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FURTHER RESULTS ON DICTATORIAL DOMAINS

Anup Pramanik

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

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Anup Pramanik [*]

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Abstract

This paper generalizes the results in Aswal et al. (2003) on dictatorial domains. This is done in two ways. In the first, the notion of connections between pairs of alternatives in Aswal et al. (2003) is weakened to *weak connectedness*. This notion requires the specification of *four* preference orderings for every alternative pair. Domains that are *linked* in the sense of Aswal et al. (2003) with weak connectedness replacing connectedness, are shown to be dictatorial. In the second, the notion of connections for alternative pairs is *strengthened* relative to its counterpart in Aswal et al. (2003). However, a domain is shown to be dictatorial if the induced graph is merely connected rather than linked. This result generalizes the result in Sato (2010) on circular domains.

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1 Introduction

The incompatibility between strategy-proofness and non-dictatorship is a major issue in social choice. The seminal result of Gibbard (1977) and Satterthwaite (1975) states that a surjective and strategy-proof social choice function (scf) with a range of at least three alternatives, defined over the complete domain, is dictatorial. Aswal et al. (2003) show that the assumption of a complete domain is far from being necessary for this result. They show that a large class of domains (including several that are "small") are *dictatorial* - i.e. domains with the property that all strategy-proof and unanimous scfs (with a range of at least three) defined over such domains, are dictatorial. A complete characterization of dictatorial domains is a natural objective but appears to difficult to provide. Our goal in this paper is to generalize the sufficiency result of Aswal et al. (2003) and unify existing results in the area.

It will be helpful to briefly recount the result of Aswal et al. (2003). Fix an arbitrary domain. They say that two alternatives a and b are *connected* if there exists a preference in the domain where a is ranked first and b, second and another preference where the reverse is true. They consider the following graph: each alternative is a vertex and there is an edge between a pair of vertices if the two alternatives represented by the vertices, are connected. A domain is *linked* if this graph is "sufficiently dense". Specifically, there should exist an arrangement of the vertices such that the first three are mutually connected and each vertex is connected to at least two in the set of vertices that precedes it. Their main result is that every linked domain is dictatorial. They show the existence of a variety of linked domains including those that are linear in the number of alternatives. However, this result is far from a characterization - for instance, the circular domains defined in Sato (2010) and are not linked.

We generalize the linked domain result in two ways. The first way is to weaken the notion of connectedness between a pair of alternatives to *weak connectedness* while retaining the "connection structure" of the induced graph as in linkedness. The second way is to strengthen the notion of connectedness to *strong connectedness* but weakening the "connection structure" on the induced graph.

Two alternatives a and b are weakly connected if there exists a (possibly empty) set of alternatives B and *four* orderings in the domain such that there is a *reversal* between B and b when a is top-ranked and there is a reversal between B and a when b is top-ranked. Reversality requires alternatives between a and b to belong to B in the case where B is better than b. Similarly, alternatives between b and a to belong to B in the case where B is better than a. A domain is called a β domain if we can arrange all the alternatives (vertices in the induced graph) in a way that the first three are mutually weakly connected and each alternative is weakly connected to at least two in the set of alternatives (vertices) that precedes it. Our first result is that β domains are dictatorial. These domains are obviously supersets of linked domains - it is also possible to find β domains that are smaller than any linked domain.

Strong connectedness between a and b requires the following "intermediateness" property in addition to weak connectedness: for any alternative c other than a and b, there exists two orderings in the domain, one where c is above b while a at the top and another where c is above a while b at the top. A domain is called a γ domain if its induced graph is connected in the usual graph-theoretic sense, i.e. there exists a path between any two alternatives(vertices). Our second result is that all γ domains whose induced graph *is not* a star-graph, are dictatorial domains. The same result holds in the star-graph case with mild additional conditions. These results generalize results on circular domains in Sato (2010) and Chatterji and Sen (2011). Finally, we apply our result to a facility location problem in a restricted environment.

The paper is organized as followed. Section 2 contains a description of the model. Sections 3 and 4 contain the results on β and γ domains respectively. Section 5 provides an application while Section 6 concludes.

2 Basic notation and definitions

Let $A = \{a_1, \ldots, a_m\}$ denote a finite set of alternatives with $m \geq 3$. Let $I = \{1, 2, \ldots, n\}$, $n \geq 2$ be a finite set of agents. Let $\mathbb{P}$ denote the set of strict orderings [1] of the elements of A. An admissible domain is a set $\mathbb{D} \subset \mathbb{P}$. A typical preference orderings will be denoted by P_i where aP_ib will signify that a is preferred (strictly) to b under P_i. A preference profile is an element of the set $\mathbb{D}^n$. Preference profiles will be denoted by $P, \bar{P}, P'$ etc and their i^{th} components as $P_i, \bar{P}_i, P'_i$ respectively with $i = 1, 2, \ldots, n$. Let $(\bar{P}_i, P_{-i})$ denote the preference profile where the i^{th} component of the profile P is replaced by $\bar{P}_i$.

Given $P_i \in \mathbb{D}$, let $r_k(P_i)$ denote the k^{th} ranked alternative in P_i, $k = 1, \ldots, m$, i.e., $[r_k(P_i) = a_j] \Rightarrow [|\{a_k \in A : a_kP_ia_j\}| = k - 1]$. For an ordering $P_i \in \mathbb{D}$ and $a_j \in A$, we let $B(a_j, P_i)$ denote the set of alternatives that are strictly better than a_j according to P_i, while $W(a_j, P_i)$ denotes the set of alternatives that are strictly worse than a_j according to P_i. Let $M(a_j, a_k, P_i)$ be the set alternatives that are strictly worse than a_j and strictly better than a_k according to P_i.

Definition 1 *A social choice function (scf) f is a mapping $f : \mathbb{D}^n \longmapsto A$.*

Some familiar properties of scfs are stated below.

Definition 2 *A scf f satisfies unanimity, if for all $P \in \mathbb{D}^n$, $f(P) = a$ whenever $a = r_1(P_i)$ for all $i \in I$.*

If an alternative is top-ranked by all voters, the scf must pick that alternative.

A scf is strategy-proof if no voter can obtain a strictly better alternative by misrepresenting her preferences for any announcements of preferences of the other voters.

[1] A strict ordering is a complete, transitive and antisymmetric binary relation.

Definition **3** *A scf $f : \mathbb{P} \to A$ is manipulable by agent i at a profile $P \in \mathbb{P}$ via P_i' if*

$$f(P_i', P_{-i}) P_i f(P).$$

A scf f is strategy-proof if it is not manipulable by any agent at any profile.

A scf is a dictatorship if a particular voter always gets her best alternative.

Definition **4** *A scf f is dictatorial if there is an individual $i \in I$ such that $f(P) = r_1(P_i)$ for all $P \in \mathbb{D}^n$*

The following well-known result provides a full characterization of strategy-proof scfs for the domain $\mathbb{P}$.

Theorem **1** *(Gibbard (1977),Satterthwaite (1975)) A scf $f : \mathbb{P}^n \to A$ is strategy-proof and satisfies unanimity if and only if it is dictatorial.*

Unfortunately, there is a large class of preference domains where strategy-proofness implies dictatorship, so that there is no escape from this unpleasant dilemma. These domains which we define formally below, are the objects of our study.

