

GENERALIZED CONTINUITY EQUATION AND MODIFIED NORMALIZATION IN PT-SYMMETRIC QUANTUM MECHANICS

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Abstract

The continuity equation relating the change in time of the position probability density to the gradient of the probability current density is generalized to PT-symmetric quantum mechanics. The normalization condition of eigenfunctions is modified in accordance with this new conservation law and illustrated with some detailed examples.

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1 Introduction

From time to time, non-Hermitian Hamiltonians have found applications in several research areas, such as nuclear physics, quantum field theory, condensed matter physics, and biology. Since the pioneering works of Bessis [1] and Bender and Boettcher [2], a subclass of such Hamiltonians, containing operators invariant under joint action of parity (P: $x \rightarrow -x$, $p \rightarrow -p$) and time reversal (T: $x \rightarrow x$, $p \rightarrow -p$, $i \rightarrow -i$), has become a subject matter of considerable interest. One of the main reasons for this is that there is a strong analytical and numerical evidence supporting the conjecture that, except when PT symmetry is spontaneously broken, the bound-state eigenvalues of these Hamiltonians are real (see e.g. [2]–[12]).

PT-symmetric quantum mechanics is often seen as a testing bench for nonconventional quantum field theories wherein PCT invariance (or PT invariance in scalar theories) plays a leading role (see e.g. [13, 14]). This may be compared to the introduction of supersymmetric quantum mechanics by Witten [15] as a testing ground for non-perturbative methods of achieving supersymmetry breaking in field theory.

Both the field-theoretic interpretation and the phenomenological relevance of complex potentials in quantum mechanics make it necessary to better understand some fundamental issues connected with the replacement of Hermiticity by PT-symmetry, such as the physical interpretation of the Hamiltonian eigenfunctions [16, 17, 18, 19, 20].

The present work is intended as a contribution to such an understanding. Our purpose is twofold. Firstly, we will generalize to PT-symmetric quantum mechanics the conservation law of standard quantum mechanics [21] (here formulated for one-dimensional systems)

$$\frac{\partial P(x, t)}{\partial t} + \frac{\partial J(x, t)}{\partial x} = 0, \quad (1)$$

similar to the continuity equation of classical hydrodynamics and relating the change in time of the position probability density $P(x, t) = |\psi(x, t)|^2$ to the gradient of the probability current density $J(x, t) = (\hbar/2mi) [\psi^*(x, t)(\partial\psi(x, t)/\partial x) - \psi(x, t)(\partial\psi^*(x, t)/\partial x)]$. Secondly, we will discuss with some detailed examples a proposal for modifying the definition of the normalization condition of bound-state eigenfunctions in accordance with this new conservation law.

2 Generalized Continuity Equation and Some of Its Consequences

Let us start with the time-dependent Schrödinger equation for a single particle in one dimension,

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x)\psi(x, t), \quad (2)$$

where the potential $V(x)$ is assumed to be complex and PT-symmetric, i.e., $PTV(x)PT = V^*(-x) = V(x)$. As a consequence, the function $\psi^*(-x, t)$ satisfies the equation

$$-i\hbar \frac{\partial \psi^*(-x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*(-x, t)}{\partial x^2} + V(x)\psi^*(-x, t), \quad (3)$$

where the potential is the same as in (2).

Consider now eq. (2) for some solution $\psi_1(x, t)$ and eq. (3) for some other solution $\psi_2(x, t)$, which may be equal to or different from $\psi_1(x, t)$. Multiplying (2) by $\psi_2^*(-x, t)$, (3) by $\psi_1(x, t)$, and subtracting, we obtain

$$\frac{\partial}{\partial t} [\psi_2^*(-x, t)\psi_1(x, t)] + \frac{\hbar}{2mi} \frac{\partial}{\partial x} \left[\psi_2^*(-x, t) \frac{\partial \psi_1(x, t)}{\partial x} - \psi_1(x, t) \frac{\partial \psi_2^*(-x, t)}{\partial x} \right] = 0. \quad (4)$$

