

# Extended supersymmetry in Dirac action with extra dimensions

Yukihiro Fujimoto,<sup>a\*</sup> Kouhei Hasegawa,<sup>b†</sup> Kenji Nishiwaki,<sup>c‡</sup>  
Makoto Sakamoto,<sup>b§</sup> Kentaro Tatsumi,<sup>b¶</sup> Inori Ueba,<sup>b||</sup>

<sup>a</sup>*National Institute of Technology, Oita College, Oita 870-0152, Japan*

<sup>b</sup>*Department of Physics, Kobe University, Kobe 657-8501, Japan*

<sup>c</sup>*School of Physics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea*

## Abstract

We investigate a new realization of extended quantum-mechanical supersymmetry. We first show that an  $\mathcal{N} = 2$  quantum-mechanical supersymmetry is hidden in the four-dimensional (4D) spectrum of the Kaluza–Klein decomposition for the higher dimensional Dirac field, that is, Kaluza–Klein mode functions of 4D right-handed spinors and 4D left-handed ones form  $\mathcal{N} = 2$  supermultiplets. In addition to  $\mathcal{N} = 2$  supersymmetry, we discover that an  $\mathcal{N}$ -extended supersymmetry ( $\mathcal{N} = d + 2$  ( $d + 1$ ) for  $d = \text{even}$  ( $\text{odd}$ ) extra dimensions) is further hidden in the 4D spectrum. The extended symmetry can explain additional degeneracy of the spectrum. Furthermore, we show that a superpotential can be introduced into the  $\mathcal{N}$ -extended supercharges and clarify the condition to preserve the supersymmetry. The partial breaking of the supersymmetry is also demonstrated.

Keywords: quantum-mechanical supersymmetry, extended supersymmetry, extra dimensions

PACS: 03.65.-w, 11.30.Pb, 12.60.-i, 12.90.+b, 14.80.Rt

KIAS-P18035, KOBE-TH-18-01

---

\*E-mail address: y-fujimoto@oita-ct.ac.jp

†E-mail address: kouhei@phys.sci.kobe-u.ac.jp

‡E-mail address: nishiken@kias.re.kr

§E-mail address: dragon@kobe-u.ac.jp

¶E-mail address: kentaro@stu.kobe-u.ac.jp

||E-mail address: i-ueba@stu.kobe-u.ac.jp

# 1 Introduction

Quantum-mechanical supersymmetry (QM SUSY) has a wide range of applicable topics, e.g. exactly solvable systems in quantum mechanics [1–5], black holes and AdS/CFT [6–10], Sachdev–Ye–Kitaev model [11–15], and so on. Thus it is interesting and important to investigate new realizations of QM SUSY. In particular, it is worth while constructing  $\mathcal{N}$ -extended supersymmetry, since not so many models with arbitrary large  $\mathcal{N}$ -extended one are known.

The  $\mathcal{N}$ -extended supersymmetry has  $\mathcal{N}$  independent supercharges corresponding to the square root of a Hamiltonian, and supercharges relate degenerate eigenstates of the Hamiltonian. In order to find a new realization of  $\mathcal{N}$ -extended QM SUSY, we will investigate fermions in the higher dimensional space-time, because the 4D spectra in gauge/gravity theories with extra dimensions have been found to be governed by  $\mathcal{N} = 2$  QM SUSYs [16–19]. We then expect that  $\mathcal{N} = 2$  QM SUSY will be hidden also in the 4D spectrum of higher dimensional fermions. However, the degeneracy of such 4D spectrum is found to be, in general, much larger than that expected by the  $\mathcal{N} = 2$  supersymmetry. This may imply that there should exist some symmetries in addition to the  $\mathcal{N} = 2$  supersymmetry. The main purpose of this letter is to show that an  $\mathcal{N}$ -extended QM SUSY is hidden in the 4D spectrum of the higher dimensional Dirac action and can explain the degeneracy of the 4D spectrum.

In this paper, we show that there generally exists an  $\mathcal{N} = 2$  QM SUSY hidden in the 4D spectrum of a higher dimensional Dirac action after the compactification of extra space dimensions. The symmetry turns out to connect the 4D right-handed spinors and the 4D left-handed ones. Furthermore, we find an  $\mathcal{N}$ -extended QM SUSY with  $\mathcal{N} = d + 2$  ( $d + 1$ ) for  $d = \text{even}$  ( $\text{odd}$ ) extra dimensions. We then show that the  $\mathcal{N}$ -extended supersymmetry explains the degeneracy of the 4D spectrum in the higher dimensional Dirac action. Explicit constructions of  $\mathcal{N}$ -extended supersymmetry algebras with higher  $\mathcal{N}$ , have been investigated [20–27], but what we found in this paper gives a new realization of the  $\mathcal{N}$ -extended supersymmetry algebra.

This paper is organized as follows: In Section 2, we clarify the properties of the mode functions of the Dirac field providing the 4D mass eigenstates. In Section 3, we reveal a hidden  $\mathcal{N} = 2$  QM SUSY on such mode functions. We show a hidden  $\mathcal{N}$ -extended QM SUSY in Section 4. It is shown that the whole degenerate mode functions on the hyperrectangular internal space (extra dimensions) can be explained by the  $\mathcal{N}$ -extended supersymmetry. In section 5, we introduce a superpotential into the extended QM SUSY. Then, we clarify the conditions that the superpotential can preserve or (partially) break the extended supersymmetry. Section 6 is devoted to summary and discussion.

## 2 Higher dimensional Dirac action and KK decomposition

In this section, we discuss the Kaluza-Klein (KK) decomposition of a higher dimensional Dirac field and clarify the properties of the mode functions of the KK decomposition. The decomposition is executed in such a way that the induced Dirac action at four dimensions (4D) is constructed in terms of the 4D mass eigenstates.

Let us consider the action of a free Dirac field on the direct product of the 4D Minkowski space-time  $M^4$  and a  $d$ -dimensional flat internal space  $\Omega$  given as<sup>1</sup>

$$S = \int_{M^4} d^4x \int_{\Omega} d^d y \bar{\Psi}(x, y) (i\Gamma^\mu \partial_\mu + i\Gamma^{y_k} \partial_{y_k} - M) \Psi(x, y). \quad (1)$$

---

<sup>1</sup>For earlier works on higher dimensional spinors, see e.g. [28–30].