Definition **5** *The domain $\mathbb{D} \subset \mathbb{P}$ is dictatorial if, for all scfs $f : \mathbb{D}^n \longmapsto A$ is strategyproof and satisfies unanimity implies f is dictatorial.*

Throughout the paper, we shall restrict attention to domains that are *minimally rich*.

Definition **6** *A domain $\mathbb{D}$ is minimally rich if, for all $a \in A$, there exists $P_i \in \mathbb{D}$ such that $r_1(P_i) = a$.*

The minimal richness assumption guarantees that every alternative is top-ranked for some ordering in the domain. This is a standard assumption in the literature, for instance Aswal et al. (2003).

3 β Domains

We first introduce the notion of weak connectedness. In what follows, we fix a domain $\mathbb{D} \subset \mathbb{P}$.

Definition **7** *A pair of alternatives a_j, a_k is weakly connected, denoted by $a_j \overset{w}{\sim} a_k$ if there exists $B \subset A$ (possibly empty) and $P_i, \bar{P}_i, P_i', P_i'' \in \mathbb{D}$ such that*

1. *$r_1(P_i) = r_1(\bar{P}_i) = a_j$ and $r_1(P_i') = r_1(P_i'') = a_k$.*

2. *$B = M(a_j, a_k, P_i)$ and $B \subset W(a_k, \bar{P}_i)$.*

3. *$B = M(a_k, a_j, P_i')$ and $B \subset W(a_j, P_i'')$.*

P_i	$\bar{P}_i$	P'_i	P''_i
a_j	a_j	a_k	a_k
B	.	B	.
a_k	a_k	a_j	a_j
.	B	.	.
.	.	.	B
.	.	.	.

Table 1: Weak connectedness

The weak connectedness concept is illustrated below.

The idea is quite simple. There exists a set B such that there is a *reversal* between B and a_k when a_j is top-ranked and there is a reversal between B and a_j when a_k is top-ranked. Reversality requires *all* alternatives between a_j and a_k to belong to B in the case where B is better than a_k. Similarly, *all* alternatives between a_k and a_j to belong to B in the case where B is better than a_j.

Observation **1** In case B is the empty set, weak connectedness reduces to connectedness in the sense of Aswal et al. (2003).

A β *domain* can be defined in the same way that a linked domain was defined in Aswal et al. (2003).

Definition **8** *Let $B \subset A$ and let $a_j \notin B$. Then a_j is linked to B if there exists $a_k, a_r \in B$ such that $a_j \overset{w}{\sim} a_k$ and $a_j \overset{w}{\sim} a_r$.*

Definition **9** *The domain $\mathbb{D}$ is called a β domain if there exists a one to one function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that*

(i) $a_{\sigma(1)} \overset{w}{\sim} a_{\sigma(2)}$

(ii) a_j *is linked to* $\{a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(j-1)}\}$, $j = 3, \ldots, m$.

By virtue of Observation 1, linked domains are β domain. However, the converse is not true as the example below shows.

Example **1** Let $A = \{a_1, a_2, a_3, a_4\}$ and let $\bar{\mathbb{D}}$ be the domain in Table 2. It is clear that a_1 is connected to a_2 and a_3, a_2 is connected to a_3, but a_4 is not connected to any other alternatives. Therefore $\bar{\mathbb{D}}$ is not linked. But it is a β domain because $a_4 \overset{w}{\sim} a_1$ and $a_4 \overset{w}{\sim} a_2$.

It is helpful to interpret a β domain in terms of the graphs induced by weak connectedness. Let $\mathbb{D}$ be a domain. The graph $G(\mathbb{D})$ is defined as follows: the vertices of the graph are the alternatives and two vertices have an edge iff the alternatives represented by the vertices are weakly connected. The graph induced by the domain in Example 1 is shown in Figure 1.

Our first Theorem shows that the linked domain result in Aswal et al. (2003) can be generalized to β domains.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4
a_2	a_3	a_1	a_3	a_1	a_2	a_1	a_2
a_4	a_4	a_4	a_4	a_2	a_1	a_2	a_1
a_3	a_2	a_3	a_1	a_4	a_4	a_3	a_3

Table 2: The domain $\bar{\mathbb{D}}$

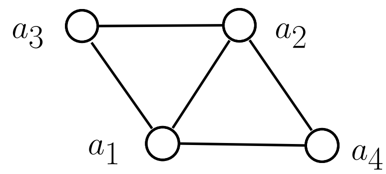

Figure 1: The graph $G(\bar{\mathbb{D}})$

Theorem **2** *A β domain is a dictatorial domain.*

Proof: Let $\mathbb{D}$ be a β domain and assume without loss of generality that the function σ in definition 9 is the identity function. For every non-empty $X \subset A$, we let $\mathbb{D}^X = \{P_i \in \mathbb{D} | r_1(P_i) \in X\}$. Similarly, for any alternative $a_i \in A$, we let $\mathbb{D}^{a_i} = \{P_i \in \mathbb{D} | r_1(P_i) = a_i\}$.

In view of Proposition 3.1 of Aswal et al. (2003) and our assumption of minimal richness, it suffices to show that if $f : \mathbb{D}^2 \to A$ is strategy-proof and unanimous, then f is dictatorial.

The following Lemma is very general.

Lemma **1** *Let $\mathbb{D}$ be an arbitrary domain and let a, b be arbitrary alternatives with $a \overset{w}{\sim} b$. If $f : \mathbb{D}^2 \to A$ satisfies strategy-proofness and unanimity, then $f(P) \in \{a, b\}$ for all $P \in \mathbb{D}^2$ such that $r_1(P_1), r_1(P_2) \in \{a, b\}$.*

Proof: Suppose not. Let a and b be the first ranked outcomes according to P_1 and P_2 respectively with $f(P) = c$ where $c \neq a, b$. Note that a and b must also be distinct from each other, otherwise we immediately contradict unanimity. Since $a \overset{w}{\sim} b$, there exists $B \subset A$ and $P_1', P_1'', P_2', P_2'' \in \mathbb{D}$ such that (i) $r_1(P_1') = r_1(P_1'') = a$ and $r_1(P_2') = r_1(P_2'') = b$, (ii) $B = M(a, b, P_1')$ and $B \subset W(b, P_1'')$, (iii) $B = M(b, a, P_2')$ and $B \subset W(a, P_2'')$. We consider two cases.

Case 1: $B = \emptyset$. By replicating the arguments in Claim A in Sen (2001), we can show that $f(P) \in \{a, b\}$. This leads to a contradiction.

Case 2: $B \neq \emptyset$. Observe that $f(P_1, P_2')$ cannot be b because 2 would manipulate at P via P_2'. Also note that $f(P_1, P_2') \notin W(a, P_2')$. Otherwise 2 would manipulate via an ordering where a is ranked first, thereby obtaining the outcome a (unanimity). We consider the following two cases.

Case 2.1: $f(P_1, P_2') = a$. Strategy-proofness implies that $f(P_1', P_2') = a$.

