GENERATING COMPLEX POTENTIALS WITH REAL EIGENVALUES IN SUPERSYMMETRIC QUANTUM MECHANICS

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Abstract

In the framework of SUSYQM extended to deal with non-Hermitian Hamiltonians, we analyze three sets of complex potentials with real spectra, recently derived by a potential algebraic approach based upon the complex Lie algebra sl(2,$\mathbb{C}$). This extends to the complex domain the well-known relationship between SUSYQM and potential algebras for Hermitian Hamiltonians, resulting from their common link with the factorization method and Darboux transformations. In the same framework, we also generate for the first time a pair of elliptic partner potentials of Weierstrass $\wp$ type, one of them being real and the other imaginary and PT symmetric. The latter turns out to be quasiexactly solvable with one known eigenvalue corresponding to a bound state. When the Weierstrass function degenerates to a hyperbolic one, the imaginary potential becomes PT non-symmetric and its known eigenvalue corresponds to an unbound state.

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1   Introduction

Recently there has been some interest [1, 2, 3, 4, 5, 6, 7] in studying PT-symmetric quantum mechanical systems. In quantum mechanics, the Hamiltonian of the underlying system is usually assumed Hermitian ensuring a real energy spectrum. However it has been conjectured [1] that under less restrictive situations, namely by requiring the Schrödinger Hamiltonian to be invariant under the joint action of parity (P) and time reversal (T) transformations, one can still have a real spectrum of energy eigenvalues. Moreover, the overall normalizability of wave functions in many cases is not affected. In the literature, PT-symmetric schemes have been explored with respect to the complexification of several well-known potentials. Further, new ones have been searched for using a variety of techniques [3, 4, 5, 6, 7].

In this paper, we shall present results from a supersymmetric point of view. We shall show that the constraints furnished by the commutation relations of the sl(2,$\mathbb{C}$) algebra admit of solutions that are consistent with supersymmetric intertwining relations. We shall also exploit the latter to obtain elliptic solutions of Weierstrass $\wp$ type. Indeed we shall report here for the first time a PT-symmetric potential defined in terms of Weierstrass $\wp$ function.

2 $N = 2$ SUSYQM and Its Complexification Procedure

In order to set the notations, it would be useful to briefly recall the key features of the one-dimensional $N = 2$ SUSY quantum mechanics (SUSYQM). As is well known [8], the latter involves a pair of supercharges Q and $Q^\dagger$, related by Hermitian conjugation, and in terms of which the governing Hamiltonian H_s is expressed as

$$H_s = \left\{ Q, Q^\dagger \right\}. \tag{2.1}$$

The supercharges Q and $Q^\dagger$ are fermionic in character and commute with H_s:

$$\begin{aligned} Q^2 &= \left(Q^\dagger \right)^2 = 0, \\ [Q, H_s] &= \left[Q^\dagger, H_s \right] = 0. \end{aligned} \tag{2.2}$$

A convenient way to deal with Q and $Q^\dagger$ is to adopt the representations

$$Q = A \otimes \sigma_-, \qquad Q^\dagger = A^\dagger \otimes \sigma_+, \tag{2.3}$$

where A is some linear differential operator and $\sigma_\pm$ are combinations $\sigma_\pm = \sigma_1 \pm i\sigma_2$ of the Pauli matrices. A first-derivative realization of A and $A^\dagger$, namely

$$A = \frac{d}{dx} + W(x), \qquad A^\dagger = -\frac{d}{dx} + W(x), \tag{2.4}$$

yields the forms

$$Q = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + W & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & -\frac{d}{dx} + W \\ 0 & 0 \end{pmatrix}, \tag{2.5}$$

where $W(x)$ is the so-called superpotential of the system. Note that (2.5) renders H_s diagonal,

$$H_s = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}. \tag{2.6}$$

As such we can factorize $H_\pm$ in the manner

$$\begin{aligned} H_+ &= A^\dagger A = -\frac{d^2}{dx^2} + V^{(+)}(x) - E, \\ H_- &= AA^\dagger = -\frac{d^2}{dx^2} + V^{(-)}(x) - E, \end{aligned} \tag{2.7}$$

at some arbitrary factorization energy E, with the partner potentials $V^{(\pm)}$ related to $W(x)$ through

$$V^{(\pm)} = W^2 \mp W' + E. \tag{2.8}$$

It is easy to be convinced that the spectra of H_+ and H_- are alike except possibly for the ground state. In the exact SUSY case to which we shall restrict ourselves here, the ground state at vanishing energy is nondegenerate and is associated with H_+ or H_-,

$$H_+\psi_0^{(+)}(x) = 0, \qquad \psi_0^{(+)}(x) = K \exp\left(-\int^x W(t)dt\right), \tag{2.9}$$

or

$$H_-\psi_0^{(-)}(x) = 0, \qquad \psi_0^{(-)}(x) = K \exp\left(\int^x W(t)dt\right), \tag{2.10}$$

according to whether $\int^x W(t)dt \to +\infty$ or $-\infty$ as $x \to \pm\infty$. In (2.9) and (2.10), K is some normalization constant.