The  $x^\mu$  ( $\mu = 0, 1, 2, 3$ ) are the coordinates of the Minkowski space-time  $M^4$  and  $y_k$  ( $k = 1, 2, \dots, d$ ) are the coordinates of the internal space  $\Omega$ . We take the metric as  $\eta_{NM} = \eta^{NM} = \text{diag}(-1, +1, \dots, +1)$  where  $N, M = 0, 1, \dots, y_d$ . The gamma matrices  $\Gamma^N$  satisfy the Clifford algebra,

$$\{\Gamma^N, \Gamma^M\} = -2\eta^{NM} \mathbf{1}_{2^{\lfloor d/2 \rfloor}}. \quad (2)$$

The  $\Psi(x, y)$  is the  $(d + 4)$ -dimensional Dirac spinor which has  $2^{\lfloor d/2 + 2 \rfloor}$  components with mass  $M$ . The symbol  $\lfloor d/2 + 2 \rfloor$  is the maximum integer less than or equal to  $d/2 + 2$ .  $\mathbf{1}_n$  represents the  $n \times n$  unit matrix. The Dirac conjugate is defined as  $\bar{\Psi}(x, y) = \Psi^\dagger(x, y)\Gamma^0$ . It turns out to be very convenient for our analysis to use the following representation of the gamma matrices  $\Gamma^N$ , based on the 4D chiral representation:

$$\begin{aligned} \Gamma^0 &= \sigma^1 \otimes \mathbf{1}_{2^{\lfloor d/2 \rfloor}} \otimes \mathbf{1}_2, \\ \Gamma^a &= i\sigma^2 \otimes \mathbf{1}_{2^{\lfloor d/2 \rfloor}} \otimes \sigma^a \quad (a = 1, 2, 3), \\ \Gamma^{y_k} &= -\sigma^3 \otimes \gamma^{y_k} \otimes \mathbf{1}_2 \quad (k = 1, 2, \dots, d), \end{aligned} \quad (3)$$

where  $\sigma^a$  denote the Pauli matrices and  $\gamma^{y_k}$  ( $k = 1, 2, \dots, d$ ) correspond to the internal space gamma matrices which satisfy  $\{\gamma^{y_k}, \gamma^{y_h}\} = -2\delta^{kh} \mathbf{1}_{2^{\lfloor d/2 \rfloor}}$  with  $(\gamma^{y_k})^\dagger = -\gamma^{y_k}$  (see Eq. (2)).<sup>2</sup> We define the 4D chiral matrix  $\Gamma_5$  and the internal space chiral matrix  $\Gamma_{d+1}$  (for even  $d$ ) by<sup>3</sup>

$$\begin{aligned} \Gamma_5 &= i\Gamma^0 \dots \Gamma^3 = -\sigma^3 \otimes \mathbf{1}_{2^{\lfloor d/2 \rfloor}} \otimes \mathbf{1}_2, \\ \Gamma_{d+1} &= i^{d/2} \Gamma^{y_1} \dots \Gamma^{y_d} = \mathbf{1}_2 \otimes \gamma_{d+1} \otimes \mathbf{1}_2 \quad (\text{for even } d), \end{aligned} \quad (4)$$

with  $\gamma_{d+1} = i^{d/2} \gamma^{y_1} \dots \gamma^{y_d}$ . The 4D chiral projection matrices  $P_{R/L}$  are given by

$$\begin{aligned} P_L &= \frac{\mathbf{1} - \Gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1}_{2^{\lfloor d/2 \rfloor}} \otimes \mathbf{1}_2, \\ P_R &= \frac{\mathbf{1} + \Gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{1}_{2^{\lfloor d/2 \rfloor}} \otimes \mathbf{1}_2. \end{aligned} \quad (5)$$

The eigenstates of the 4D chiral matrix  $\Gamma_5$  with the eigenvalue  $+1$  ( $-1$ ) are called right-handed (left-handed).

In terms of the 4D left-handed (right-handed) two component chiral spinors  $\phi_\alpha^{(n)}(x)$  ( $\bar{\chi}_\alpha^{(n)}(x)$ ),<sup>4</sup> the KK decomposition of the  $(d + 4)$ -dimensional Dirac field  $\Psi(x, y)$  will be given by

$$\begin{aligned} \Psi(x, y) &= \sum_n \sum_\alpha \left\{ \mathbf{e}_L \otimes \mathbf{f}_\alpha^{(n)}(y) \otimes \phi_\alpha^{(n)}(x) + \mathbf{e}_R \otimes \mathbf{g}_\alpha^{(n)}(y) \otimes \bar{\chi}_\alpha^{(n)}(x) \right\} \\ &= \sum_n \sum_\alpha \begin{pmatrix} \mathbf{f}_\alpha^{(n)}(y) \otimes \phi_\alpha^{(n)}(x) \\ \mathbf{g}_\alpha^{(n)}(y) \otimes \bar{\chi}_\alpha^{(n)}(x) \end{pmatrix}, \end{aligned} \quad (6)$$

where  $\mathbf{e}_L = (1, 0)^\top$  and  $\mathbf{e}_R = (0, 1)^\top$  characterize the 4D chirality. The index  $n$  represents the  $n$ -th level of the KK modes and  $\alpha$  denotes the index that distinguishes the degeneracy of the  $n$ -th KK modes (if exist). The mode functions  $\mathbf{f}_\alpha^{(n)}(y)$  ( $\mathbf{g}_\alpha^{(n)}(y)$ ) have  $2^{\lfloor d/2 \rfloor}$  components and are assumed to form a complete set with respect to the internal space associated with the 4D left-handed (right-handed) chiral spinors  $\phi_\alpha^{(n)}(x)$  ( $\bar{\chi}_\alpha^{(n)}(x)$ ).

<sup>2</sup>For the case of  $d = 1$ ,  $\gamma^{y_1}$  is defined as  $i$ .

<sup>3</sup>In odd dimensions, there is no internal chiral matrix corresponding to  $\Gamma_{d+1}$  or  $\gamma_{d+1}$ .

<sup>4</sup>For two component spinors, we follow the notation adopted in [31], which includes the followings:  $\sigma^\mu = \{\mathbf{1}_2, \sigma^a\}$ ,  $\bar{\sigma}^\mu = \{\mathbf{1}_2, -\sigma^a\}$ ,  $(\phi_\alpha^{(n)}(x))^\dagger = \bar{\phi}_\alpha^{(n)}(x)$ , and  $(\bar{\chi}_\alpha^{(n)}(x))^\dagger = \chi_\alpha^{(n)}(x)$ .