Observe that $f(P_1', P_2) \neq a$ because then 1 would manipulate at P via P_1'. Also $f(P_1', P_2) \notin W(b, P_1')$, otherwise 1 would manipulate via an ordering where b is ranked first, thereby obtaining the outcome b (unanimity). Therefore, $f(P_1', P_2) \in B \cup b$. If $f(P_1', P_2) \in B \cup b$, then 2 will manipulate at (P_1', P_2') via P_2 - a contradiction.

Case 2.2: $f(P_1, P_2') \in B$. Let $f(P_1, P_2') = d$. Then it must be the case that aP_1dP_1b. First we show that $f(P_1', P_2') \in B$. If $f(P_1', P_2') \in W(b, P_1') \bigcup \{b\}$, then 1 would manipulate at (P_1', P_2') via P_1. If $f(P_1', P_2') = a$, then 1 would manipulate at (P_1, P_2') via P_1'. Therefore $f(P_1', P_2') \in B$. Next we show that $f(P_1', P_2'') = a$. If $f(P_1', P_2'') = b$, then 2 would manipulate at (P_1', P_2') via P_2''. If $f(P_1', P_2'') \in B$, then 2 would manipulate at (P_1', P_2'') via an ordering where a is ranked first, thereby obtaining the outcome a (unanimity). If $f(P_1', P_2'') \in W(b, P_1')$, then 1 would manipulate at (P_1', P_2'') via an ordering where b is ranked first, thereby obtaining the outcome b (unanimity). Therefore $f(P_1', P_2'') = a$. At (P_1'', P_2''), $f(P_1'', P_2'') = a$, otherwise 1 would manipulate at (P_1'', P_2'') via P_1'. Finally we show that $f(P_1'', P_2') = a$. Note that $f(P_1'', P_2') \neq b$, otherwise 2 would manipulate at (P_1'', P_2'') via P_2'. Also $f(P_1'', P_2') \notin B$, otherwise 1 would manipulate at (P_1'', P_2') via an ordering where b is ranked first, thereby obtaining the outcome b (unanimity). If $f(P_1'', P_2') \in W(a, P_1')$, Then 2 would manipulate at (P_1'', P_2') via an ordering where a is ranked first, thereby obtaining the outcome a (unanimity). Therefore $f(P_1'', P_2') = a$. Note that 1 would manipulate at (P_1', P_2') via P_1'' because earlier we have shown $f(P_1', P_2') \in B$. This leads to a contradiction. ■

Our proof consists in establishing the following two steps.

Step 1: Let $X = \{a_1, a_2, a_3\}$. There exists $j \in \{1, 2\}$ such that $f(P) = r_1(P_j)$ for all $P \in \mathbb{D}^X \times \mathbb{D}^X$.

Step 2: Let $\bar{X} = \{a_1, a_2, \ldots, a_{l-1}\}$ and $X^* = \{a_1, a_2, \ldots, a_l\}$, $l = 4, \ldots, m$. If $f(P) = r_1(P_j)$ for all $P \in \mathbb{D}^{\bar{X}} \times \mathbb{D}^{\bar{X}}$, then $f(P) = r_1(P_j)$ for all $P \in \mathbb{D}^{X^*} \times \mathbb{D}^{X^*}$.

We proceed to establish Step 1 through a sequence of claims. First note that since $\mathbb{D}$ is a β domain and σ is the identity function, we have $a_1 \overset{w}{\sim} a_2$, $a_2 \overset{w}{\sim} a_3$ and $a_3 \overset{w}{\sim} a_1$. By Lemma 1, either $f(P_1, P_2) = a_1$ or $f(P_1, P_2) = a_2$ for all $P \in \mathbb{D}^2$ such that $r_1(P_1) = a_1$ and $r_1(P_2) = a_2$. Let $\bar{P}_1$ and $\bar{P}_2$ be such that $r_1(\bar{P}_1) = a_1$, $r_1(\bar{P}_2) = a_2$ and w.l.o.g. we assume that $f(\bar{P}_1, \bar{P}_2) = a_1$. We complete Step 1 by showing that agent 1 is the dictator. By Lemma 1 and strategy-proofness, $f(P_1, P_2) = a_1$ for all $P \in \mathbb{D}^2$ where $r_1(P_1) = a_1$ and $r_1(P_2) = a_2$. The following pair of claims are required to establish Step 1.

Claim **1** *For all $P \in \mathbb{D}^2$ where $r_1(P_1) = a_3$ and $r_1(P_2) = a_2$, $f(P_1, P_2) = a_3$.*

Proof: Suppose not. Then, there exists (P_1', P_2') with $r_1(P_1') = a_3$ and $r_1(P_2') = a_2$ such that $f(P_1', P_2') \neq a_3$. Lemma 1 implies that $f(P_1', P_2') = a_2$. By lemma 1 and strategy-proofness,

$f(P_1, P_2) = a_2$ for all $(P_1, P_2) \in \mathbb{D}^2$ with $r_1(P_1) = a_3$ and $r_1(P_2) = a_2$. Since $a_3 \overset{w}{\sim} a_1$, there exists $B \subset A$, P_1'', and P_1^* such that $r_1(P_1'') = r_1(P_1^*) = a_3$, $B = M(a_3, a_1, P_1'')$ and $B \subset W(a_1, P_1^*)$. Note that either $a_2 \in B$ or $a_2 \notin B$. If $a_2 \in B$, then $f(P_1^*, P_2') \neq a_2$. Otherwise, agent 1 would manipulate via an ordering where a_1 is ranked first - a contradiction. If $a_2 \notin B$, then $f(P_1'', P_2') \neq a_2$. Otherwise, agent 1 would manipulate via an ordering where a_1 is ranked first - a contradiction. ■

Claim **2** *For all $P \in \mathbb{D}^2$ where $r_1(P_1) = a_1$ and $r_1(P_2) = a_3$, $f(P_1, P_2) = a_1$.*

Proof: Suppose not. Then, there exists (P_1', P_2') with $r_1(P_1') = a_1$ and $r_1(P_2') = a_3$ such that $f(P_1', P_2') \neq a_1$. Lemma 1 implies that $f(P_1', P_2') = a_3$. By Lemma 1 and strategy-proofness, $f(P_1, P_2) = a_3$ for all $(P_1, P_2) \in \mathbb{D}^2$ with $r_1(P_1) = a_1$ and $r_1(P_2) = a_3$. Since $a_3 \overset{w}{\sim} a_2$, there exists $B \subset A$, P_2'', and P_2^* such that $r_1(P_2'') = r_1(P_2^*) = a_2$, $B = M(a_2, a_3, P_2'')$ and $B \subset W(a_3, P_2^*)$. Note that either $a_1 \in B$ or $a_1 \notin B$. If $a_1 \in B$, then $f(P_1', P_2^*) = a_2$. Otherwise if $f(P_1', P_2^*) = a_1$, agent 2 would manipulate via an ordering where a_3 is ranked first - a contradiction. If $a_1 \notin B$, then $f(P_1', P_2'') = a_2$. Otherwise, agent 2 would manipulate via an ordering where a_3 is ranked first - a contradiction. ■

Using the arguments used in the proof of Claim 1 and 2, it is straightforward to show that $f(P) = r_1(P_1)$ for all $P \in \mathbb{D}^X \times \mathbb{D}^X$. This establishes Step 1.