The double degeneracy of the spectrum of H_s can also be summarized by intertwining H_+ and H_- according to

$$AH_+ = H_-A, \qquad H_+A^\dagger = A^\dagger H_-. \tag{2.11}$$

These relations follow from (2.7).

To generate non-Hermitian potentials within SUSYQM [2], it is instructive to decompose the underlying superpotential $W(x)$, the partner potentials $V^{(\pm)}(x)$, and the factorization energy E into a real and an imaginary part, namely

$$W(x) = f(x) + ig(x), \tag{2.12}$$

$$V^{(+)}(x) = V_R^{(+)}(x) + iV_I^{(+)}(x), \tag{2.13}$$

$$V^{(-)}(x) = V_R^{(-)}(x) + iV_I^{(-)}(x), \tag{2.14}$$

$$E = E_R + iE_I, \tag{2.15}$$

where f, g, $V_R^{(\pm)}$, $V_I^{(\pm)}$, E_R, and $E_I \in \mathbb{R}$. All this leads to SUSY without Hermiticity: in particular, the supercharges are no longer related by Hermitian conjugation. As will be evident below, this presents no difficulty in so far as developing a theoretical framework is concerned.

From (2.8) and (2.12)–(2.15), it follows that

$$V_R^{(+)} = f^2 - g^2 - f' + E_R, \tag{2.16}$$

$$V_I^{(+)} = 2fg - g' + E_I, \tag{2.17}$$

$$V_R^{(-)} = f^2 - g^2 + f' + E_R, \tag{2.18}$$

$$V_I^{(-)} = 2fg + g' + E_I. \tag{2.19}$$

These expressions are consistent with intertwining relationships.

Since we will be interested only in a real energy spectrum, we can set $E_I = 0$. As such our basic relations correspond to

$$V_R^{(+)} = f^2 - g^2 - f' + E_R, \tag{2.20}$$

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$$V_I^{(+)} = 2fg - g', \tag{2.21}$$

$$V_R^{(-)} = f^2 - g^2 + f' + E_R, \tag{2.22}$$

$$V_I^{(-)} = 2fg + g'. \tag{2.23}$$

Observe that $V_R^{(+)}$ and $V_R^{(-)}$ are related by $f \to -f$, while $V_I^{(+)}$ and $V_I^{(-)}$ are linked through $g \to -g$. In the following our task will be to analyze plausible solutions for the functions f and g pertaining to the set (2.20)–(2.23), including the PT-symmetric ones. The main point to be noted is that by imposing an additional PT-symmetric restriction the Hermitian character of the intertwined Hamiltonians is lost.

We shall first of all seek connections with the sl(2,$\mathbb{C}$) potential algebraic approach to the construction of non-Hermitian Hamiltonians with real spectra. To this end we will show that our potentials (2.20)–(2.23) fit into such a scheme of complex potentials. In the next section, we therefore make a few remarks about the realization of the potential algebra sl(2,$\mathbb{C}$).

3 sl(2,$\mathbb{C}$) Potential Algebra

In Ref. [6], we made a detailed study of the sl(2,$\mathbb{C}$) algebra. Its underlying commutation relations are

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0. \tag{3.1}$$

The generators J_0 and $J_\pm$ can be given a differential realization

$$J_0 = -\mathrm{i}\frac{\partial}{\partial\phi}, \qquad J_\pm = e^{\pm \mathrm{i}\phi}\left[\pm\frac{\partial}{\partial x} + \left(\mathrm{i}\frac{\partial}{\partial\phi} \mp \frac{1}{2}\right)F(x) + G(x)\right], \tag{3.2}$$

where the auxiliary variable $\phi \in [0, 2\pi)$ facilitates their closure and the two functions $F(x), G(x) \in \mathbb{C}$ are subjected to constraints of the form

$$\frac{dF}{dx} = 1 - F^2, \qquad \frac{dG}{dx} = -FG, \qquad x \in \mathbb{R}. \tag{3.3}$$

Note that we have here $J_- \neq J_+^\dagger$, thereby inducing an sl(2,$\mathbb{C}$) algebra rather than so(2,1), which is consistent with $J_- = J_+^\dagger$.