Substituting this expansion (6) into the action (1), we find

$$S = \int_{M_4} d^4x \sum_{n,m} \sum_{\alpha,\beta} \left[ \langle \mathbf{f}_\alpha^{(n)} | \mathbf{f}_\beta^{(m)} \rangle \bar{\phi}_\alpha^{(n)}(x) i\bar{\sigma}^\mu \partial_\mu \phi_\beta^{(m)}(x) + \langle \mathbf{g}_\alpha^{(n)} | \mathbf{g}_\beta^{(m)} \rangle \chi_\alpha^{(n)}(x) i\sigma^\mu \partial_\mu \bar{\chi}_\beta^{(m)}(x) \right. \\ \left. + \langle \mathbf{f}_\alpha^{(n)} | A \mathbf{g}_\beta^{(m)} \rangle \bar{\phi}_\alpha^{(n)}(x) \bar{\chi}_\beta^{(m)}(x) + \langle \mathbf{g}_\alpha^{(n)} | A^\dagger \mathbf{f}_\beta^{(m)} \rangle \chi_\alpha^{(n)}(x) \phi_\beta^{(m)}(x) \right], \quad (7)$$

where  $\langle \mathbf{X} | \mathbf{Y} \rangle = \int_\Omega d^d y \mathbf{X}^\dagger(y) \mathbf{Y}(y)$  and  $A = +i\gamma^{jk} \partial_{y_k} - M$ ,  $A^\dagger = -i\gamma^{jk} \partial_{y_k} - M$ . We note that  $A$  and  $A^\dagger$  are the off-diagonal components of the Dirac operator  $\Gamma^0(i\Gamma^{jk} \partial_{y_k} - M)$  written as

$$\Gamma^0(i\Gamma^{jk} \partial_{y_k} - M) = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \otimes \mathbf{1}_2. \quad (8)$$

We require that  $\phi_\alpha^{(n)}(x)$  and  $\bar{\chi}_\alpha^{(n)}(x)$  are mass eigenstates of the 4D spectrum and that the action (7) should be written into the form

$$S = \int_{M^4} dx \left\{ \sum_\alpha \sum_n \bar{\psi}_\alpha^{(n)}(x) (i\gamma^\mu \partial_\mu + m_n) \psi_\alpha^{(n)}(x) \right. \\ \left. + \sum_\alpha \bar{\phi}_\alpha^{(0)}(x) i\bar{\sigma}^\mu \partial_\mu \phi_\alpha^{(0)}(x) + \chi_\alpha^{(0)}(x) i\sigma^\mu \partial_\mu \bar{\chi}_\alpha^{(0)}(x) \right\}, \quad (9)$$

where  $\psi_\alpha^{(n)}(x) = (\phi_\alpha^{(n)}(x), \bar{\chi}_\alpha^{(n)}(x))^T$  are the  $n$ -th KK 4D Dirac fields with mass  $m_n$  and  $\phi_\alpha^{(0)}(x)$  ( $\bar{\chi}_\alpha^{(0)}(x)$ ) in the second line are massless left-handed (right-handed) 4D chiral fields. This can be realized, provided the mode functions  $\mathbf{f}_\alpha^{(n)}(y)$  and  $\mathbf{g}_\alpha^{(n)}(y)$  satisfy the following orthonormality relations:

$$\langle \mathbf{f}_\alpha^{(n)} | \mathbf{f}_\beta^{(m)} \rangle = \langle \mathbf{g}_\alpha^{(n)} | \mathbf{g}_\beta^{(m)} \rangle = \delta_{\alpha\beta} \delta^{nm}, \\ \langle \mathbf{f}_\alpha^{(n)} | A \mathbf{g}_\beta^{(m)} \rangle = \langle \mathbf{g}_\alpha^{(n)} | A^\dagger \mathbf{f}_\beta^{(m)} \rangle = m_n \delta_{\alpha\beta} \delta^{nm}. \quad (10)$$

### 3 $\mathcal{N} = 2$ quantum-mechanical supersymmetry

In this section, we show that the orthonormality relations (10) imply that the mode functions  $\mathbf{f}_\alpha^{(n)}(y)$  and  $\mathbf{g}_\alpha^{(n)}(y)$  form supersymmetric partners (except for zero modes) in an  $\mathcal{N} = 2$  supersymmetric quantum mechanics (SQM).

Since the mode functions  $\mathbf{f}_\alpha^{(n)}(y)$  and  $\mathbf{g}_\alpha^{(n)}(y)$  are assumed to form complete sets on the internal space, Eq. (10) leads to the relations

$$A^\dagger \mathbf{f}_\alpha^{(n)}(y) = m_n \mathbf{g}_\alpha^{(n)}(y), \quad A \mathbf{g}_\alpha^{(n)}(y) = m_n \mathbf{f}_\alpha^{(n)}(y). \quad (11)$$

It immediately follows that  $\mathbf{f}_\alpha^{(n)}(y)$  and  $\mathbf{g}_\alpha^{(n)}(y)$  satisfy the eigenvalue equations

$$(-\partial_y^2 + M^2) \mathbf{f}_\alpha^{(n)}(y) = m_n^2 \mathbf{f}_\alpha^{(n)}(y), \\ (-\partial_y^2 + M^2) \mathbf{g}_\alpha^{(n)}(y) = m_n^2 \mathbf{g}_\alpha^{(n)}(y), \quad (12)$$

and that the above relations (11) and (12) can be naturally embedded in a system of an  $\mathcal{N} = 2$  SQM, as explained below.

Let us introduce the supercharge  $Q$  and the ‘‘fermion’’ number operator  $(-1)^F$  such as

$$Q = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \quad (-1)^F = \begin{pmatrix} -\mathbf{1}_{2^{\lfloor d/2 \rfloor}} & 0 \\ 0 & \mathbf{1}_{2^{\lfloor d/2 \rfloor}} \end{pmatrix}, \quad (13)$$

and then define the Hamiltonian by

$$H = Q^2. \quad (14)$$

This system is known as the  $\mathcal{N} = 2$  SQM (see reviews [1]).<sup>5</sup> It should be noticed that the supercharge  $Q$  and the fermion number operator  $(-1)^F$  are represented as

$$\Gamma^0(i\Gamma^y\partial_y - M) = Q \otimes \mathbf{1}_2, \quad \Gamma_5 = (-1)^F \otimes \mathbf{1}_2. \quad (15)$$

It follows that  $Q$  and  $(-1)^F$  can be interpreted as the Dirac operator on the internal space and the 4D chiral operator, respectively.