We now turn to Step 2. Pick an integer l in the set $\{4, \ldots, m\}$. We state our induction hypothesis below.

Induction Hypothesis (IH): $f(P) = r_1(P_1)$ for all $P \in \mathbb{D}^{\bar{X}} \times \mathbb{D}^{\bar{X}}$.

Our objective is to show that $f(P) = r_1(P_1)$ for all $P \in \mathbb{D}^{X^*} \times \mathbb{D}^{X^*}$.

Statement*: Since a_l is linked to $\{a_1, \ldots, a_{l-1}\}$, there exists $a_i, a_j \in \{a_1, \ldots, a_{l-1}\}$ such that $a_l \overset{w}{\sim} a_i$ and $a_l \overset{w}{\sim} a_j$.

Claim **3** *For all $(P_1, P_2) \in \mathbb{D}^2$ such that $P_1 \in \mathbb{D}^{a_l}$ and $P_2 \in \mathbb{D}^{\{a_i, a_j\}}$, $f(P) = r_1(P_1)$ (a_i and a_j are as specified in (*)).*

Proof: Suppose not. There exists an $(\bar{P}_1, \bar{P}_2) \in \mathbb{D}^2$ such that $\bar{P}_1 \in \mathbb{D}^{a_l}$, $\bar{P}_2 \in \mathbb{D}^{\{a_i, a_j\}}$ and $f(\bar{P}_1, \bar{P}_2) \neq a_l$. Therefore by Lemma 1, $f(\bar{P}_1, \bar{P}_2) = r_1(\bar{P}_2)$. Let $r_1(\bar{P}_2) = a_i$ - a similar argument holds if $r_1(\bar{P}_2) = a_j$. Since $a_l \overset{w}{\sim} a_j$, there exists an ordering P_1^* such that (i) $r_1(P_1^*) = a_l$ and (ii) $a_j P_1^* a_i$. By Lemma 1 and strategy-proofness $f(P_1^*, \bar{P}_2) = a_i$. Note that agent 1 would manipulate at $(P_1^*, \bar{P}_2)$ via an ordering P_1' where $r_1(P_1') = a_j$ because by induction hypothesis $f(P_1', \bar{P}_2) = a_j$ - a contradiction. ■

Claim **4** *For all $(P_1, P_2) \in \mathbb{D}^2$ such that $P_1 \in \mathbb{D}^{a_l}$ and $P_2 \in \mathbb{D}^{\bar{X}}$, $f(P) = r_1(P_1)$.*

Proof: In the view of Claim 3, we need to consider only the case where $P_2 \in \mathbb{D}^{a_r}$ where $a_r \in \{a_1, \ldots, a_{l-1}\}$ and $a_r \neq a_i, a_j$. Suppose there exists $(\bar{P}_1, \bar{P}_2)$ such that $r_1(\bar{P}_1) = a_l$, $r_1(\bar{P}_2) = a_r$ and $f(\bar{P}_1, \bar{P}_2) \neq a_l$. Since $a_l \overset{w}{\sim} a_i$, there exists $B \subset A$, P_1' and P_2'' such that (i) $r_1(P_1') = a_l$ and $r_1(P_2'') = a_i$, (ii) $B = M(a_l, a_i, P_1')$ and $B = M(a_i, a_l, P_2'')$. By strategy-proofness and IH, $f(P_1', \bar{P}_2) = B \cup a_i$. Claim 3 implies that $f(P_1', P_2'') = a_l$. Therefore, agent 2 would manipulate at (P_1', P_2'') via $\bar{P}_2$, contradicting the assumption of strategy-proofness.

■

Claim **5** *For all* $(P_1, P_2) \in \mathbb{D}^2$ *such that* $P_1 \in \mathbb{D}^{\{a_i, a_j\}}$ *and* $P_2 \in \mathbb{D}^{a_l}$*,* $f(P) = r_1(P_1)$ *(here too,* a_i *and* a_j *are as specified in (*)).*

Proof: Suppose not. There exists an $(\bar{P}_1, \bar{P}_2) \in \mathbb{D}^2$ such that $\bar{P}_1 \in \mathbb{D}^{\{a_i, a_j\}}$, $\bar{P}_2 \in \mathbb{D}^{a_l}$ and $f(\bar{P}_1, \bar{P}_2) \neq r_1(\bar{P}_1)$. Therefore by Lemma 1, $f(\bar{P}_1, \bar{P}_2) = a_l$. Let $r_1(\bar{P}_1) = a_i$ - a similar argument holds if $r_1(\bar{P}_1) = a_j$. Since $a_l \overset{w}{\sim} a_j$, there exists P_2' such that $r_1(P_2') = a_j$ and $a_l P_2' a_i$. Since, $f(\bar{P}_1, P_2') = a_i$ by IH, agent 2 would manipulate at $(\bar{P}_1, P_2')$ via $\bar{P}_2$ - a contradiction.

■

Claim **6** *Let* $a_r \overset{w}{\sim} a_s$ *and* $a_r, a_s \in \{a_1, a_2, \ldots, a_{l-1}\}$*. If* $f(P) = a_r$ *for all* $(P_1, P_2) \in \mathbb{D}^2$ *such that* $P_1 \in \mathbb{D}^{a_r}$ *and* $P_2 \in \mathbb{D}^{a_l}$*, then* $f(P) = a_s$ *for all* $(P_1, P_2) \in \mathbb{D}^2$ *such that* $P_1 \in \mathbb{D}^{a_s}$ *and* $P_2 \in \mathbb{D}^{a_l}$*.*

Proof: Suppose not. There exists an $(\bar{P}_1, \bar{P}_2) \in \mathbb{D}^2$ such that $\bar{P}_1 \in \mathbb{D}^{a_s}$, $\bar{P}_2 \in \mathbb{D}^{a_l}$ and $f(\bar{P}_1, \bar{P}_2) \neq a_s$. Since $a_r \overset{w}{\sim} a_s$, there exists $B \subset A$, P_1' and P_2' such that (i) $r_1(P_1') = a_s$ and $r_1(P_2') = a_r$, (ii) $B = M(a_s, a_r, P_1')$ and $B = M(a_r, a_s, P_2')$. By strategy-proofness and our assumption, $f(P_1', \bar{P}_2) \in B \cup a_r$. Since $f(P_1', P_2') = a_s$ by IH, 2 would manipulate at (P_1', P_2') via $\bar{P}_2$ - a contradiction. ■

Claim **7** *For all* $a_r \in \{a_1, a_2, \ldots, a_{l-1}\}$*,* $P_1 \in \mathbb{D}^{a_r}$ *and* $P_2 \in \mathbb{D}^{a_l}$*,* $f(P_1, P_2) = a_r$*.*

