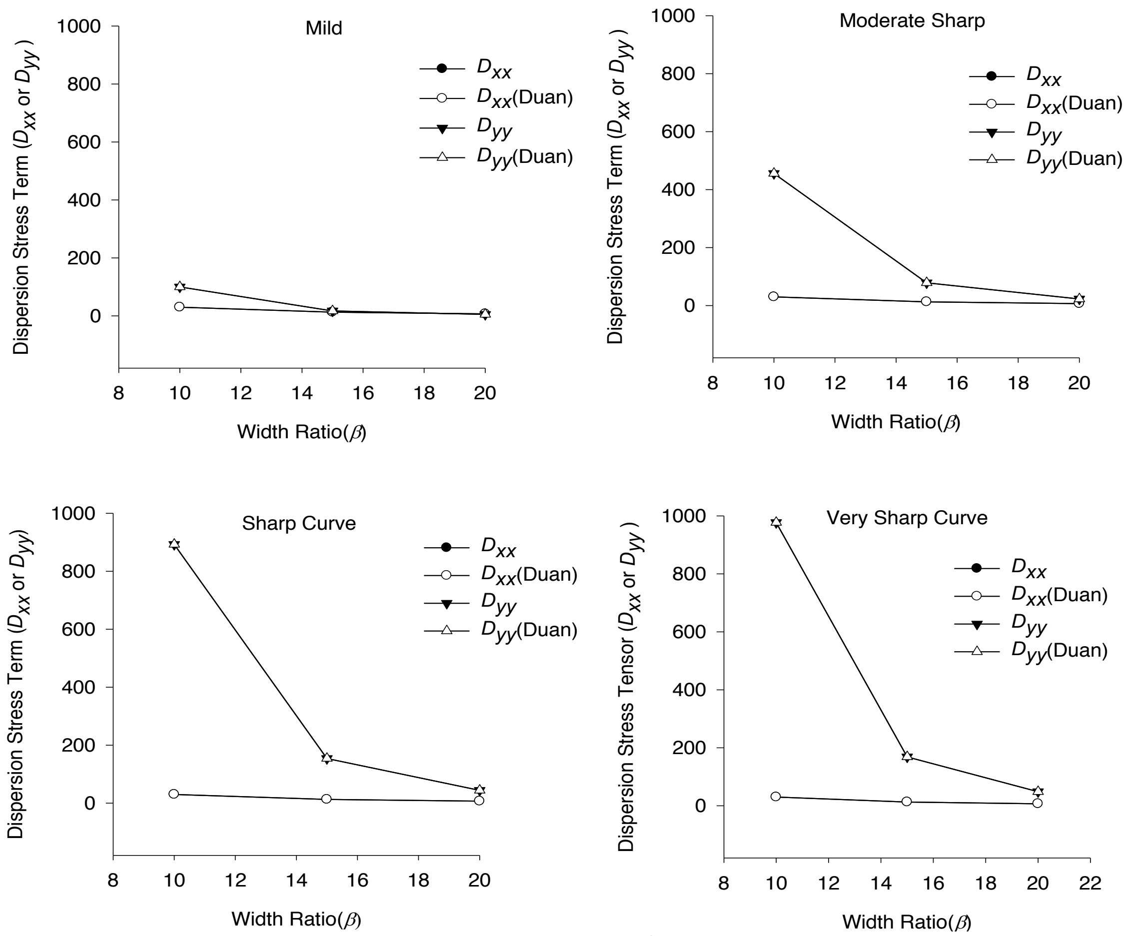


Fig. 3 Variation of $D^c_{xx}$, and $D^c_{yy}$ with width ratio ($\beta$) for *super-critical flow condition*

## 5. Conclusions

New expressions for dispersion stress tensor are proposed. A comparison between these terms and one given by Duan, indicates the conditions in which there is a good agreement between the two. An insight is provided to estimate one of the unknown parameters in the Duan's dispersion stress tensor. Compared to Duan's model, two of the three components of dispersion stress tensor namely $D^c_{xx}$, $D^c_{xy}$ are considerably simplified in the mathematical representation.

## Notation

$\beta$ = width ratio; $C_d$ =frictional stress coefficient; $D^c_{xx}$ , $D^c_{xy}$ and $D^c_{yy}$=components of dispersion coefficient tensor in curvilinear coordinates;$D_{xx}$, $D_{xy}$ and $D_{yy}$ =components of dispersion coefficient tensor in Cartesian coordinates; $g$ =acceleration of gravity; $H$ =water surface elevation; $h$ =flow depth; $\kappa_s$ =roughness height; $n$ =Manning's roughness coefficient; $\eta_0$=dimensionless zero bed elevation; $\xi$=stream-wise direction; $\eta$=transverse direction; $\theta_s$ ,$\theta_n$=angles between the stream-wise and transverse directions towards outer bank and the positive $x$ axis; $\varkappa$ = *Von Karman's* constant; $\nu$=kinetic viscosity; $\nu_t$ =eddy viscosity; $\varrho$=density of flow; $r$=radius of curvature; $t$=time; $\widehat{u_m}$ =depth averaged velocity in curvilinear coordinate system; $U$ =magnitude of depth averaged velocity; $U^{*}$= shear velocity; $U_x$, $U_y$ =depth averaged velocities in Cartesian coordinates; $u_n$, $U_n$=transverse actual and depth averaged velocities; $u_s$, $U_s$= Stream-wise actual and depth averaged velocities; $x$, $y$ = horizontal Cartesian coordinate; $z$=vertical coordinate; $z_0$=zero velocity level.

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