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In the case of either sl(2,$\mathbb{C}$) or so(2,1), the irreducible representations are furnished by [9]

$$
\begin{aligned}
J_0|km\rangle &= m|km\rangle, \qquad m = k, k+1, k+2, \ldots, \\
J^2|km\rangle &= k(k-1)|km\rangle,
\end{aligned}
\tag{3.4}
$$

which are essentially of the type D_k^+. The Casimir operator J^2 corresponds to

$$
\begin{aligned}
J^2 &= J_0^2 \mp J_0 - J_\pm J_\mp \\
&= \frac{\partial^2}{\partial x^2} - \left(\frac{\partial^2}{\partial \phi^2} + \frac{1}{4} \right) F' + 2\mathrm{i} \frac{\partial}{\partial \phi} G' - G^2 - \frac{1}{4}.
\end{aligned}
\tag{3.5}
$$

Looking for representations that are

$$
|km\rangle = \Psi_{km}(x, \phi) = \psi_{km}(x) \frac{e^{\mathrm{i}m\phi}}{\sqrt{2\pi}},
\tag{3.6}
$$

where $k > 0$ and $m = k + n$, $n = 0, 1, 2, \ldots$, we find $\psi_{km}(x)$ to satisfy the Schrödinger equation

$$
-\psi_{km}'' + V_m \psi_{km} = -\left(k - \tfrac{1}{2} \right)^2 \psi_{km}.
\tag{3.7}
$$

In (3.7), the one-parameter family of potentials V_m is given by

$$
\begin{aligned}
V_m &= -\left(m - \tfrac{1}{2} \right) \left(m + \tfrac{1}{2} \right) F' + 2mG' + G^2 \\
&= -\left(m - \tfrac{1}{2} \right) \left(m + \tfrac{1}{2} \right) \left(1 - F^2 \right) - 2mFG + G^2,
\end{aligned}
\tag{3.8}
$$

where Eq. (3.3) has been used. These potentials share the same real energy eigenvalues

$$
E_n^{(m)} = -\left(m - n - \tfrac{1}{2} \right)^2,
\tag{3.9}
$$

thus producing a potential algebra.

Equations (3.3) can be solved for the functions F and G to get a quite complete realization of the sl(2,$\mathbb{C}$) algebra. The results obtained by us may be summarized as follows:

$$
\begin{array}{lll}
\mathrm{I}: & F(x) = \tanh(x - c - \mathrm{i}\gamma), & G(x) = b\,\mathrm{sech}(x - c - \mathrm{i}\gamma), \\
\mathrm{II}: & F(x) = \coth(x - c - \mathrm{i}\gamma), & G(x) = b\,\mathrm{cosech}(x - c - \mathrm{i}\gamma), \\
\mathrm{III}: & F(x) = \pm 1, & G(x) = b e^{\mp x},
\end{array}
\tag{3.10}
$$

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where $b = b_R + ib_I$, b_R, $b_I \in \mathbb{R}$, and $-\frac{\pi}{4} \le \gamma < \frac{\pi}{4}$. These lead to potentials

$$\text{I}: \quad V_m = \left[(b_R + ib_I)^2 - m^2 + \tfrac{1}{4}\right] \text{sech}^2(x - c - i\gamma)$$
$$- 2m(b_R + ib_I) \text{sech}(x - c - i\gamma) \tanh(x - c - i\gamma), \qquad (3.11)$$

$$\text{II}: \quad V_m = \left[(b_R + ib_I)^2 + m^2 - \tfrac{1}{4}\right] \text{cosech}^2(x - c - i\gamma)$$
$$- 2m(b_R + ib_I) \text{cosech}(x - c - i\gamma) \coth(x - c - i\gamma), \quad (3.12)$$

$$\text{III}: \quad V_m = (b_R + ib_I)^2 e^{\mp 2x} \mp 2m(b_R + ib_I)e^{\mp x}. \qquad (3.13)$$

In this way the group theoretical approach of the potential algebras can be extended to non-Hermitian Hamiltonians (a subclass of which forms the PT-symmetric ones) by a simple complexification of the real algebras considered for Hermitian Hamiltonians.