Proof: Pick $a_r \in \{a_1, a_2, \ldots, a_{l-1}\}$. Since $\mathbb{D}$ is a β domain, there must exist a sequence $b_0, b_1, \ldots, b_t \in \{a_1, a_2, \ldots, a_{l-1}\}$ such that $b_0 = a_j$, $b_t = a_r$ and $b_0 \overset{w}{\sim} b_1, b_1 \overset{w}{\sim} b_2, \ldots, b_{t-1} \overset{w}{\sim} b_t$. By Claim 5, $f(P) = a_j$ for all $P \in \mathbb{D}^2$ where $P_1 \in \mathbb{D}^{a_j}$ and $P_2 \in \mathbb{D}^{a_l}$. Applying Claim 6 repeatedly, it follows that $f(P) = a_r$ for all $P \in \mathbb{D}^2$ where $P_1 \in \mathbb{D}^{a_r}$ and $P_2 \in \mathbb{D}^{a_l}$. ■

Claims 3-7 establish Step 2. This completes the proof of the Theorem. ■

Observation **2** Aswal et al. (2003) proved that linked domains are dictatorial. Since linked domains are β domain, Theorem 2 clearly generalizes that of Aswal et al. (2003). We note that β domain can be much smaller than linked domains. For instance, the domain in Example 1 has eight orderings while the minimal linked domain with four alternatives has ten orderings. In fact, the size of a minimal dictatorial domain is $2m$, the bound that is obtained by β domains in the case where $m = 4$.

4 γ Domains

In this section, we consider a strengthening of the notion of weak connectedness. This generates new conditions for dictatorial domains where the induced graph on alternatives has fewer edges.

We introduce the notion of strong connectedness formally below.

Definition **10** *A pair of alternatives a_j, a_k is strongly connected, denoted by $a_j \approx a_k$ if $a_j \overset{w}{\sim} a_k$ and for all $a_r \neq a_j, a_k$ there exists $P_i, P_i' \in \mathbb{D}$ such that*

1. $r_1(P_i) = a_j$ *and* $a_r P_i a_k$.
2. $r_1(P_i') = a_k$ *and* $a_r P_i' a_j$.

In addition to weak connectedness, strong connectedness requires the following "intermediateness" property : for any alternative a_r other than a_j and a_k, there exist two orderings in the domain, one where a_r is above a_k while a_j at the top and another where a_r is above a_j while a_k at the top.

Fix a domain $\mathbb{D}$. The graph induced by strong connectedness $\bar{G}(\mathbb{D})$ is constructed in the same way as $G(\mathbb{D})$ with weak connectedness replaced by strong connectedness. In other words, the set of vertices in $\bar{G}(\mathbb{D})$ is A and there is an edge $\{a_j, a_k\}$ in $\bar{G}(\mathbb{D})$ if and only if $a_j \approx a_k$.

The objective of this paper is to show that $\bar{G}(\mathbb{D})$ requires "fewer" edges than $G(\mathbb{D})$ in order to be dictatorial. In particular, we will only require $\bar{G}(\mathbb{D})$ to be *connected.* [2]

Definition **11** *A domain $\mathbb{D}$ is a γ domain if $\bar{G}(\mathbb{D})$ is connected.*

A γ domain may not be a β domain as the example below shows.

Example **2** Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let $\hat{\mathbb{D}}$ be the domain in Table 3.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8	P_i^9	P_i^{10}
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4	a_5	a_5
a_2	a_4	a_1	a_4	a_1	a_5	a_5	a_2	a_4	a_1
a_3	a_3	a_3	a_5	a_2	a_4	a_3	a_1	a_3	a_2
a_4	a_5	a_4	a_3	a_4	a_2	a_2	a_3	a_1	a_3
a_5	a_2	a_5	a_1	a_5	a_1	a_1	a_5	a_2	a_4

Table 3: The domain $\hat{\mathbb{D}}$

The domain $\hat{\mathbb{D}}$ is a γ domain. The induced graph $\bar{G}(\hat{\mathbb{D}})$ (shown in Figure 2) is connected.

[2]This is the standard notion of a connected graph, i.e. a graph where there is a path between any two vertices. A complete definition can be found inWest (2001).

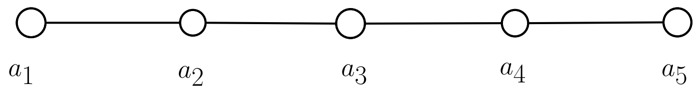

Figure 2: The graph $\bar{G}(\hat{\mathbb{D}})$

If a domain is a β domain, then for every $a_j \in A$, there exists a_k and a_r such that $a_j \overset{w}{\sim} a_k$ and $a_j \overset{w}{\sim} a_r$. However a_5 is not weakly connected to a_1 or a_2 or a_3. Therefore $\hat{\mathbb{D}}$ is not a β domain.

Example **3** Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let $\mathbb{D}^*$ be the domain in Table 4. The graph induced by $\mathbb{D}^*$ is a star graph[3] (shown in Figure 3). Since the star graph is connected, $\mathbb{D}^*$ is a γ domain.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8	P_i^9	P_i^{10}	P_i^{11}	P_i^{12}
a_1	a_1	a_2	a_2	a_3	a_3	a_3	a_4	a_4	a_4	a_5	a_5
a_4	a_3	a_4	a_5	a_1	a_2	a_2	a_1	a_3	a_3	a_4	a_3
a_2	a_5	a_1	a_3	a_2	a_4	a_5	a_5	a_2	a_5	a_2	a_1
a_5	a_2	a_3	a_1	a_5	a_1	a_1	a_2	a_1	a_1	a_1	a_4
a_3	a_4	a_5	a_4	a_4	a_5	a_4	a_3	a_5	a_2	a_3	a_2

Table 4: The domain $\mathbb{D}^*$

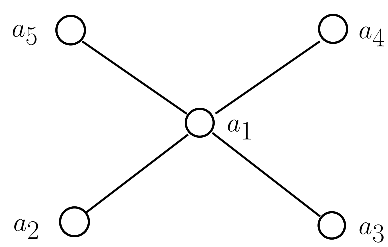

Figure 3: The star graph $\bar{G}(\mathbb{D}^*)$

The the following we show that the *circular domain* introduced in Sato (2010), is a γ domain.

Definition **12** *A domain is called a circular domain ($\mathbb{D}^c$) if the elements of A can be indexed $a_1, a_2, \ldots, a_m$ so that for each $k \in \{1, 2, \ldots, m\}$, there exist two preferences P_i and P_i' in $\mathbb{D}^c$ such that*

(i) $r_1(P_i) = a_k$, $r_2(P_i) = a_{k+1}$ *and* $r_m(P_i) = a_{k-1}$.

[3] A graph $G = (N, E)$ is a *star graph* if there exists a vertex $a \in N$ (the center of the star) such that (i) for all $b \in N \setminus \{a\}$, $\{a, b\}$ is an edge in G and (ii) for all $b, c \in N \setminus \{a\}$, $\{b, c\}$ is not an edge in G.

(ii) $r_1(P_i') = a_k$, $r_2(P_i') = a_{k-1}$ *and* $r_m(P_i') = a_{k+1}$.