For $\psi_2(x, t) = \psi_1(x, t) = \psi(x, t)$, it is clear that eq. (4) is a natural generalization of the continuity equation (1) of standard quantum mechanics, namely

$$\frac{\partial P_{PT}(x, t)}{\partial t} + \frac{\partial J_{PT}(x, t)}{\partial x} = 0, \quad (5)$$

with

$$P_{PT}(x, t) = \psi^*(-x, t)\psi(x, t), \quad J_{PT}(x, t) = \frac{\hbar}{2mi} \left[\psi^*(-x, t) \frac{\partial \psi(x, t)}{\partial x} - \psi(x, t) \frac{\partial \psi^*(-x, t)}{\partial x} \right]. \quad (6)$$

If $\psi_1(x, t) \rightarrow 0$ and $\psi_2(x, t) \rightarrow 0$ for $x \rightarrow \pm\infty$, as is normally required for bound-state wave functions, integration of (4) over the entire real line leads to the conservation law

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \psi_2^*(-x, t)\psi_1(x, t) = 0, \quad (7)$$

or for $\psi_2(x, t) = \psi_1(x, t) = \psi(x, t)$,

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx P_{PT}(x, t) = 0. \quad (8)$$

Note that for Hermitian Hamiltonians, the result corresponding to (8) is interpreted in standard quantum mechanics as the conservation of norm in time [21].

From now on, we are going to restrict ourselves to energy eigenfunctions

$$\psi_1(x, t) = u_1(x)e^{-\frac{i}{\hbar}E_1t}, \quad \psi_2(x, t) = u_2(x)e^{-\frac{i}{\hbar}E_2t}, \quad (9)$$

associated with (possibly complex) energies E_1, E_2 , respectively. On inserting (9) into (7), we obtain

$$(E_1 - E_2^*)e^{-\frac{i}{\hbar}(E_1 - E_2^*)t} \int_{-\infty}^{+\infty} dx u_2^*(-x)u_1(x) = 0. \quad (10)$$

To study the consequences of (10), we have to distinguish between three cases.

If the two eigenvalues E_1 and E_2 are real, then $E_2 \neq E_1$ implies the vanishing of the integral on the left-hand side of (10), namely

$$\int_{-\infty}^{+\infty} dx u_2^*(-x)u_1(x) = 0, \quad (11)$$

whereas for $E_2 = E_1 = E$, eq. (10) is automatically satisfied. In such a case, if E is nondegenerate, $u_2(x)$ is proportional to $u_1(x) = u(x)$ and following (8) we may propose

$$\int_{-\infty}^{+\infty} dx u^*(-x)u(x) \quad (12)$$

as a counterpart of the standard normalization integral in PT-symmetric quantum mechanics. In the same way, the left-hand side of (11) can be seen as a counterpart of the scalar product. It should be noted that in giving up the Hermiticity condition of the Hamiltonian and replacing it by PT symmetry, some properties of norms and scalar products have been lost. For instance, it is obvious that the integral in (12) is not positive-definite and may therefore be called pseudo-norm [17].

If instead only one of the eigenvalues (for instance E_1) is real and the other (E_2) is complex, eq. (11) again follows from (10).

Finally, if both eigenvalues E_1 and E_2 are complex, eq. (10) implies condition (11) if $E_2 \neq E_1^*$. This is true in particular for $E_2 = E_1 = E$. In such a case, if E is nondegenerate, thence $u_2(x)$ is proportional to $u_1(x) = u(x)$ and condition (11) becomes

$$\int_{-\infty}^{+\infty} dx u^*(-x)u(x) = 0. \quad (13)$$

On the contrary, if $E_2 = E_1^* = E^*$, then eq. (10) is automatically satisfied and we may take

$$\int_{-\infty}^{+\infty} dx u_2^*(-x)u_1(x) \quad (14)$$

as a counterpart of the standard normalization integral. Note that in (14), $u_1(x)$ and $u_2(x)$ correspond to E and E^* , respectively. It is remarkable that in going from real to complex eigenvalues, the roles of (11) and (12) are interchanged. Such results agree with the analysis carried out in [17].