To rewrite the relations (11) and (12) in the language of the  $\mathcal{N} = 2$  SQM, we introduce

$$\Phi_{\alpha,+}^{(n)}(y) = \begin{pmatrix} 0 \\ \mathbf{g}_\alpha^{(n)}(y) \end{pmatrix}, \quad \Phi_{\alpha,-}^{(n)}(y) = \begin{pmatrix} \mathbf{f}_\alpha^{(n)}(y) \\ 0 \end{pmatrix}. \quad (16)$$

Then, in terms of  $\Phi_{\alpha,\pm}^{(n)}(y)$ , the relations (11) and (12) can be rewritten as

$$Q\Phi_{\alpha,\pm}^{(n)}(y) = m_n\Phi_{\alpha,\mp}^{(n)}(y), \quad (17)$$

$$H\Phi_{\alpha,\pm}^{(n)}(y) = m_n^2\Phi_{\alpha,\pm}^{(n)}(y), \quad (18)$$

with

$$(-1)^F\Phi_{\alpha,\pm}^{(n)}(y) = \pm\Phi_{\alpha,\pm}^{(n)}(y). \quad (19)$$

We call Eqs. (11) and (17), supersymmetric relations. The mode functions  $\Phi_{\alpha,+}^{(n)}(y)$  (or  $\mathbf{g}_\alpha^{(n)}(y)$ ) and  $\Phi_{\alpha,-}^{(n)}(y)$  (or  $\mathbf{f}_\alpha^{(n)}(y)$ ) are (at least) doubly degenerate in the spectrum except for zero energy states (see Figure 1), and they form  $\mathcal{N} = 2$  supermultiplets. Thus, the  $\mathcal{N} = 2$  supersymmetry turns out to be hidden in the 4D spectrum of the higher dimensional Dirac theory, as well as higher dimensional gauge/gravity theories [16–19].

Here, a question arises about the degeneracy of the spectrum. Although the  $\mathcal{N} = 2$  supersymmetry assures a pair of eigenstates in the spectrum, as seen in Figure 1, the degeneracy of the spectrum in the higher dimensional Dirac action is found to be much larger than that expected by the  $\mathcal{N} = 2$  supersymmetry (see Figure 2), in general. This fact suggests that there should exist some symmetries in addition to the  $\mathcal{N} = 2$  supersymmetry. To reveal hidden symmetries on the degeneracy of the spectrum is the purpose of the next section.

Before closing this section, we should give a comment on the Hermiticity property of the supercharge. The supercharge  $Q$  has to be Hermitian in the  $\mathcal{N} = 2$  SQM but the Hermiticity of  $Q$  would not be trivial if the internal space  $\Omega$  has boundaries. To verify the Hermiticity of  $Q$ , let us examine the variational principle of the action,  $\delta S = 0$ , which leads to the  $(d + 4)$ -dimensional Dirac equation and also the condition for the surface integral, i.e.

$$\int_{\partial\Omega} d^{d-1}y \begin{pmatrix} \mathbf{f}_{\tilde{\alpha}}^{(\tilde{n})}(y) \\ \mathbf{g}_{\tilde{\beta}}^{(\tilde{m})}(y) \end{pmatrix}^\dagger \begin{pmatrix} 0 & in_{y_k}\gamma^{y_k} \\ -in_{y_k}\gamma^{y_k} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f}_\alpha^{(n)}(y) \\ \mathbf{g}_\beta^{(m)}(y) \end{pmatrix} = 0, \quad (20)$$

for any  $n, m, \alpha, \beta$  and  $\tilde{n}, \tilde{m}, \tilde{\alpha}, \tilde{\beta}$ . The  $n_{y_k}$  is a normal vector at each point of the boundary  $\partial\Omega$ . It turns out that the supercharge  $Q$  is Hermitian, provided that boundary conditions on  $\mathbf{f}_\alpha^{(n)}(y)$  and  $\mathbf{g}_\alpha^{(n)}(y)$  are chosen to satisfy Eq. (20).

<sup>5</sup>If one defines  $Q_1 = Q$  and  $Q_2 = i(-1)^F Q$ , then they form the  $\mathcal{N} = 2$  supersymmetry algebra  $\{Q_j, Q_k\} = 2H\delta_{jk}$  for  $j, k = 1, 2$ .

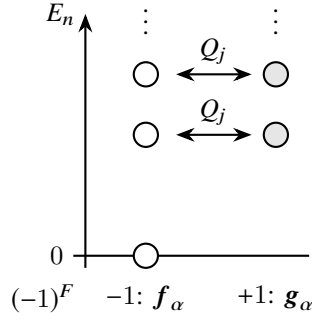


Figure 1: The one dimensional SQM model (known as a Witten model [32]) has the doubly degenerated spectrum except for the zero energy state ( $j = 1, 2$ ).

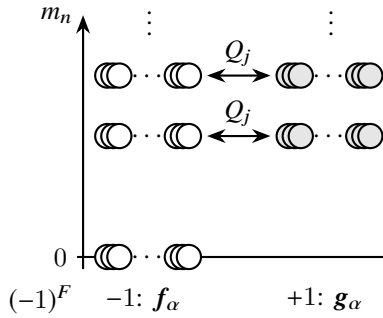


Figure 2: The eigenfunctions  $f_\alpha^{(n)}(y)$  and  $g_\alpha^{(n)}(y)$  of the higher dimensional Dirac field are, in general, multiply degenerate at each KK level. An  $\mathcal{N}$ -extended supersymmetry with the supercharges  $Q_j$  ( $j = 1, 2, \dots, d+2$  ( $j = 1, 2, \dots, d+1$ ) for even (odd) dimensions) is hidden in the spectrum.