(Let $a_{m+1} = a_1$ *and* $a_0 = a_m$*.)*

PROPOSITION **1** $\mathbb{D}^c$ *is a* γ *domain.*

Proof: First we show that for each $k \in \{1, 2, \ldots, m\}$, $\{a_k, a_{k+1}\}$ is an edge in $\bar{G}(\mathbb{D}^c)$. Since there exists $P', P'' \in \mathbb{D}^c$ where $r_1(P_i') = a_k$, $r_2(P_i') = a_{k+1}$, $r_1(P_i'') = a_{k+1}$ and $r_2(P_i') = a_k$, a_k and a_{k+1} are weakly connected. Pick an alternative $b \neq a_k, a_{k+1}$. Since there exists $P_i^1, P_i^2 \in \mathbb{D}^c$ where $r_1(P_i^1) = a_k$, $r_m(P_i^1) = a_{k+1}$, $r_1(P_i^2) = a_{k+1}$ and $r_m(P_i^2) = a_k$, b is ranked above a_{k+1} in P_i^1 and also above a_k in P_i^2. Therefore, $a_k \approx a_{k+1}$ and $\{a_k, a_{k+1}\}$ is an edge in $\bar{G}(\mathbb{D}^c)$. Since for each $k \in \{1, 2, \ldots, m\}$, $\{a_k, a_{k+1}\}$ is an edge in $\bar{G}(\mathbb{D}^c)$, it is connected. ■

OBSERVATION **3** $\bar{G}(\mathbb{D}^c)$ is not a star graph.

OBSERVATION **4** A circular domain may or may not be a β domain. Chatterji and Sen (2011) introduced a more restricted class of circular domains (which they also called circular domains). These domains are β domain.

OBSERVATION **5** A β domain may not be a γ domain. For instance the domain in Example 1, is not a γ domain. The induced graph (shown in Figure 4) is not connected.

Figure 4: The graph $\bar{G}(\bar{\mathbb{D}})$

Our main result in this section shows that any $\mathbb{D}$ for which $\bar{G}(\mathbb{D})$ is connected, is dictatorial. Unfortunately, some extra conditions are needed in the very special case when $\bar{G}(\mathbb{D})$ is a star graph. We are unable to show the dictatorial result for the star graph without additional conditions but we conjecture that the additional conditions are not required. In Parts B and C of the Theorem below, we provide two independent conditions for the star-graph case that ensure dictatoriality.

THEOREM **3** *Let* $\mathbb{D}$ *be a* γ *domain.*

A. If $\bar{G}(\mathbb{D})$ *is not a star graph, then* $\mathbb{D}$ *is dictatorial.*

B. Let $\bar{G}(\mathbb{D})$ *be a star graph and let* a *be the center of the star. If there exists* $b, c \in A \setminus \{a\}$ *such that* $b \overset{w}{\sim} c$*, then* $\mathbb{D}$ *is dictatorial.*

C. *Let $\bar{G}(\mathbb{D})$ be a star graph and let a be the center of the star. If there exists $P_i^1, P_i^2, P_i^3, P_i^4 \in \mathbb{D}$ such that (i) $r_1(P_i^1) = r_1(P_i^2) = b \neq a$ and $r_1(P_i^3) = r_1(P_i^4) = c \neq a$ and (ii) $M(b, a, P_i^1) = W(a, P_i^2) = M(c, a, P_i^3) = W(a, P_i^4)$, then $\mathbb{D}$ is dictatorial.*

Proof: Let $\mathbb{D}$ be a γ domain and let $f : \mathbb{D}^2 \to A$ be a strategy-proof and unanimous scf [4].

Let $\bar{G}(\mathbb{D})$ be the induced connected graph. We will say a pair of alternatives $a, b \in A$ are neighbors if $\{a, b\}$ is an edge in the graph $\bar{G}(\mathbb{D})$. Agent $i \in \{1, 2\}$ is said to be decisive over $a \in A$ if for any $P \in \mathbb{D}^2$ with $r_1(P_i) = a$, $f(P) = a$. Agent $i \in \{1, 2\}$ is dictator if i is decisive over all alternatives in A.

Lemma **2** *Let a and b be neighbors. For all $i, j \in \{1, 2\}$, if i is not decisive over a, then j is decisive over b.*

Proof: We assume that agent i is not decisive over a. If agent i is not decisive over a, then we argue that $f(\bar{P}_i, \bar{P}_j) \neq a$, where $r_1(\bar{P}_i) = a$ and $r_1(\bar{P}_j) = b$. If $f(\bar{P}_i, \bar{P}_j) = a$, then applying Lemma 1, $f(P_i, P_j) = a$ for all $(P_i, P_j) \in \mathbb{D}^2$ such that $r_1(P_i) = a$ and $r_1(P_j) = b$. In that case, we argue that agent i is decisive over a. Suppose not. Then there exists a profile $P' \in \mathbb{D}^2$, such that $r_1(P'_i) = a$ and $f(P') = c \neq a$. Since $a \approx b$, there exist P''_j with $r_1(P''_j) = b$ and $cP''_j a$. Therefore, agent j can manipulate at (P'_i, P''_j) via P'_j.

By Lemma 1 and our assumption, $f(\bar{P}_i, \bar{P}_j) = b$. If $f(\bar{P}_i, \bar{P}_j) = b$, then applying Lemma 1, $f(P_i, P_j) = b$ for all $(P_i, P_j) \in \mathbb{D}^2$ such that $r_1(P_i) = a$ and $r_1(P_j) = b$. Finally, we argue that agent j is decisive over b. Suppose not. Then there exists a profile $P' \in \mathbb{D}^2$, such that $r_1(P'_j) = b$ and $f(P') = c \neq b$. Since $a \approx b$, there exist P''_i with $r_1(P''_i) = a$ and $cP''_j b$. Therefore, agent i can manipulate at (P''_i, P'_j) via P'_i. ■

Lemma **3** *For any distinct a and b in A, it is impossible that agent 1 is decisive over a and agent 2 is decisive over b.*

Proof: Pick a profile $P \in \mathbb{D}^2$ such that $r_1(P_1) = a$ and $r_1(P_2) = b$. Since agent 1 is decisive over a, $f(P) = a$. But $f(P) = b$, because agent 2 is decisive over c. Therefore, the single-valuedness of f is contradicted. ■

Proof of Part A: Suppose $\bar{G}(\mathbb{D})$ is not a star graph. We show that f is dictatorial. First we show the following claim.

Claim **8** *For any $a \in A$, either agent 1 is decisive over a or agent 2 is decisive over a.*

[4] In view of Proposition 3.1 of Aswal et al. (2003) and the fact that a γ domain is minimally rich, it suffices to show that if $f : \mathbb{D}^2 \to A$ is strategy-proof and unanimous, then f is dictatorial.

Proof: Suppose not. There exists an alternative $a \in A$ such that either both the agents are decisive over a or none of them are decisive over a. We consider the following two cases.

Case 1: Suppose both the agents are decisive over a. Since $\bar{G}(\mathbb{D})$ is connected and not a star, there exists two edges $\{a, b\}$ and $\{b, c\}$ where $a \neq c$. By Lemma 3, both the agents are not decisive over b. By Lemma 2, both the agents are decisive over c, because b and c are neighbors. Since agent 1 is decisive over a and 2 is decisive over b, Lemma 3 is contradicted.