4   sl(2,$\mathbb{C}$) Potentials in SUSYQM

To realize sl(2,$\mathbb{C}$) potentials from the supersymmetry-inspired Eqs. (2.20)–(2.23), we notice that V_m can be considered as a special case of the complex potential $V^{(+)} = W^2 - W' + E$ given by (2.8), corresponding to the choice of the complex superpotential

$$W = \left(m - \tfrac{1}{2}\right) F - G, \qquad (4.1)$$

and the real energy

$$E = E_R = -\left(m - \tfrac{1}{2}\right)^2. \qquad (4.2)$$

Inserting (4.1) and (4.2) into the definition of $V^{(+)}$, we indeed get

$$V^{(+)} \equiv V_m = \left[\left(m - \tfrac{1}{2}\right) F - G\right]^2 - \left[\left(m - \tfrac{1}{2}\right)\left(1 - F^2\right) + FG\right] - \left(m - \tfrac{1}{2}\right)^2$$
$$= -\left(m - \tfrac{1}{2}\right)\left(m + \tfrac{1}{2}\right)\left(1 - F^2\right) - 2mFG + G^2, \qquad (4.3)$$

which coincides with the sl(2,$\mathbb{C}$) form (3.8).

The potential $V^{(-)} = W^2 + W' + E$ of the superpartner is now

$$V^{(-)} = \left[\left(m - \tfrac{1}{2}\right) F - G\right]^2 + \left[\left(m - \tfrac{1}{2}\right)\left(1 - F^2\right) + FG\right] - \left(m - \tfrac{1}{2}\right)^2$$
$$= -\left(m - \tfrac{3}{2}\right)\left(m - \tfrac{1}{2}\right)\left(1 - F^2\right) - 2(m - 1)FG + G^2$$
$$= V_{m-1}. \qquad (4.4)$$

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From (3.9), it is obvious that we are here in the case where H_- has one level less than H_+ and Eq. (2.9) applies.

In Table 1 we have displayed the various forms of the complex superpotential W for different solutions of F and G summarized in (3.10).

It is interesting to discuss the results corresponding to the choice $\gamma = 0$. While for $b_I = 0$, the superpotential along with the partner potentials reduce to their real forms which is only expected, the possibility $b_R = 0$ is worth taking a look. For case I, W simply becomes

$$W = \left(m - \tfrac{1}{2}\right)\tanh(x - c) - ib_I\,\mathrm{sech}(x - c), \tag{4.5}$$

leading to

$$V^{(+)} \equiv V_m = \left(-b_I^2 - m^2 + \tfrac{1}{4}\right)\mathrm{sech}^2(x - c) - 2imb_I\,\mathrm{sech}(x - c)\tanh(x - c). \tag{4.6}$$

Its superpartner can be read off readily from (4.4) and is

$$V^{(-)} \equiv V_{m-1} = \left[-b_I^2 - (m - 1)^2 + \tfrac{1}{4}\right]\mathrm{sech}^2(x-c) - 2i(m-1)b_I\,\mathrm{sech}(x-c)\tanh(x-c). \tag{4.7}$$

Note that both $V^{(+)}$ and $V^{(-)}$ turn out to be PT symmetric. For completeness we give the solutions for f, g, $V_R^{(\pm)}$, and $V_I^{(\pm)}$. These are

$$f = \left(m - \tfrac{1}{2}\right)\tanh(x - c), \tag{4.8}$$

$$g = -b_I\,\mathrm{sech}(x - c), \tag{4.9}$$

$$V_R^{(+)} = \left(-b_I^2 - m^2 + \tfrac{1}{4}\right)\mathrm{sech}^2(x - c), \tag{4.10}$$

$$V_I^{(+)} = -2mb_I\,\mathrm{sech}(x - c)\tanh(x - c), \tag{4.11}$$

$$V_R^{(-)} = \left[-b_I^2 - (m - 1)^2 + \tfrac{1}{4}\right]\mathrm{sech}^2(x - c), \tag{4.12}$$

$$V_I^{(-)} = -2(m - 1)b_I\,\mathrm{sech}(x - c)\tanh(x - c). \tag{4.13}$$

A particular case of the above scheme corresponding to $m = 1$ was derived by Bagchi and Roychoudhury [4], who showed that the PT-symmetric combination of (4.10) and (4.11) has energy levels that are negative semi-definite and, except for the zero-energy state, coincide with those of the sech^2 potential resulting from (4.12).