## 4 $\mathcal{N}$ -extended quantum-mechanical supersymmetry

Interestingly, the previous  $\mathcal{N} = 2$  supersymmetry can be enlarged to an  $\mathcal{N}$ -extended supersymmetry describing the additional degeneracy of the spectrum. The supersymmetry has  $d+2$  ( $d+1$ ) supercharges for  $d = \text{even}$  ( $\text{odd}$ ) extra dimensions. The supercharges form the  $\mathcal{N}$ -extended supersymmetry algebra such as

$$\{Q_i, Q_j\} = 2H\delta_{ij}, \quad [Q_i, H] = 0, \quad (21)$$

where  $i, j = 1, 2, \dots, d+2$  ( $i, j = 1, 2, \dots, d+1$ ) for even (odd) dimensions. The Hamiltonian  $H$  is the same as that given in the  $\mathcal{N} = 2$  supersymmetry algebra. The supercharges are defined by the composition of the reflections operators as

$$\begin{aligned} Q_k &= i(-1)^F (\mathbf{1}_2 \otimes \gamma_{d+1} \gamma^{y_k}) R_k Q & (k = 1, \dots, d), \\ Q_{d+1} &= i(-1)^F (\mathbf{1}_2 \otimes \gamma_{d+1}) P Q, \\ Q_{d+2} &= Q, \end{aligned} \quad (22)$$

for even dimensions, and

$$\begin{aligned} Q_k &= i(-1)^F (\mathbf{1}_2 \otimes i\gamma^{y_d} \gamma^{y_k}) R_d R_k Q & (k = 1, \dots, d-1), \\ Q_d &= i(-1)^F (\mathbf{1}_2 \otimes i\gamma^{y_d}) R_d P Q, \\ Q_{d+1} &= Q, \end{aligned} \quad (23)$$

for odd dimensions. The  $R_k$  ( $k = 1, 2, \dots, d$ ) denotes the reflection operator for the  $y_k$  direction, i.e.  $(R_k f)(y_1, \dots, y_k, \dots, y_d) \equiv f(y_1, \dots, -y_k, \dots, y_d)$  for any function  $f(y_1, \dots, y_d)$ .<sup>6</sup>  $P = \prod_{k=1}^d R_k$  represents the point reflection (or parity) operator of the internal space. Here (when  $d$  is more than one), the element  $i(-1)^F Q$  is not included in this algebra since it is commutative with the other components except for  $Q$ .

To demonstrate that the supercharges  $Q_j$  can explain the degeneracy of the spectrum at each KK level with  $m_n^2 > 0$ , let us consider, as an example, the  $d$ -dimensional hyperrectangle internal space:

$$\Omega = [-L_1/2, L_1/2] \times \dots \times [-L_d/2, L_d/2], \quad (24)$$

where  $L_k$  ( $k = 1, 2, \dots, d$ ) is the length of the  $k$ -th side of the hyperrectangle with the Dirichlet boundary condition on  $\mathbf{g}_\alpha^{(n)}(y)$ , i.e.

$$\mathbf{g}_\alpha^{(n)}(y) = 0 \quad (\text{or } \Phi_{\alpha,+}^{(n)}(y) = 0) \quad \text{on } \partial\Omega. \quad (25)$$

Then, the supersymmetric relations (11) (or (17)) lead to the following boundary condition on  $\mathbf{f}_\alpha^{(n)}(y)$  [33, 34]:

$$A^\dagger \mathbf{f}_\alpha^{(n)}(y) = 0 \quad (\text{or } Q\Phi_{\alpha,-}^{(n)}(y) = 0) \quad \text{on } \partial\Omega. \quad (26)$$

It should be emphasized that the boundary conditions (25) and (26) are compatible with the  $\mathcal{N}$ -extended supersymmetry.

The  $n$ -th mode functions  $\mathbf{g}_\alpha^{(n)}(y)$  and  $\mathbf{f}_\alpha^{(n)}(y)$  with a KK mass  $m_n^2 > 0$  are found to be of the form

$$\mathbf{g}_\alpha^{(n)}(y) = h^{(n)}(y)\mathbf{e}_\alpha, \quad (27)$$

$$\mathbf{f}_\alpha^{(n)}(y) = \frac{1}{m_n} A \mathbf{g}_\alpha^{(n)}(y) \quad \left( \text{or } \Phi_{\alpha,-}^{(n)}(y) = \frac{1}{m_n} Q \Phi_{\alpha,+}^{(n)}(y) \right), \quad (28)$$

where  $h^{(n)}(y)$  is a scalar function. Hereafter, we adopt the following variables for representing  $\alpha$  concretely, where the basis vectors of the spinor space  $\mathbf{e}_\alpha = \mathbf{e}_{s_1 s_2 \dots s_{\lfloor d/2 \rfloor}}$  ( $s_p = \pm$  for  $p = 1, 2, \dots, \lfloor d/2 \rfloor$ ) are the eigenvectors of  $\gamma_{(p)} = i\gamma^{2p-1} \gamma^{2p}$  [35, 36], i.e.

$$\gamma_{(p)} \mathbf{e}_{s_1 \dots s_p \dots s_{\lfloor d/2 \rfloor}} = s_p \mathbf{e}_{s_1 \dots s_p \dots s_{\lfloor d/2 \rfloor}}, \quad (29)$$

for all  $p = 1, 2, \dots, \lfloor d/2 \rfloor$ .<sup>7</sup>  $s_p$  represents an eigenvalue of the  $p$ -th internal chirality of  $\gamma_{(p)}$ . Thus, the degeneracy at each KK level is  $2 \times 2^{\lfloor d/2 \rfloor}$ . (The additional factor 2 corresponds to the degeneracy between  $\mathbf{g}_\alpha^{(n)}$  and  $\mathbf{f}_\alpha^{(n)}$ .) The scalar function  $h^{(n)}(y)$  is the eigenmode of the Hamiltonian  $H = -\partial_y^2 + M^2$  with the eigenvalue  $m_n^2 = M^2 + \sum_{k=1}^d (n_k \pi)^2 / L_k^2$  where  $n_k = 1, 2, \dots$  ( $k = 1, 2, \dots, d$ ), and is explicitly given by

$$h^{(n)}(y) = \prod_{k=1}^d \sqrt{\frac{2}{L_k}} \sin\left(\frac{n_k \pi}{L_k} \left(y_k + \frac{L_k}{2}\right)\right). \quad (30)$$

It is easy to verify that the  $\mathcal{N}$ -extended supercharges  $Q_j$  can explain the degeneracy of the mode functions, as we expected. We can derive the following general relationships with  $k' = 2p - 1$  or  $2p$  ( $p = 1, 2, \dots, \lfloor d/2 \rfloor$ ) for  $d$  being even,

$$Q_{k'} \Phi_{s_1 \dots s_p \dots s_{\lfloor d/2 \rfloor}, \pm}^{(n)}(y) \propto \Phi_{s_1 \dots (-s_p) \dots s_{\lfloor d/2 \rfloor}, \mp}^{(n)}(y), \quad (31)$$

$$Q_{d+1, d+2} \Phi_{s_1 \dots s_p \dots s_{\lfloor d/2 \rfloor}, \pm}^{(n)}(y) \propto \Phi_{s_1 \dots s_p \dots s_{\lfloor d/2 \rfloor}, \mp}^{(n)}(y), \quad (32)$$

<sup>6</sup> We note that the reflection operator  $R_k$  acts on  $\partial_l$  as  $R_k \partial_{y_l} R_k^{-1} = (1 - 2\delta_{kl}) \partial_{y_l}$ .