Case 2: Suppose none of the agents are decisive over a. Since $\bar{G}(\mathbb{D})$ is connected, there exists an edge $\{a, b\}$ in $\bar{G}(\mathbb{D})$. By Lemma 2, both the agents are decisive over b. Arguments in case 1 can now be replicated with alternative a replaced by b to show a contradiction. ■

Claim **9** *There exists an agent who is decisive over all alternatives in A.*

Proof: Let a be any element of A. By Claim 8, either agent 1 is decisive over a or agent 2 is decisive over a. W.l.o.g we assume that agent 1 is decisive over a. We complete the proof by showing that 1 is decisive over all alternatives in A. Let b be any element of $A \setminus \{a\}$. We show that agent 1 is decisive over b. Since $\bar{G}(\mathbb{D})$ is connected, there exists a path $(a = a_1, a_2, \ldots, a_{k-1}, a_k = b)$ in $\bar{G}(\mathbb{D})$ from a to b. First, we show that if agent 1 is decisive over a_i then agent 1 is decisive over a_{i+1} for all $i \in \{1, 2, \ldots, k-1\}$ and applying this fact again and again we conclude that agent 1 is decisive over b. Note that a_i and a_{i+1} are neighbors in $\bar{G}(\mathbb{D})$. By Lemma 3, if agent 1 is decisive over a_i, then agent 2 is not decisive over a_{i+1}. By Claim 8, agent 1 is decisive over a_{i+1}. Therefore we conclude that agent 1 is decisive over b. Because b was arbitrary, agent 1 is decisive over all alternatives in A. ■

Claim 8 and Claim 9 establish Part A of the Theorem 3. ■

Proof of Part B: Suppose $\bar{G}(\mathbb{D})$ is a star graph and let a be the center of the star. Let $b, c \in A \setminus \{a\}$ be such that $b \overset{w}{\sim} c$. We show that f is dictatorial. First we show the following claim.

Claim **10** *It is impossible that agent 1 and 2 are both decisive over a.*

Proof: Since $b \overset{w}{\sim} c$, there exists $B \subset A$ and $P_1, P_2 \in \mathbb{D}$ such that (i) $r_1(P_1) = b$ and $r_1(P_2) = c$, (ii) $B = M(b, c, P_1)$ and (iii) $B = M(c, b, P_2)$. By Lemma 1, $f(P_1, P_2)$ is either b or c. Let $f(P_1, P_2) = b$ - a similar arguments works if $f(P_1, P_2) = c$. If $a \in B$, 2 would manipulate at (P_1, P_2) via an ordering where a is ranked first because 2 is decisive over a - a contradiction.

Suppose $a \notin B$. Since a and c are neighbors, $a \overset{w}{\sim} c$. Therefore, there exists $B' \subset A$ and $P_2', P_2'' \in \mathbb{D}$ such that (i) $r_1(P_2') = r_1(P_2'') = c$, (ii) $B' = M(c, a, P_2')$ and (iii) $B' \subset W(a, P_2'')$. Lemma 1 and strategy-proofness imply that $f(P_1, P_2') = f(P_1, P_2'') = b$. If $b \in B'$, 2 would manipulate at (P_1, P_2'') via an ordering where a is ranked first because 2 is decisive over a. Similarly, if $b \notin B'$, 2 would manipulate at (P_1, P_2') via an ordering where a is ranked first. This completes the proof of the Claim. ■

Claim **11** *For any $d \in A$, either agent 1 is decisive over d or agent 2 is decisive over d.*

Proof: Suppose not. Therefore, there exists an alternative $d \in A$ such that either both the agents are decisive over d or none of them are decisive over d. We consider following two cases.

Case 1: Suppose both the agents are decisive over d. Claim 10 implies that d is not the center of $\bar{G}(\mathbb{D})$. Since d and a are neighbors, Lemma 3 implies that agent 1 and 2 are not decisive over a. By Lemma 2, both agents are decisive over c ($\neq a, d$), because a and c are neighbors. Since agent 1 is decisive over d and 2 is decisive over c, Lemma 3 is contradicted.

Case 2: In this case we consider that agent 1 and 2 are not decisive over d. First we argue that d is the center of $\bar{G}(\mathbb{D})$. If d is not the center, Lemma 2 implies that agent 1 and 2 are decisive over a - Claim 10 is contradicted. Therefore, d is the center of $\bar{G}(\mathbb{D})$. By Lemma 2, both agents are decisive over two distinct non-central alternatives b and c. Since agent 1 is decisive over b and 2 is decisive over c, Lemma 3 is contradicted. ■

Claim **12** *There exists an agent who is decisive over all alternatives in A.*

Proof: Replacing Claim 8 by Claim 11 in the proof of Claim 9 we can establish this Claim. ■

Claim 10-12 establish Part A of the Theorem 3. ■

Proof of Part C: Suppose $\bar{G}(\mathbb{D})$ is a star graph and let a be the center of the star. Let $P_1^1, P_1^2, P_2^3, P_2^4 \in \mathbb{D}$ be such that (i) $r_1(P_1^1) = r_1(P_1^2) = b \neq a$ and $r_1(P_2^3) = r_1(P_2^4) = c \neq a$ and (ii) $M(b, a, P_1^1) = W(a, P_1^2) = M(c, a, P_2^3) = W(a, P_2^4)$. We show that f is dictatorial. First we show the following claim.

Claim **13** *It is impossible that agent 1 and 2 are both decisive over a.*

Proof: Suppose not, i.e. agent 1 and 2 are decisive over a. Let $M(b, a, P_1^1) = W(a, P_1^2) = M(c, a, P_2^3) = W(a, P_2^4) = B$. By our assumption, $b, c \notin B$. Now consider the preference profile (P_1^1, P_2^4). We show that $f(P_1^1, P_2^4) = b$. Note that $f(P_1^1, P_2^4) \notin B$, otherwise 2 will manipulate via an ordering where a is first-ranked. Since bP_2^4a, $f(P_1^1, P_2^4) \neq a$, otherwise 2 will manipulate via an ordering where b is first-ranked. Since 1 is decisive over a, $f(P_1^1, P_2^4) \notin W(a, P_1^1)$. Therefore $f(P_1^1, P_2^4) = b$.

Strategy-proofness implies that $f(P_1^2, P_2^4) = b$. We complete the proof of the claim by showing that $f(P_1^2, P_2^3) \notin A$, because it contradicts with the fact that f is a function. Note that $f(P_1^2, P_2^3) \neq c$, otherwise 2 will manipulate at (P_1^2, P_2^4) via P_2^3. Also $f(P_1^2, P_2^3) \notin B$, otherwise 1 will manipulate via an ordering where a is first-ranked. Since $cP_1^2 a$, $f(P_1^2, P_2^3) = a$, otherwise 1 will manipulate via an ordering where c is first-ranked. Since 2 is decisive over a, $f(P_1^2, P_2^3) \notin W(a, P_2^3)$. This completes the proof of the Claim. ■

Claim **14** *For any $d \in A$, either agent 1 is decisive over d or agent 2 is decisive over d.*

Proof: Replacing Claim 10 by Claim 12 in the proof of Claim 11, we can establish this Claim. ■

Claim **15** *There exists an agent who is decisive over all alternatives in A.*

Proof: Replacing Claim 8 by Claim 14 in the proof of Claim 9, we can establish this Claim. ■

Claim 13-15 establish Part C of Theorem 3. ■

This completes the proof of the Theorem. ■

Observation **6** Since the graph induced by a *circular domain* is connected and not a star graph (Observation 3), Part A of Theorem 3 generalizes the results in Sato (2010). Moreover, the minimal size of a γ domain is $2m$ (Example 2).