Let us consider in more detail the case of an eigenfunction $u(x)$ corresponding to some real, nondegenerate eigenvalue E . It satisfies the time-independent Schrödinger equation $-(\hbar^2/2m)d^2u(x)/d^2x + V(x)u(x) = Eu(x)$, and so is the case of $u^*(-x)$. Since E is nondegenerate, we must have $u^*(-x) = cu(x)$ for some complex constant c . From this relation, it follows that $u^*(x) = cu(-x) = |c|^2u^*(x)$, hence $|c| = 1$. We conclude that

$$u^*(-x) = e^{i\phi}u(x), \quad 0 \leq \phi < 2\pi. \quad (15)$$

If $\phi = 0$ (resp. $\phi = \pi$) in (15), then $u^*(-x) = u(x)$ (resp. $u^*(-x) = -u(x)$) or, in other words, $u(x)$ is PT-symmetric (resp. PT-antisymmetric). If however $\phi \neq 0, \pi$, it is always possible to convert $u(x)$ into such a function by multiplying it by some appropriate phase factor. The functions

$$v_\sigma(x) \equiv e^{\frac{i}{2}(\phi - \frac{\pi}{2} + \sigma\frac{\pi}{2})}u(x), \quad \sigma = \pm 1, \quad (16)$$

indeed satisfy the relations $v_\sigma^*(-x) = \sigma v_\sigma(x)$, $\sigma = \pm 1$ and are PT-symmetric ($\sigma = +1$) or PT-antisymmetric ($\sigma = -1$).

We propose to normalize $u(x)$, or the corresponding $v_\sigma(x)$, according to the prescription

$$\int_{-\infty}^{+\infty} dx u^*(-x)u(x) = \sigma \int_{-\infty}^{+\infty} dx [v_\sigma(x)]^2 = \sigma. \quad (17)$$

In the following, we illustrate the use of the above condition by considering some specific cases of PT-symmetric model potentials, namely, PT-symmetric oscillator, generalized Poschl-Teller, and Scarf II potentials. In particular, we show that for this rule to be meaningful, we have to tell which PT-parity σ , i.e., which function $v_\sigma(x)$, we are going to associate to any given eigenfunction $u(x)$ with real eigenvalue.

3 Modified Normalization for the PT-Symmetric Oscillator

The potential for the PT-symmetric oscillator is given by [6]

$$V^{(\alpha)}(x) = (x - ic)^2 + \frac{\alpha^2 - \frac{1}{4}}{(x - ic)^2}, \quad (18)$$

where $\alpha > 0$. It can be obtained from the usual three-dimensional radial harmonic oscillator potential by effecting a complex shift of coordinate $x \rightarrow x - ic$, $c > 0$, and replacing l by $\alpha - \frac{1}{2}$. Although it is beset with a centrifugal-like core, the shift of the singularity off the integration path makes the corresponding Hamiltonian exactly solvable on the entire real line for any $\alpha > 0$, like the linear harmonic oscillator to which it reduces for $\alpha = \frac{1}{2}$ and $c = 0$.

If α is different from an integer (which we shall assume here), the PT-symmetric oscillator Hamiltonian has a double series of energy eigenvalues $E_{qn}^{(\alpha)}$, which may be distinguished by a quantum number $q = \pm 1$ [6]. For $\hbar = 2m = 1$, they are given by

$$E_{qn}^{(\alpha)} = 4n + 2 - 2q\alpha, \quad n = 0, 1, 2, \dots \quad (19)$$

The corresponding eigenfunctions are expressible in terms of generalized Laguerre polynomials:

$$u_{qn}^{(\alpha)}(x) = \mathcal{N}_{qn}^{(\alpha)} e^{-\frac{1}{2}(x-ic)^2} (x - ic)^{-q\alpha + \frac{1}{2}} L_n^{(-q\alpha)}[(x - ic)^2]. \quad (20)$$

Here $\mathcal{N}_{qn}^{(\alpha)}$ is some yet undetermined normalization coefficient. Although the Laguerre polynomials in (20) are in principle defined only for $-q\alpha > -1$, i.e., for α values in the interval $0 < \alpha < 1$, their definition can be extended to any values of α . In the interval $0 < \alpha < 1$ (and only in it), however, the energies (19) are ordered according to increasing values of $N = 2n + \frac{1}{2}(1 - q) = 0, 1, 2, \dots$, and alternatively correspond to $q = +1$ and $q = -1$. In the special case of the linear harmonic oscillator ($\alpha = \frac{1}{2}$ and $c = 0$), q becomes the parity of the eigenfunctions. For this reason, q is called quasi-parity of the PT-symmetric oscillator.