<sup>7</sup> Note that  $\mathbf{g}_{s_1 s_2 \dots s_{\lfloor d/2 \rfloor}}^{(n)}$  are eigenvectors of  $\gamma_{(p)}$ , while  $\mathbf{f}_{s_1 s_2 \dots s_{\lfloor d/2 \rfloor}}^{(n)}$  are not. The forms of  $\mathbf{f}_{s_1 s_2 \dots s_{\lfloor d/2 \rfloor}}^{(n)}$  are determined by the corresponding states of  $\mathbf{g}_{s_1 s_2 \dots s_{\lfloor d/2 \rfloor}}^{(n)}$  through the relation (28).

and for  $d$  being odd,

$$Q_{k'} \Phi_{s_1 \dots s_p \dots s_{[d/2], \pm}}^{(n)}(y) \propto \Phi_{s_1 \dots (-s_p) \dots s_{[d/2], \mp}}^{(n)}(y), \quad (33)$$

$$Q_{d,d+1} \Phi_{s_1 \dots s_p \dots s_{[d/2], \pm}}^{(n)}(y) \propto \Phi_{s_1 \dots s_p \dots s_{[d/2], \mp}}^{(n)}(y), \quad (34)$$

where flips are observed in one of the internal chiralities for the cases of (31) and (33). Here, we adopt the notation (being similar to that in Eq. (16))

$$\begin{aligned} \Phi_{s_1 \dots s_p \dots s_{[d/2], +}}^{(n)}(y) &= \begin{pmatrix} 0 \\ \mathbf{g}_{s_1 \dots s_p \dots s_{[d/2]}}^{(n)}(y) \end{pmatrix}, \\ \Phi_{s_1 \dots s_p \dots s_{[d/2], -}}^{(n)}(y) &= \begin{pmatrix} \mathbf{f}_{s_1 \dots s_p \dots s_{[d/2]}}^{(n)}(y) \\ 0 \end{pmatrix}. \end{aligned} \quad (35)$$

We summarize the above statements as follows. Once a single mode function  $\mathbf{g}_\alpha^{(n)}(y)$  (or  $\mathbf{f}_\alpha^{(n)}(y)$ ) is given (for fixed  $\alpha$ ), the whole mode functions can be obtained by successively operating the  $\mathcal{N}$ -extended supercharges on  $\mathbf{g}_\alpha^{(n)}(y)$  (or  $\mathbf{f}_\alpha^{(n)}(y)$ ) as shown at Figure 3.

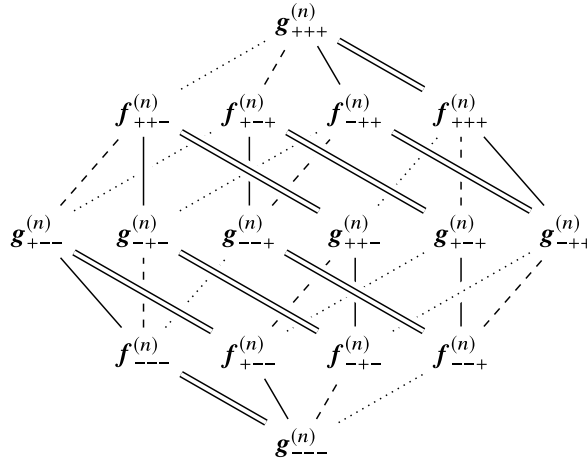


Figure 3: The  $\mathbf{g}_{s_1 s_2 s_3}^{(n)}$  and  $\mathbf{f}_{s_1 s_2 s_3}^{(n)}$  ( $s_1, s_2, s_3 = \pm$ ) are the degenerate  $n$ -th mode functions on the six-dimensional hyperrectangle in the ten-dimensional space-time. The  $\mathcal{N}$ -extended supercharges generate the whole mode functions. The lines indicate the relationships through the  $\mathcal{N}$ -extended supercharges, where the double continuous, single continuous, dashed, and dotted lines correspond to  $Q_7$  or  $Q_8$ ,  $Q_1$  or  $Q_2$ ,  $Q_3$  or  $Q_4$ , and  $Q_5$  or  $Q_6$ , respectively.

## 5 Extension of supersymmetry with superpotential

Let us consider the extension of the supersymmetry with a superpotential. To this end, we replace the bulk mass  $M$  with a scalar function  $W(y)$ . Then, we can show that the  $\mathcal{N} = 2$  supersymmetry given in Section 3 holds with the supercharge

$$Q = \begin{pmatrix} 0 & i\gamma^{y_k} \partial_{y_k} - W(y) \\ -i\gamma^{y_k} \partial_{y_k} - W(y) & 0 \end{pmatrix}. \quad (36)$$



It is, however, non-trivial for the whole of the extended supersymmetry to preserve with the introduction of the superpotential  $W(y)$ .

For  $d = \text{even}$ , the conditions for a realization of the full  $\mathcal{N} = d + 2$  extended supersymmetry are found to be given by

$$(R_k W)(y) = W(y) \quad \text{for } k = 1, \dots, d, \quad (37)$$

or equivalently

$$W(y_1, \dots, -y_k, \dots, y_d) = W(y_1, \dots, y_k, \dots, y_d) \quad \text{for } k = 1, \dots, d. \quad (38)$$

If the superpotential does not satisfy the above conditions, the extended supersymmetry is broken. For instance, let us consider the case that the superpotential  $W(y)$  satisfies the relations (37) only for the directions  $y_{k_1}, \dots, y_{k_m}$  ( $m < d$ ), i.e.

$$(R_k W)(y) = W(y) \quad \text{for } k = k_1, \dots, k_m. \quad (39)$$

Then, the supercharges  $Q_k$  ( $k \neq k_1, \dots, k_m$ ) and  $Q_{d+1}$  in Eq.(22) become ill-defined and should be removed from the supersymmetry algebra. Thus, the supercharges  $Q_k$  ( $k = k_1, \dots, k_m$ ) and  $Q_{d+2}$  in Eq.(22) form the  $\mathcal{N} = m + 1$  extended supersymmetry for  $m = \text{odd}$ . Interestingly, it turns out that the  $\mathcal{N} = m + 1$  extended supersymmetry can be enlarged to the  $\mathcal{N} = m + 2$  extended one for  $m = \text{even}$  with an additional supercharge

$$Q_{(k_1 \dots k_m)} \equiv i(-1)^F (\mathbf{1}_2 \otimes i^{m/2} \gamma^{y_{k_1}} \dots \gamma^{y_{k_m}}) R_{k_1} \dots R_{k_m} Q. \quad (40)$$

The same argument can be also applied to the odd  $d$ -dimensional case.