Similarly we can deal with cases II and III, both of which are PT non-symmetric, for the choice of parameters $\gamma = 0$ and $b_R = 0$. While case II gives the coth, cosech version of (4.8)–(4.13), case III leads to the complexified Morse potential. The results for W, f, and g are

Case II for $\gamma = 0$, $b_R = 0$:

$$
\begin{aligned}
W &= \left(m - \tfrac{1}{2}\right)\coth(x - c) - ib_I\,\text{cosech}(x - c), \\
f &= \left(m - \tfrac{1}{2}\right)\coth(x - c), \\
g &= -b_I\,\text{cosech}(x - c),
\end{aligned}
\tag{4.14}
$$

Case III for $\gamma = 0$, $b_R = 0$:

$$
\begin{aligned}
W &= \pm\left(m - \tfrac{1}{2}\right) - ib_I e^{\mp x}, \\
f &= \pm\left(m - \tfrac{1}{2}\right), \\
g &= -b_I e^{\mp x}.
\end{aligned}
\tag{4.15}
$$

5 A PT-Symmetric Potential in Terms of Weierstrass $\wp$ Function

In this section we make a specific attempt to search for a PT-symmetric potential described by Weierstrass $\wp$ function. In Ref. [2], Andrianov *et al.* took $V_R^{(+)} = 0$, $V_I^{(+)} = 0$ to analyze complex transparent potentials belonging to the set (2.20)–(2.23). Here we consider an equally viable possibility by setting $V_R^{(-)} = 0$, $V_I^{(+)} = 0$. This case is indeed nontrivial since other possibilities are either related to Andrianov *et al.* conjecture or the present one under $f \to -f$ and $g \to -g$.

While $V_I^{(+)} = 0$ results in

$$
f = \frac{g'}{2g},
\tag{5.1}
$$

leading to

$$
V_R^{(+)} = \frac{3g'^2}{4g^2} - \frac{g''}{2g} - g^2 + E_R,
\tag{5.2}
$$

$$
V_R^{(-)} = -\frac{g'^2}{4g^2} + \frac{g''}{2g} - g^2 + E_R,
\tag{5.3}
$$

$$
V_I^{(-)} = 2g',
\tag{5.4}
$$

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$V_R^{(-)} = 0$ gives us the solution for g in terms of the differential equation

$$\frac{dg}{\sqrt{g\left(\frac{4}{3}g^3 - 4E_R g + a\right)}} = \pm dx, \tag{5.5}$$

where a represents a constant of integration, which we take to be non-zero.

Writing $y(g) = g\left(\frac{4}{3}g^3 - 4E_R g + a\right)$ in the form $y(g) = a_0 g^4 + 4a_1 g^3 + 6a_2 g^2 + 4a_3 g + a_4$, we have $a_0 = \frac{4}{3}$, $a_1 = 0$, $a_2 = -\frac{2}{3}E_R$, $a_3 = \frac{1}{4}a$, $a_4 = 0$. We next define quantities g_2 and g_3 as

$$g_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2 = \frac{4}{3}E_R^2, \tag{5.6}$$

$$g_3 = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4 = \frac{8}{27}E_R^3 - \frac{1}{12}a^2. \tag{5.7}$$

Let

$$z = \int_{g0}^{g} [y(t)]^{-1/2} dt, \tag{5.8}$$

where g_0 is any root of the equation $y(g) = 0$. We identify g_0 as $g_0 = 0$.

Let us substitute $t = \frac{1}{\tau}$ and $g = \frac{1}{\xi}$ to rewrite (5.8) as

$$z = \int_{\xi}^{\infty} \left(4a_3 \tau^3 + 6a_2 \tau^2 + a_0\right)^{-1/2} d\tau. \tag{5.9}$$

The second term in the integrand can be removed by effecting the transformations

$$\tau = \frac{\sigma - \frac{1}{2}a_2}{a_3} = \frac{4}{a}\left(\sigma + \frac{1}{3}E_R\right), \tag{5.10}$$

$$\xi = \frac{s - \frac{1}{2}a_2}{a_3} = \frac{4}{a}\left(s + \frac{1}{3}E_R\right). \tag{5.11}$$

As a result, z turns out to be given by

$$z = \int_{s}^{\infty} \left(4\sigma^3 - g_2 \sigma - g_3\right)^{-1/2} d\sigma. \tag{5.12}$$

Now from the theory of elliptic functions [10], we can read off

$$s = \wp(z; g_2, g_3), \tag{5.13}$$

where $\wp(z; g_2, g_3)$ is Weierstrass elliptic function with g_2 and g_3 as invariants. Equation (5.11) yields

$$g = \frac{a}{4}\left[\wp(z; g_2, g_3) + \frac{1}{3}E_R\right]^{-1}. \tag{5.14}$$

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Hence we deduce

$$V_I^{(-)} = -\frac{a}{2}\frac{\wp'(z)}{\left[\wp(z) + \frac{1}{3}E_R\right]^2}, \tag{5.15}$$

$$V_R^{(+)} = 2\left[\wp(z) + \frac{1}{3}E_R\right] - \frac{a^2}{12\left[\wp(z) + \frac{1}{3}E_R\right]^2}. \tag{5.16}$$

In deriving (5.16), use has been made of the differential equation satisfied by $\wp(z)$,

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3, \tag{5.17}$$

and of its consequence

$$\wp''(z) = 6\wp^2(z) - \tfrac{1}{2}g_2. \tag{5.18}$$

We notice that while $V^{(+)} = V_R^{(+)}$ is pure real, $V^{(-)} = iV_I^{(-)}$ is a pure imaginary potential.