It is straightforward to see that $[u_{qn}^{(\alpha)}(-x)]^*$ is related to $u_{qn}^{(\alpha)}(x)$ as shown in (15), where the phase is given by $\phi = \pi(-q\alpha + \frac{1}{2}) - 2\nu$ (up to a multiple of 2π). Here ν denotes the phase of the normalization coefficient $\mathcal{N}_{qn}^{(\alpha)} = |\mathcal{N}_{qn}^{(\alpha)}|e^{i\nu}$.

We now choose to identify the PT-parity σ of the functions $v_\sigma(x)$ in (16) with the quasi-parity q of the eigenfunctions (20). As a consequence, the functions

$$v_{qn}^{(\alpha)}(x) = e^{iq\frac{\pi}{2}(-\alpha+\frac{1}{2})} |\mathcal{N}_{qn}^{(\alpha)}| e^{-\frac{1}{2}(x-ic)^2} (x-ic)^{-q\alpha+\frac{1}{2}} L_n^{(-q\alpha)}[(x-ic)^2] \quad (21)$$

are PT-symmetric or PT-antisymmetric for $q = +1$ or $q = -1$, respectively.

Let us normalize the functions $u_{qn}^{(\alpha)}(x)$ and $v_{qn}^{(\alpha)}(x)$ according to the rule (17), where σ is replaced by q . From (20), we obtain

$$\int_{-\infty}^{+\infty} dx [u_{qn}^{(\alpha)}(-x)]^* u_{qn}^{(\alpha)}(x) = e^{i\pi(-q\alpha+\frac{1}{2})} |\mathcal{N}_{qn}^{(\alpha)}|^2 \mathcal{I}_{qn}^{(\alpha)}, \quad (22)$$

where

$$\mathcal{I}_{qn}^{(\alpha)} = \int_{-\infty}^{+\infty} dx e^{-(x-ic)^2} (x-ic)^{-2q\alpha+1} \left\{ L_n^{(-q\alpha)}[(x-ic)^2] \right\}^2. \quad (23)$$

To calculate $\mathcal{I}_{qn}^{(\alpha)}$, we consider the function of the complex variable z given by $f(z) = e^{-z^2} z^{-2q\alpha+1} [L_n^{(-q\alpha)}(z^2)]^2$. It is analytic everywhere except at $z = 0$, which is a branch point. Let us take the negative real axis as the branch cut. As the integral $\mathcal{I}_{qn}^{(\alpha)}$ can be seen as the limit for $R \rightarrow \infty$ of $\int_{-R}^{+R} dz f(z)$ for $z = x - ic$ (i.e., the integral on segment AB in Fig. 1), we close a contour $\Gamma \equiv ABCDEF$ in the complex plane, avoiding the singularity by using a semi-circle C_ρ of radius ρ . Since $f(z)$ is analytic within and on Γ , by Cauchy's theorem, we have $\int_\Gamma dz f(z) = 0$. Evaluating this integral and taking the limits $R \rightarrow \infty$ and $\rho \rightarrow 0$, we obtain that $\mathcal{I}_{qn}^{(\alpha)}$ converges if $q\alpha < 1$, which is consistent with $q = \pm 1$ and $0 < \alpha < 1$.