## 6 Summary and discussion

In this paper, we have succeeded in constructing the new realization of the  $\mathcal{N}$ -extended quantum-mechanical supersymmetry. First we considered the hidden  $\mathcal{N} = 2$  quantum-mechanical supersymmetry in the KK decomposition of the higher dimensional Dirac field. The  $\mathcal{N} = 2$  supersymmetry shows that the mode function  $f_\alpha^{(n)}$  for the left-handed chiral spinor  $\phi_\alpha^{(n)}$  and  $g_\alpha^{(n)}$  for the right-handed chiral spinor  $\bar{\chi}_\alpha^{(n)}$  in Eq. (6) form a pair of  $\mathcal{N} = 2$  supermultiplets.

Surprisingly, we further found that the  $\mathcal{N}$ -extended quantum-mechanical supersymmetry with multiple supercharges is hidden in the spectrum. It turns out that once a single mode function is given, the supercharges in the  $\mathcal{N}$ -extended supersymmetry can generate the whole degenerate mode functions of the KK spectrum on the hyperrectangle extra dimensions with the boundary conditions (25) and (26).

Furthermore we revealed the extended supersymmetry with the superpotential. Then, we clarified the conditions for the superpotential to preserve or partially preserve the extended supersymmetry.

Our analysis is far from being complete. We have demonstrated only the case of the boundary conditions given in Eq. (25) and (26). Although the boundary conditions (25) and (26) are compatible with the  $\mathcal{N}$ -extended supersymmetry, all the supercharges given in Eq. (22) or (23) are not necessarily well-defined for other types of boundary conditions. For example, boundary conditions with no reflection symmetry of  $y_k \rightarrow -y_k$ , the supercharge  $Q_k$  will become ill-defined because  $Q_k$  includes the reflection operator  $R_k$  in the definition of Eq. (22) or (23). Thus, it would be of importance to determine how the  $\mathcal{N}$ -extended supersymmetry is broken by boundary conditions.

Allowed boundary conditions on a rectangle (as two dimensional extra dimensions) have been classified in Refs. [33, 34]. However, general boundary conditions for Dirac fields in arbitrary higher dimensions have

not been obtained yet, because it is nontrivial to find general solutions to Eq. (20). So, it will be interesting to find a class of boundary conditions consistent with Eq. (20).

Although we have introduced a superpotential into the supercharges, we have not solved any mass spectrum of concrete models in this paper. It would be interesting to try to find a new class of exactly solvable quantum-mechanical models with the  $\mathcal{N}$ -extended supersymmetry. The issues mentioned in this section will be reported in a future work.

## Acknowledgement

This work is supported in part by Grants-in-Aid for Scientific Research [No. 15K05055 and No. 25400260 (M.S.)] from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) in Japan.

## References

- [1] F. Cooper, A. Khare, U. Sukhatme, Supersymmetry and quantum mechanics, *Physics Reports* 251 (5-6) (1995) 267–385. [arXiv:hep-th/9405029](#), [doi:10.1016/0370-1573\(94\)00080-M](#).
- [2] C. Quesne, V. M. Tkachuk, Dirac oscillator with nonzero minimal uncertainty in position, *J. Phys. A* 38 (2005) 1747–1766. [arXiv:math-ph/0412052](#), [doi:10.1088/0305-4470/38/8/011](#).
- [3] B. Bagchi, C. Quesne, R. Roychoudhury, Isospectrality of conventional and new extended potentials, second-order supersymmetry and role of PT symmetry, *Pramana* 73 (2009) 337–347. [arXiv:0812.1488](#), [doi:10.1007/s12043-009-0126-4](#).
- [4] S. Odake, R. Sasaki, Infinitely many shape invariant potentials and new orthogonal polynomials, *Phys. Lett. B* 679 (2009) 414–417. [arXiv:0906.0142](#), [doi:10.1016/j.physletb.2009.08.004](#).
- [5] D. J. Fernandez C., Supersymmetric Quantum Mechanics, *AIP Conf. Proc.* 1287 (2010) 3–36. [arXiv:0910.0192](#), [doi:10.1063/1.3507423](#).
- [6] P. Claus, M. Derix, R. Kallosh, J. Kumar, P. K. Townsend, A. Van Proeyen, Black holes and superconformal mechanics, *Phys. Rev. Lett.* 81 (1998) 4553–4556. [arXiv:hep-th/9804177](#), [doi:10.1103/PhysRevLett.81.4553](#).
- [7] G. W. Gibbons, P. K. Townsend, Black holes and Calogero models, *Phys. Lett. B* 454 (1999) 187–192. [arXiv:hep-th/9812034](#), [doi:10.1016/S0370-2693\(99\)00266-X](#).
- [8] R. Britto-Pacumio, J. Michelson, A. Strominger, A. Volovich, Lectures on Superconformal Quantum Mechanics and Multi-Black Hole Moduli Spaces, *NATO Sci. Ser. C* 556 (2000) 255–284. [arXiv:hep-th/9911066](#), [doi:10.1007/978-94-011-4303-5\\_6](#).
- [9] S. Bellucci, A. Galajinsky, E. Ivanov, S. Krivonos, AdS(2) / CFT(1), canonical transformations and superconformal mechanics, *Phys. Lett. B* 555 (2003) 99–106. [arXiv:hep-th/0212204](#), [doi:10.1016/S0370-2693\(03\)00040-6](#).
- [10] I. Bakas, Energy-momentum/Cotton tensor duality for AdS(4) black holes, *JHEP* 01 (2009) 003. [arXiv:0809.4852](#), [doi:10.1088/1126-6708/2009/01/003](#).