Comparing (5.5) with (5.8), we see that $z = \pm x + c$, where c is an integration constant. Since $\wp(z)$ and $\wp'(z)$ are respectively even and odd functions of z, the two solutions obtained by taking either sign correspond to each other by a mere change of signs of the integration constants a and c. Thus without any loss of generality we can take $z = x + c$ only.

We may distinguish between the non-degenerate and degenerate cases of Weierstrass $\wp$ function [10].

In the non-degenerate case, the roots of the cubic equation

$$4\sigma^3 - g_2\sigma - g_3 = 0 \tag{5.19}$$

are all distinct and the corresponding discriminant

$$D = g_2^3 - 27g_3^2 = \frac{a^2}{48}\left(64E_R^3 - 9a^2\right) \tag{5.20}$$

is non-vanishing. It is either positive or negative according to whether $|a| < \frac{8}{3}E_R^{3/2}$ or $|a| > \frac{8}{3}E_R^{3/2}$, the former case occurring only for $E_R > 0$.

By using numerical studies, we showed that in the $D < 0$ case, wherein the Weierstrass function has a pair of complex conjugate primitive periods 2ω, $2\omega' =$

$2\omega^*$, $V_R^{(+)}$ and $V_I^{(-)}$ go to $-\infty$ for some z values because the denominators in (5.15) and (5.16) vanish for such values. On the contrary, in the $D > 0$ case, wherein the Weierstrass function has a pair of primitive periods 2ω, $2\omega'$ with ω real and ω' imaginary, we obtain well-behaved potentials defined on the interval $0 < z < 2\omega$ or $-c < x < 2\omega - c$. The potential $V_R^{(+)}$ is a single-well potential, singular at $z \to 0$ and $z \to 2\omega$ (where it behaves as $1/z^2$ and $1/(z-2\omega)^2$, respectively), and symmetric around its minimum at $z = \omega$, whereas $V_I^{(-)}$ vanishes at $z \to 0$ and $z \to 2\omega$, and is antisymmetric around $z = \omega$. Hence, the potential $V^{(-)} = iV_I^{(-)}$ is PT symmetric provided the parity operation is defined with respect to a mirror placed at $z = \omega$ or $x = \omega - c$.

In Fig. 1, the functions $V_R^{(+)}$ and $V_I^{(-)}$ are displayed in terms of z for $E_R = \sqrt{3}$ and $a = 4\sqrt{2/\sqrt{3}}$, corresponding to the invariants $g_2 = 4$, $g_3 = 0$. For such values, the cubic equation (5.19) has the three real roots $e_1 = 1$, $e_2 = 0$, $e_3 = -1$, and the real primitive period is given by $2\omega = 2x_*$ where $x_* = \sqrt{\pi}\,\Gamma(5/4)/\Gamma(3/4) \simeq 1.311$ [11]. The minimum of $V_R^{(+)}$ is equal to $6\left(1 - \frac{1}{\sqrt{3}}\right)$.

In the degenerate case, at least two of the roots of (5.19) are equal, meaning that $D = 0$. This condition imposes that $E_R > 0$, $a = \pm\frac{8}{3}E_R^{3/2}$, $g_2 = \frac{4}{3}E_R^2$, and $g_3 = -\frac{8}{27}E_R^3$. We are then in a case where the real period becomes infinite and $\wp(z)$ reduces to [10]

$$\wp(z) = E_R\left[\tfrac{1}{3} + \mathrm{cosech}^2\left(\sqrt{E_R}\,z\right)\right]. \tag{5.21}$$

In consequence we have

$$g = \pm\frac{\sqrt{E_R}}{1 + \frac{3}{2}\,\mathrm{cosech}^2\left(\sqrt{E_R}\,z\right)}, \tag{5.22}$$

from which we obtain

$$V_R^{(+)} = \frac{4}{3}E_R\left\{1 + \frac{3}{2}\,\mathrm{cosech}^2\left(\sqrt{E_R}\,z\right) - \frac{1}{\left[1 + \frac{3}{2}\,\mathrm{cosech}^2\left(\sqrt{E_R}\,z\right)\right]^2}\right\}, \tag{5.23}$$

$$V_I^{(-)} = \pm 6E_R\frac{\mathrm{cosech}^2\left(\sqrt{E_R}\,z\right)\coth\left(\sqrt{E_R}\,z\right)}{\left[1 + \frac{3}{2}\,\mathrm{cosech}^2\left(\sqrt{E_R}\,z\right)\right]^2}, \tag{5.24}$$