As a result, eq. (22) becomes

$$\int_{-\infty}^{+\infty} dx [u_{qn}^{(\alpha)}(-x)]^* u_{qn}^{(\alpha)}(x) = |\mathcal{N}_{qn}^{(\alpha)}|^2 \cos \pi(-q\alpha + \frac{1}{2}) \frac{\Gamma(-q\alpha + n + 1)}{n!}, \quad (24)$$

if $0 < \alpha < 1$. Since the cosine on the right-hand side of (24) is positive or negative according to whether $q = +1$ or $q = -1$, the result in (24) is consistent with condition (17). The latter therefore leads to

$$|\mathcal{N}_{qn}^{(\alpha)}| = \left(\frac{n!}{\Gamma(-q\alpha + n + 1) \cos \pi(-\alpha + \frac{1}{2})} \right)^{1/2}. \quad (25)$$

As compared with the standard harmonic oscillator, the presence of a cosine on the right-hand side of (25) is a new feature. Such a cosine disappears for $\alpha = \frac{1}{2}$, so eq. (25) agrees with the known normalization coefficients of the linear oscillator ($\alpha = \frac{1}{2}$, $q = \pm 1$) and of the three-dimensional radial oscillator ($\alpha = \frac{1}{2}$, $l = 0$, $q = -1$).

4 Modified Normalization for the PT-Symmetric Generalized Pöschl-Teller Potential

The PT-symmetric generalized Pöschl-Teller potential is given by [10]

$$V^{(A,B)}(x) = \left[B^2 + A(A+1) \right] \operatorname{cosech}^2 \tau - B(2A+1) \operatorname{cosech} \tau \coth \tau, \quad \tau \equiv x - i\gamma, \quad (26)$$

where $B > A + \frac{1}{2} > 0$ and $-\frac{\pi}{4} \leq \gamma < 0$ or $0 < \gamma < \frac{\pi}{4}$. By using an $\operatorname{sl}(2, \mathbb{C})$ algebraic framework, it was recently shown [10] that whenever $B - A - \frac{1}{2}$ is different from an integer, the corresponding Hamiltonian has two series of real energy eigenvalues $E_{qn}^{(A,B)}$, which may be distinguished by a label $q = \pm 1$ (here called quasi-parity for convenience sake) and which for $\hbar = 2m = 1$ are given by

$$E_{+n}^{(A,B)} = - \left(B - \frac{1}{2} - n \right)^2, \quad n = 0, 1, \dots, n_{+max},$$

$$B - \frac{3}{2} \leq n_{+max} < B - \frac{1}{2}, \quad (27)$$

$$E_{-n}^{(A,B)} = - (A - n)^2, \quad n = 0, 1, \dots, n_{-max},$$

$$A - 1 \leq n_{-max} < A, \quad (28)$$

where $B > \frac{1}{2}$ and $A > 0$. The accompanying eigenfunctions read

$$u_{qn}^{(A,B)}(x) = \mathcal{N}_{qn}^{(A,B)} (y-1)^{\frac{1}{2}(\lambda+\frac{1}{2})} (y+1)^{\frac{1}{2}(\mu+\frac{1}{2})} P_n^{(\lambda,\mu)}(y)$$

$$= \mathcal{N}_{qn}^{(A,B)} 2^{\frac{1}{2}(\lambda+\mu+1)} \left(\sinh \frac{\tau}{2} \right)^{\lambda+\frac{1}{2}} \left(\cosh \frac{\tau}{2} \right)^{\mu+\frac{1}{2}} P_n^{(\lambda,\mu)}(\cosh \tau), \quad (29)$$

where $\lambda = q(A - B + \frac{1}{2})$, $\mu = -A - B - \frac{1}{2}$, $y = \cosh \tau$, $P_n^{(\lambda,\mu)}(y)$ is a Jacobi polynomial, and $\mathcal{N}_{qn}^{(A,B)}$ some normalization coefficient.

In the limit $\gamma \rightarrow 0$, the real generalized Pöschl-Teller potential is recovered and only the eigenvalues $E_{-n}^{(A,B)}$ and corresponding eigenfunctions $u_{-n}^{(A,B)}(x)$ survive [22]. Since this

potential is singular, it has to be restricted to the half-line $(0, +\infty)$. The complexified potential, as given above, gets regularized on performing the shift $x \rightarrow x - i\gamma$ and so may be considered on the entire real line, as it was the case for the PT-symmetric harmonic oscillator in the previous section.