- [11] W. Fu, D. Gaiotto, J. Maldacena, S. Sachdev, Supersymmetric Sachdev-Ye-Kitaev models, *Phys. Rev. D* 95 (2) (2017) 026009. [arXiv:1610.08917](#), [doi:10.1103/PhysRevD.95.069904](#), [10.1103/PhysRevD.95.026009](#).
- [12] T. Li, J. Liu, Y. Xin, Y. Zhou, Supersymmetric SYK model and random matrix theory, *JHEP* 06 (2017) 111. [arXiv:1702.01738](#), [doi:10.1007/JHEP06\(2017\)111](#).
- [13] J. Yoon, Supersymmetric SYK Model: Bi-local Collective Superfield/Supermatrix Formulation, *JHEP* 10 (2017) 172. [arXiv:1706.05914](#), [doi:10.1007/JHEP10\(2017\)172](#).
- [14] J. Murugan, D. Stanford, E. Witten, More on Supersymmetric and 2d Analogs of the SYK Model, *JHEP* 08 (2017) 146. [arXiv:1706.05362](#), [doi:10.1007/JHEP08\(2017\)146](#).
- [15] N. Hunter-Jones, J. Liu, Chaos and random matrices in supersymmetric SYK, *JHEP* 05 (2018) 202. [arXiv:1710.08184](#), [doi:10.1007/JHEP05\(2018\)202](#).
- [16] C. S. Lim, T. Nagasawa, M. Sakamoto, H. Sonoda, Supersymmetry in gauge theories with extra dimensions, *Physical Review D* 72 (6) (2005) 064006. [arXiv:hep-th/0502022](#), [doi:10.1103/PhysRevD.72.064006](#).
- [17] C. S. Lim, T. Nagasawa, S. Ohya, K. Sakamoto, M. Sakamoto, Supersymmetry in 5D gravity, *Physical Review D* 77 (4) (2008) 045020. [arXiv:0710.0170v2](#), [doi:10.1103/PhysRevD.77.045020](#).
- [18] C. S. Lim, T. Nagasawa, S. Ohya, K. Sakamoto, M. Sakamoto, Gauge fixing and residual symmetries in gauge/gravity theories with extra dimensions, *Physical Review D* 77 (6) (2008) 065009. [arXiv:0801.0845](#), [doi:10.1103/PhysRevD.77.065009](#).
- [19] T. Nagasawa, S. Ohya, K. Sakamoto, M. Sakamoto, Emergent Supersymmetry in Warped Backgrounds, *Symmetry, Integrability and Geometry: Methods and Applications* 7. [arXiv:1105.4829](#), [doi:10.3842/SIGMA.2011.065](#).
- [20] A. Pashnev, F. Toppan, On the classification of N-extended supersymmetric quantum mechanical systems, *Journal of Mathematical Physics* 42 (11) (2001) 5257. [arXiv:hep-th/0010135](#), [doi:10.1063/1.1409349](#).
- [21] Z. Kuznetsova, M. Rojas, F. Toppan, Classification of irreps and invariants of the N -extended Supersymmetric Quantum Mechanics, *Journal of High Energy Physics* 2006 (03) (2006) 098–098. [arXiv:hep-th/0511274](#), [doi:10.1088/1126-6708/2006/03/098](#).
- [22] M. Faux, S. J. Gates, Jr., Adinkras: A Graphical technology for supersymmetric representation theory, *Phys. Rev. D* 71 (2005) 065002. [arXiv:hep-th/0408004](#), [doi:10.1103/PhysRevD.71.065002](#).
- [23] M. de Crombrugghe, V. Rittenberg, Supersymmetric quantum mechanics, *Annals of Physics* 151 (1) (1983) 99–126. [doi:10.1016/0003-4916\(83\)90316-0](#).
- [24] P. Howe, S. Penati, M. Pernici, P. Townsend, Wave equations for arbitrary spin from quantization of the extended supersymmetric spinning particle, *Physics Letters B* 215 (3) (1988) 555–558. [doi:10.1016/0370-2693\(88\)91358-5](#).
- [25] V. Akulov, M. Kuznetsov, Extended supersymmetric quantum mechanics, *Phys. Lett. B* 460 (1999) 365–370. [arXiv:hep-th/9905070](#), [doi:10.1016/S0370-2693\(99\)00773-X](#).

- [26] T. Nagasawa, M. Sakamoto, K. Takenaga, Supersymmetry and discrete transformations on  $S^1$  with point singularities, *Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics* 583 (3-4) (2004) 357–363. [arXiv:hep-th/0311043](#), [doi:10.1016/j.physletb.2003.12.065](#).
- [27] T. Nagasawa, M. Sakamoto, K. Takenaga, Extended supersymmetry and its reduction on a circle with point singularities, *Journal of Physics A: Mathematical and General* 38 (37) (2005) 8053–8082. [arXiv:hep-th/0505132](#), [doi:10.1088/0305-4470/38/37/009](#).
- [28] R. Brauer, H. Weyl, Spinors in  $n$  dimensions, *Amer. J. Math.* 57 (2) (1935) 425–449. [doi:10.2307/2371218](#).
- [29] C. Wetterich, Massless Spinors in More Than Four-dimensions, *Nucl. Phys. B* 211 (1983) 177–188. [doi:10.1016/0550-3213\(83\)90191-8](#).
- [30] T. Kugo, P. K. Townsend, Supersymmetry and the Division Algebras, *Nucl. Phys. B* 221 (1983) 357–380. [doi:10.1016/0550-3213\(83\)90584-9](#).
- [31] J. Wess, J. Bagger, *Supersymmetry and Supergravity*, 2nd Edition, Princeton University Press, 1992.
- [32] E. Witten, Dynamical breaking of supersymmetry, *Nuclear Physics B* 188 (3) (1981) 513–554. [doi:10.1016/0550-3213\(81\)90006-7](#).
- [33] Y. Fujimoto, K. Hasegawa, K. Nishiwaki, M. Sakamoto, K. Tatsumi, Supersymmetry in the 6D Dirac action, *Progress of Theoretical and Experimental Physics* 2017 (7). [arXiv:1609.04565](#), [doi:10.1093/ptep/ptx092](#).
- [34] Y. Fujimoto, K. Hasegawa, K. Nishiwaki, M. Sakamoto, K. Tatsumi, 6d Dirac fermion on a rectangle; scrutinizing boundary conditions, mode functions and spectrum, *Nuclear Physics B* 922 (2017) 186–225. [arXiv:1609.01413](#), [doi:10.1016/j.nuclphysb.2017.06.024](#).
- [35] J. Polchinski, *String Theory: Volume 1, An Introduction to the Bosonic String*, Cambridge University Press, Cambridge, 1998. [doi:10.1017/CBO9780511816079](#).
- [36] J. Polchinski, *String Theory: Volume 2, Superstring Theory and Beyond*, Cambridge University Press, Cambridge, 1998. [doi:10.1017/CBO9780511618123](#).