Observation **7** For the star-graph case, there are domains for which the conditions specified in Part B and Part C do not hold - the domain in Example 3 is such an example.

Observation **8** The relationship between linked domains (Aswal et al. (2003)), circular domains (Sato (2010)), circular domains (Chatterji and Sen (2011)), β domains and γ domains is summarized as follows.

1. linked domains (Aswal et al. (2003)) $\subset$ β domains.
2. circular domains (Chatterji and Sen (2011)) $\subset$ circular domains (Sato (2010)) $\subset$ γ domains.
3. circular domains (Chatterji and Sen (2011)) $\subset$ β domains.

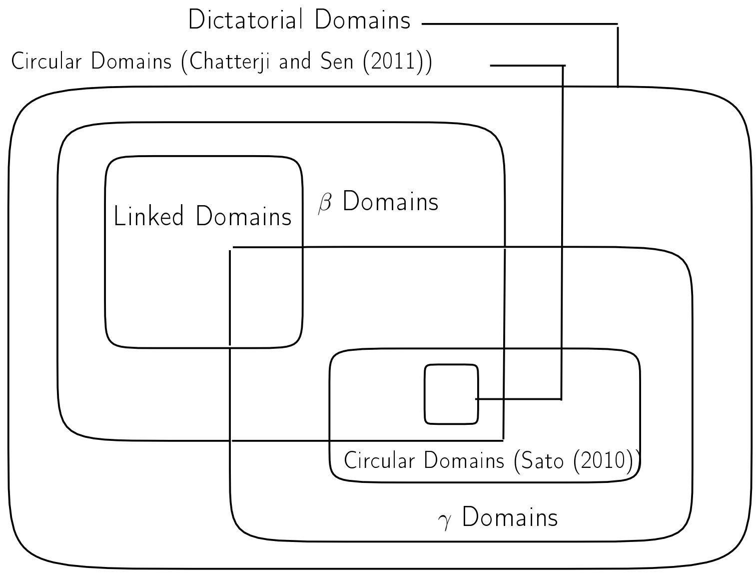

Figure 5: Diagrammatic representation of Observation 8

5 An Application

Consider a city with a hub. A finite number of citizens are located in the city but not at the hub. However, their locations are directly connected to the hub by a road. A public facility such as a hospital or a school has to be located in the city. The location decision is based on the preferences of citizens which are private information. What are the scfs that induce agents to report their preferences truthfully?

Let H be the hub of the city. Agents are located at a finite set of locations denoted by $a_1, \ldots, a_m$, $m \geq 2$. Agents' preferences are restricted in the following manner: an agent located at a_i has one of the four orderings shown in Table 5.

P_i	$\bar{P}_i$	P'_i	P''_i
a_i	a_i	H	H
H	.	a_i	.
.	.	.	.
.	.	.	.
.	H	.	a_i

Table 5: Possible Preferences of an agent located at a_i

The rationale behind the preference restrictions is as follows. Some citizens want it either at their location or at the hub - these citizens prefer proximity to the facility. Thus a_i and H take first two places in their ordering and are represented either by P_i or P'_i. Some citizens want it at any residential location rather than at the hub and most prefer it when it is located near them - these preferences are represented by $\bar{P}_i$. Finally some citizens are affected by the congestion created by the facility and are strongly averse to it being located near them. They most prefer it being located at the hub for easy access. Such preferences are represented by P''_i.

A domain with the four preference orderings in Table 5 for each a_i, will be called a *hub domain* and denoted by $\mathbb{D}^H$.

PROPOSITION **2** *A hub domain is a γ domain.*

Proof: Let $\mathbb{D}^H$ be a hub domain. First we show that for each $k \in \{1, 2, \ldots, m\}$, $\{a_k, H\}$ is an edge in $\bar{G}(\mathbb{D}^H)$. Since there exists $P', P'' \in \mathbb{D}^H$ where $r_1(P'_i) = a_k$, $r_2(P'_i) = H$, $r_1(P''_i) = H$ and $r_2(P'_i) = a_k$, a_k and a_{k+1} are weakly connected. Pick an alternative $b \neq a_k, H$. Since there exists $P^1_i, P^2_i \in \mathbb{D}^H$ where $r_1(P^1_i) = a_k$, $r_{m+1}(P^1_i) = H$, $r_1(P^2_i) = H$ and $r_{m+1}(P^2_i) = a_k$, b is ranked above H in P^1_i and also above a_k in P^2_i. Therefore, $a_k \approx a_{k+1}$ and $\{a_k, a_{k+1}\}$ is an edge in $\bar{G}(\mathbb{D}^c)$. Since for each $k \in \{1, 2, \ldots, m\}$, $\{a_k, H\}$ is an edge in $\bar{G}(\mathbb{D}^H)$, it is connected. ■

The induced graph by a hub domain may or may not be a star-graph. In either case, it is a dictatorial domain.

THEOREM **4** *A hub domain is dictatorial.*

Proof: Let $\mathbb{D}^H$ be a hub domain. By Proposition 2, $\bar{G}(\mathbb{D}^H)$ is connected. If $\bar{G}(\mathbb{D}^H)$ is not a star-graph, we are done by Part A of Theorem 3.

Suppose that $\bar{G}(\mathbb{D}^H)$ is a star-graph. Note that there exists $P^1_i, P^2_i, P^3_i, P^4_i \in \mathbb{D}^H$ such that *(i)* $r_1(P^1_i) = r_1(P^2_i) = a_i \neq H$ and $r_1(P^3_i) = r_1(P^4_i) = a_j \neq H$ and *(ii)* $M(a_i, H, P^1_i) = W(H, P^2_i) = M(a_j, H, P^3_i) = W(H, P^4_i) = \emptyset$. Therefore, by Part C of Theorem 3, $\mathbb{D}^H$ is dictatorial. ■

6 Conclusion

We have generalized the results of Aswal et al. (2003) in two different ways. Our results generate new examples of dictatorial domains and also unify existing results by covering some hitherto isolated cases.

References

ASWAL, N., S. CHATTERJI, AND A. SEN (2003): "Dictatorial Domains," *Economic Theory*, 22, 45–62.

CHATTERJI, S. AND A. SEN (2011): "Tops-Only Domains," *Economic Theory*, 46, 255–282.

Gibbard, A. (1977): "Manipulation of Voting Schemes: a General Result," *Econometrica*, 45, 439–446.

Sato, S. (2010): "Circular domains," *Review of Economic Design*, 14, 331–342.

Satterthwaite, M. (1975): "Strategy-proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory*, 10, 187–217.

Sen, A. (2001): "Another Direct proof of the Gibbard-Satterthwaite Theorem," *Economics Letters*, 70, 381–385.

West, D. (2001): "Introduction to Graph Theory," *Prentice Hall*.