The analogy between these two PT-symmetric potentials also extends to the normalization problem. The phase ϕ in (15) is now given (up to a multiple of 2π) by $\phi = \pi(\lambda + \frac{1}{2}) - 2\nu$, where ν denotes the phase of the normalization coefficient. This shows that the role of α for the PT-symmetric oscillator is now played by the constant $-q\lambda = B - A - \frac{1}{2}$. As in sect. 3, the PT-parity σ may be identified with q , leading to PT-symmetric and PT-antisymmetric eigenfunctions $v_{qn}^{(A,B)}(x)$ ($q = \pm 1$), given by

$$v_{qn}^{(A,B)}(x) = e^{i\frac{\pi}{2}(\lambda + \frac{1}{2}q)} \left| \mathcal{N}_{qn}^{(A,B)} \right| 2^{\frac{1}{2}(\lambda + \mu + 1)} \left(\sinh \frac{\tau}{2} \right)^{\lambda + \frac{1}{2}} \left(\cosh \frac{\tau}{2} \right)^{\mu + \frac{1}{2}} P_n^{(\lambda, \mu)}(\cosh \tau). \quad (30)$$

Applying again prescription (17) to normalize the eigenfunctions, we are led to the integral of the function $f(z) = (\sinh \frac{z}{2})^{2\lambda+1} (\cosh \frac{z}{2})^{2\mu+1} \left[P_n^{(\lambda, \mu)}(\cosh z) \right]^2$ on the contour Γ of Fig. 1, where c is replaced by γ (and the latter is assumed positive for simplicity's sake). In this respect, it is worth noting that among the branch points of $f(z)$ at $z = 0, \pm i\pi, \pm 2i\pi, \pm 3i\pi, \dots$, we only have to take care of the first one because of the range of γ , as stated below eq. (26).

The final result for the normalization integral reads

$$\int_{-\infty}^{+\infty} dx \left[u_{qn}^{(A,B)}(-x) \right]^* u_{qn}^{(A,B)}(x) = 2 \left| \mathcal{N}_{qn}^{(A,B)} \right|^2 \cos \left[\pi \left(\lambda + \frac{1}{2} \right) \right] I_n^{(\lambda, \mu)}, \quad (31)$$

provided λ is restricted by the condition $\lambda > -1$, which is consistent with $q = \pm 1$ and $A + \frac{1}{2} < B < A + \frac{3}{2}$. In (31), $I_n^{(\lambda, \mu)}$ is the real integral

$$I_n^{(\lambda, \mu)} = \int_1^{\infty} dt (t-1)^\lambda (t+1)^\mu \left[P_n^{(\lambda, \mu)}(t) \right]^2, \quad (32)$$

which, for $q = -1$, appears in the calculation of the normalization coefficient of the real potential eigenfunctions. It is convergent and positive for the above-mentioned restricted range of parameters and both values of q . As far as we know, no closed analytical formula

is known for it, except when $n = 0$. In such a case, the change of variable $s = 1/t$ leads to [23]

$$I_0^{(\lambda, \mu)} = 2^{\lambda + \mu + 1} \frac{\Gamma(-\lambda - \mu - 1)\Gamma(\lambda + 1)}{\Gamma(-\mu)}. \quad (33)$$

Furthermore, the cosine on the right-hand side of (31) is positive or negative according to whether $q = +1$ or $q = -1$. Hence eq. (31) is consistent with (17), from which it follows that

$$|\mathcal{N}_{qn}^{(A, B)}| = \left\{ 2 \cos[\pi(A - B + 1)] I_n^{(\lambda, \mu)} \right\}^{-1/2}. \quad (34)$$

The cosine disappears from (34) if $B = A + 1$. For this choice of parameters, potential (26) reduces to $V^{(A, A+1)} = -\frac{1}{2}(A + 1)(2A + 1) \operatorname{sech}^2 \frac{x}{2}$, which remains nonsingular in the limit $\gamma \rightarrow 0$.