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defined on the interval $0 < z < \infty$ or $-c < x < \infty$. As such $V^{(-)}$, given by $V^{(-)} = \mathrm{i}V_I^{(-)}$, is a non-PT-symmetric potential. We also note that as $z \to 0$, $V_R^{(+)} \sim 2/z^2$ while for $z \to \infty$, $V_R^{(+)} \sim 24E_R \exp(-2\sqrt{E_R}\,z)$. These are reasonable boundary conditions, the behaviour of $V_R^{(+)}$ proving to be repulsive.

An example is displayed in Fig. 2 for $E_R = \sqrt{3}$ and $a = 8/3^{1/4}$, corresponding to the invariants $g_2 = 4$ and $g_3 = -8/3^{3/2}$. For such values, the cubic equation (5.19) has the three real roots $e_1 = e_2 = 1/\sqrt{3}$ and $e_3 = -2/\sqrt{3}$.

As discussed in Sec. 2, the spectra of the supersymmetric partners H_+ and H_- are alike except for the ground state. In the non-degenerate case, whenever $E_R > 0$ and $|a| < \frac{8}{3}E_R^{3/2}$, the Hermitian Hamiltonian H_+ has an infinite number of (unknown) discrete positive-energy levels. The same is true for the PT-symmetric Hamiltonian H_-, but in addition the latter has a normalizable eigenfunction $\psi_0^{(-)}$ corresponding to $E = 0$. From (2.10), (2.12), and (5.1), it is given by

$$\psi_0^{(-)}(x) = K\sqrt{g}\exp\left(\mathrm{i}\int^x g(t)dt\right). \tag{5.25}$$

By using (5.14), its modulus can be written as

$$|\psi_0^{(-)}(x)| = \frac{|K|\sqrt{|a|}}{2}\left[\wp(z;g_2,g_3) + \frac{1}{3}E_R\right]^{-1/2}. \tag{5.26}$$

Hence it vanishes at $z = 0$ and $z = 2\omega$, and is regular in between, showing that $\psi_0^{(-)}(x)$ is indeed normalizable on $(-c, 2\omega - c)$.

Similarly, in the degenerate case, i.e., for $E_R > 0$ and $|a| = \frac{8}{3}E_R^{3/2}$, the Hamiltonians H_+ and H_- have both a continuous spectrum of unbounded positive-energy states. In addition, H_- has an unbound zero-energy state, whose wave function is still given by (5.25). From (5.14) and (5.21), we indeed obtain

$$|\psi_0^{(-)}(x)| = \frac{|K|}{2}\sqrt{\frac{|a|}{E_R}}\left[\operatorname{cosech}^2\left(\sqrt{E_R}\,z\right) + \frac{2}{3}\right]^{-1/2}, \tag{5.27}$$

which vanishes for $z = 0$ and goes to $\frac{|K|}{2}\sqrt{\frac{3|a|}{2E_R}}$ for $z \to \infty$.

As a final point, it is worth noting that the known zero-energy eigenfunction of H_- is an eigenfunction of the Hamiltonian $-\frac{d^2}{dx^2} + V^{(-)}(x)$ with energy E_R.

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6 Conclusion

In the present paper, we both constructed some new PT-preserving or non-PT-preserving complex potentials and analyzed some known ones from a SUSYQM viewpoint extended to deal with non-Hermitian Hamiltonians.

To start with, we considered three sets of complex potentials, recently derived by a potential algebraic approach based on the complex Lie algebra sl(2,$\mathbb{C}$) [6], and proved that they can be generated as well from a complex superpotential and a pair of supercharges that are not related by Hermitian conjugation. This extends to the complex domain the well-known relationship between SUSYQM and potential algebras for Hermitian Hamiltonians, resulting from their common link [12] with the factorization method [13] and Darboux transformations [14].

In a second step, we analyzed the special case of the extended SUSYQM theory [2] wherein the starting potential is real and its partner imaginary. This allowed us to build for the first time a pair of complex partner potentials defined in terms of Weierstrass elliptic function. The PT-symmetric imaginary partner potential has one known eigenvalue equal to the factorization energy E_R and corresponding to a bound state. When the Weierstrass function degenerates to a hyperbolic one, the imaginary partner potential becomes PT non-symmetric and its eigenvalue E_R corresponds to an unbound state. We have therefore constructed two new quasiexactly solvable complex potentials [5, 7].