As a final point, it is worth mentioning that the generalized Pöschl-Teller potential (26) is related to the PT-symmetric Pöschl-Teller II potential considered in [9] through a complex point canonical coordinate transformation [10], which also connects the corresponding eigenfunctions. The results of the present section can therefore easily be applied to the latter potential.

5 Modified Normalization for the PT-Symmetric Scarf II Potential

The PT-symmetric Scarf II potential is defined by [10]

$$V^{(A, B)}(x) = -\left[B^2 + A(A + 1)\right] \operatorname{sech}^2 x + iB(2A + 1) \operatorname{sech} x \tanh x, \quad (35)$$

where $A > B - \frac{1}{2} > 0$. It was shown using $\operatorname{sl}(2, \mathbb{C})$ as a tool [10] that for values of $A - B + \frac{1}{2}$ different from an integer, the corresponding Hamiltonian has a double series of energy eigenvalues $E_{qn}^{(A, B)}$, distinguished by a label $q = \pm 1$ (again called quasi-parity for convenience sake), and given by

$$\begin{aligned} E_{+n}^{(A, B)} &= -(A - n)^2, & n = 0, 1, \dots, n_{+max}, \\ & A - 1 \leq n_{+max} < A, \end{aligned} \quad (36)$$

$$\begin{aligned} E_{-n}^{(A, B)} &= -\left(B - \frac{1}{2} - n\right)^2, & n = 0, 1, \dots, n_{-max}, \\ & B - \frac{3}{2} \leq n_{-max} < B - \frac{1}{2}, \end{aligned} \quad (37)$$

if $\hbar = 2m = 1$, $A > 0$, and $B > \frac{1}{2}$. The accompanying eigenfunctions read

$$u_{qn}^{(A,B)}(x) = \mathcal{N}_{qn}^{(A,B)} (\operatorname{sech} x)^{-\frac{1}{2}(\lambda+\mu+1)} e^{-\frac{i}{2}(\lambda-\mu) \arctan(\sinh x)} P_n^{(\lambda,\mu)}(i \sinh x) \quad (38)$$

in terms of Jacobi polynomials. Here $\lambda = q(-A + B - \frac{1}{2})$, $\mu = -A - B - \frac{1}{2}$, and $\mathcal{N}_{qn}^{(A,B)}$ is some normalization coefficient.

Contrary to the potentials considered in the two previous sections, the PT-symmetric Scarf II potential is not obtained from the corresponding real potential [22] by a complex shift of coordinate, but instead by complexifying one parameter ($B \rightarrow iB$). Such a complexification is responsible for the appearance of the additional series (37) of energy eigenvalues.

Comparing (38) with (15), we find (up to a multiple of 2π) $\phi = -2\nu$, where ν is the phase of the normalization coefficient. If we identify PT-parity with quasi-parity again, we obtain

$$v_{qn}^{(A,B)}(x) = e^{i(q-1)\frac{\pi}{4}} \left| \mathcal{N}_{qn}^{(A,B)} \right| (\operatorname{sech} x)^{-\frac{1}{2}(\lambda+\mu+1)} e^{-\frac{i}{2}(\lambda-\mu) \arctan(\sinh x)} P_n^{(\lambda,\mu)}(i \sinh x), \quad (39)$$

where $\exp[i(q-1)\frac{\pi}{4}] = 1$ for PT-symmetric functions and $-i$ for PT-antisymmetric ones.

The normalization integral is now

$$\begin{aligned} & \int_{-\infty}^{+\infty} dx \left[u_{qn}^{(A,B)}(-x) \right]^* u_{qn}^{(A,B)}(x) \\ &= \left| \mathcal{N}_{qn}^{(A,B)} \right|^2 \int_{-\infty}^{+\infty} dx (\operatorname{sech} x)^{-\lambda-\mu-1} e^{-i(\lambda-\mu) \arctan(\sinh x)} \left[P_n^{(\lambda,\mu)}(i \sinh x) \right]^2. \end{aligned} \quad (40)$$

Let us consider, for instance, the $n = 0$ case corresponding to the lowest states of both series of energy levels. The integral on the right-hand side of (40) is then easily calculated by performing the change of variable $\tan y = \sinh x$. The results read [23]

$$\int_{-\infty}^{+\infty} dx \left[u_{+0}^{(A,B)}(-x) \right]^* u_{+0}^{(A,B)}(x) = \left| \mathcal{N}_{+0}^{(A,B)} \right|^2 \frac{\pi \Gamma(2A)}{2^{2A-1} \Gamma(A - B + \frac{1}{2}) \Gamma(A + B + \frac{1}{2})}, \quad (41)$$

$$\int_{-\infty}^{+\infty} dx \left[u_{-0}^{(A,B)}(-x) \right]^* u_{-0}^{(A,B)}(x) = \left| \mathcal{N}_{-0}^{(A,B)} \right|^2 \frac{\pi \Gamma(2B - 1)}{2^{2B-2} \Gamma(B - A - \frac{1}{2}) \Gamma(B + A + \frac{1}{2})}. \quad (42)$$

The right-hand side of (41) is positive for any allowed values of A and B because the arguments of all three gamma functions are positive. This is compatible with the

normalization condition (17) with $\sigma = q = +1$. On the right-hand side of (42), on the contrary, the argument of $\Gamma(B - A - \frac{1}{2})$ is negative while those of the two remaining gamma functions are positive, which means that the sign of the normalization integral for $\sigma = q = -1$ depends on the relative magnitudes of A and $B - \frac{1}{2}$. A negative sign compatible with (17) is obtained for $B - \frac{1}{2} + 2k < A < B + \frac{1}{2} + 2k$, $k = 0, 1, 2, \dots$. In particular, for $k = 0$, we get a condition $B - \frac{1}{2} < A < B + \frac{1}{2}$ rather similar to the conditions previously encountered for the PT-symmetric oscillator and generalized Pöschl-Teller potentials.

6 Conclusion

To conclude, we have derived, in the set up of a very general framework, the conservation law pertaining to PT-symmetric quantum mechanical systems. We have found, as a consequence, that the normalization integral ceases to be always positive definite. The possibility of encountering non-positive definite norms is illustrated by means of some specific examples of PT-symmetric potentials.

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Figure caption

Fig. 1. Contour Γ in the complex plane.

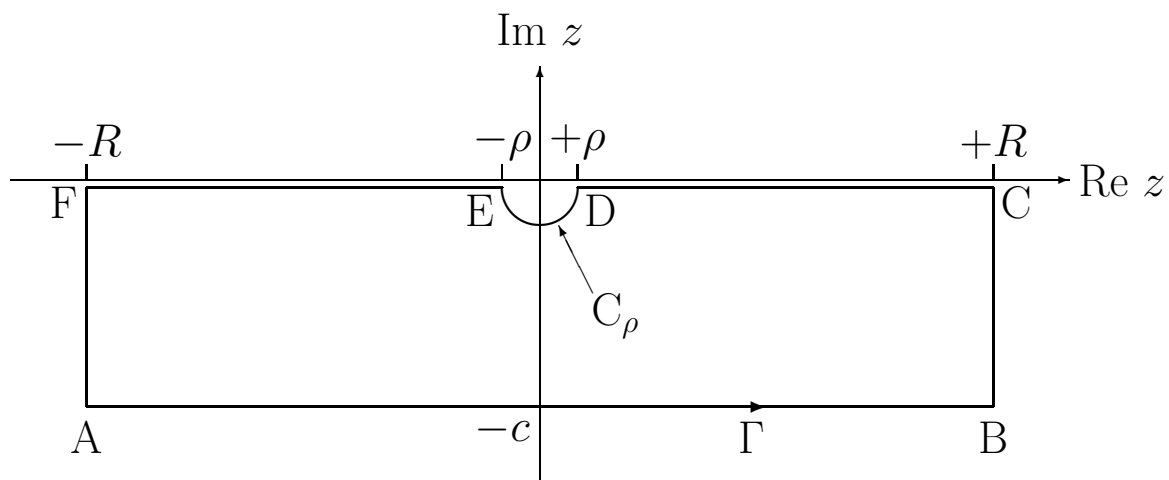


Figure 1