Acknowledgments

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Table 1: The functions F, G, and the superpotential W corresponding to the three cases considered in the text.

Cases	F	G	W
I	$\tanh(x - c - \mathrm{i}\gamma)$	$b\,\mathrm{sech}(x - c - \mathrm{i}\gamma)$	$\left(m - \frac{1}{2}\right)\tanh(x - c - \mathrm{i}\gamma)$ $-b\,\mathrm{sech}(x - c - \mathrm{i}\gamma)$
II	$\coth(x - c - \mathrm{i}\gamma)$	$b\,\mathrm{cosech}(x - c - \mathrm{i}\gamma)$	$\left(m - \frac{1}{2}\right)\coth(x - c - \mathrm{i}\gamma)$ $-b\,\mathrm{cosech}(x - c - \mathrm{i}\gamma)$
III	± 1	$be^{\mp x}$	$\pm\left(m - \frac{1}{2}\right) - be^{\mp x}$

Figure captions

Fig. 1. (a) $V_R^{(+)}$ and (b) $V_I^{(-)}$ in terms of $z = x + c$ for the non-degenerate case of Weierstrass $\wp$ function, $E_R = \sqrt{3}$, and $a = 4\sqrt{2/\sqrt{3}}$.

Fig. 2. (a) $V_R^{(+)}$ and (b) $V_I^{(-)}$ in terms of $z = x + c$ for the degenerate case of Weierstrass $\wp$ function and $E_R = \sqrt{3}$.
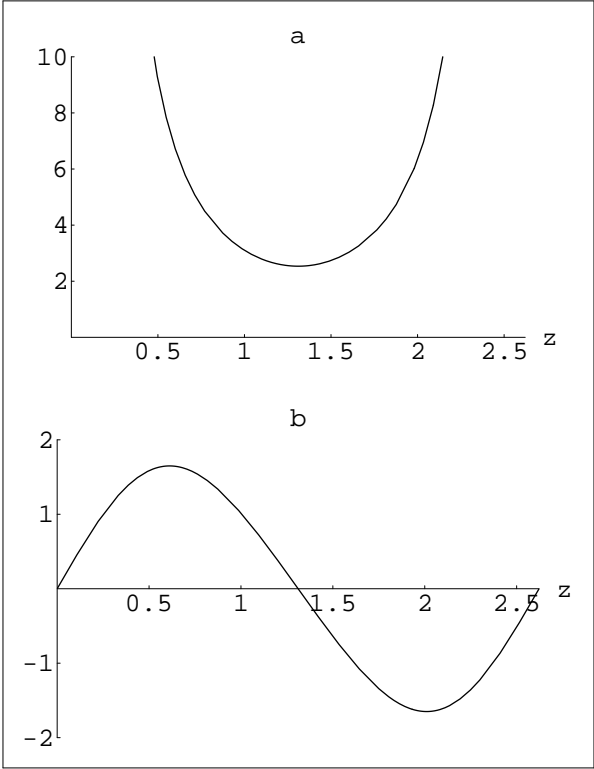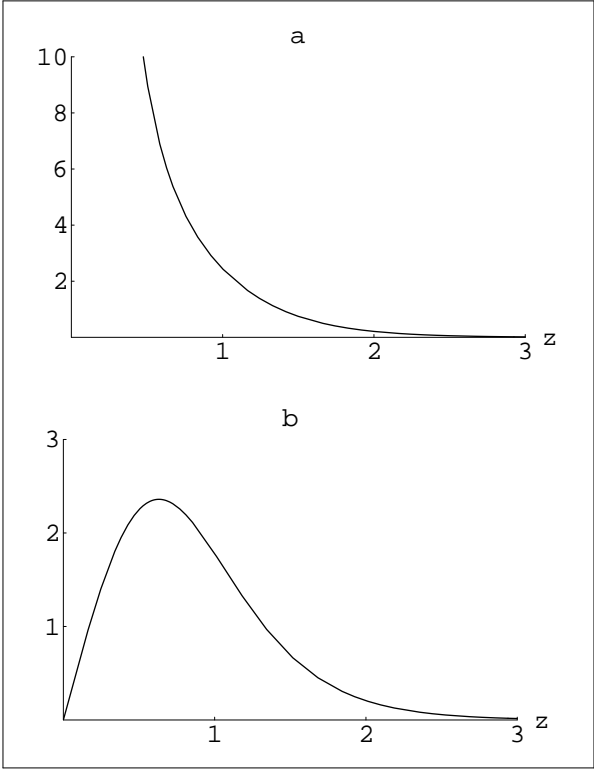
Figure 1

Figure 2