Exploring Branched Hamiltonians For A Class Of Nonlinear Systems

Bijan Bagchi*
Department of Applied Mathematics, University of Calcutta
92 Acharya Prafulla Chandra Road
Kolkata - 700009, India

Subhrajit Modak and Prasanta K. Panigrahi †
Indian Institute of Science,
Education and Research (Kolkata),
Mohanpur, West Bengal 741 246, India.

František Ruzicka and Miloslav Znojil ‡
Nuclear Physics Institute ASCR, 250 68 *Řež*
Czech Republic

Abstract

One of the less well understood ambiguities of quantization is emphasized to result from the presence of higher-order time derivatives in the Lagrangians resulting in multiple-valued Hamiltonians. We explore certain classes of branched Hamiltonians in the context of nonlinear autonomous differential equation of Liénard type. Two eligible elementary nonlinear models that emerge are shown to admit a feasible quantization along these lines.

Mathematical Classification 34C14, 70H33.

Keywords quantization; branched classical Hamiltonians; partner quantum Hamiltonians; perturbation solutions;

*E-mail: bbagchi123@gmail.com
†E-mail: modoksuvrojit@gmail.com, panigrahi.iiser@gmail.com
‡E-mail: fruzicka@gmail.com, znojil@ujf.cas.cz

1 Introduction

The Bohr's formulation of his model of atom [1] was one of the most important steps towards the subsequent birth of full-fledged quantum theory from classical mechanics. His choice of benchmark systems was insightful (quantum hydrogen with Coulomb potential $V(\vec{x}) \sim 1/|\vec{x}|$ in non-relativistic Schrödinger equation is one of not too may exactly solvable real-world systems) and lucky (indeed, the subsequent inclusion of subtleties led just to some not too essential corrections and modifications of the picture).

Today, more than 100 years later, the Bohr's emphasis on connectedness between classical and quantum worlds is still alive and inspiring. Although we are currently paying attention to multiple much more sophisticated quantum systems, the principle of correspondence and the traditional description of dynamics via certain real and local potentials $V(\vec{x})$ still guides the majority of phenomenological studies in atomic, molecular and nuclear physics.

In a broader methodical framework, say, of condensed matter physics, the traditional approach does not work that well. Incessantly, its applicability is being challenged by theoretical innovations. Just to name a few, let us mention the recent growth of popularity of the detailed study of certain non-real, complex potentials with real spectra [2] or, most recently, of their friendly and tractable non-local generalizations [3]. An emerging interest also involves various non-linear forms of the interactions which may be mediated, say, by an energy-dependence of the dynamical parameters [4]. Last but not least, let us mention the fruitful transfer of the dynamics-determining role from elementary potentials $V(x)$ to the other factors like, e.g., coordinate-dependent masses $m = m(x)$ [5, 6, 7] etc.

In this context a new class of innovations of the description and simulation of quantum dynamics recently emerged in connection with the possible specific role of the models of physical systems in which the Lagrangians possess the time derivatives in excess of quadratic powers [8, 9, 10, 11, 12, 13]. The use of these models leads, on both the classical and quantum level, to the necessity of a re-evaluation of the dynamical interpretation of the momentum p which, in principle, becomes a multiple function of velocity v. In what follows we intend to present a few results in this direction.

2 Branched Hamiltonians

A typical classical model of the above-mentioned non-quadratic type may be sampled by Lagrangian

$$L = (v-1)^{\frac{2k-1}{2k+1}} - V(x)\,. \tag{1}$$

Function $V(x)$ represents a convenient local interaction potential while the traditional kinetic-energy term is tentatively replaced by a fairly unusual function of "velocity" v.

This model was recently analyzed in Ref. [14] where the definition of the fractional powers of difference $v-1$ was adapted to the needs of possible phenomenology. In detail, the

$(2k+1)-$st root was required real and positive or negative for $v>1$ or $v<1$, respectively. Correspondingly, the quantity v turned out to be a double-valued function of p.

2.1 The problem of quantization

In a search for a quantum version or analogue of model (1), both of its latter features appear truly challenging. Firstly, the definition using multivalued functions (referring to the literature let us speak about "branched Hamiltonians") seems to require a deeper insight in its physical nature and connections.

In this note we decided to consider the subject of the branched Hamiltonians in the context of a class of nonlinear autonomous differential equations of Liénard type [15, 16] since these systems are of potential importance not only in optics [17] but also in the shallow-water-wave studies [18] and in non-Hermitian quantum mechanics [19]. Due to nonlinearity one of the main obstacles lies in the absence of superposition principle. The recent progress in the direction of seeking analytical and numerical solutions as well as of tracking down new mathematical properties has been dramatic, nevertheless. In particular, an inspiring feature of non-linear oscillations was found in the dependence of amplitudes on the frequency, etc. A useful role in this analysis was played, e.g., by the study of the class of Liénard equations

$$\ddot{x}+r(x)\dot{x}+q(x)=0 \tag{2}$$

where overdot indicates a derivative with respect to the time variable and where $r(x)$ and $q(x)$ are two continuously differentiable functions of the spatial coordinate x.

After a restriction of the model to the specific form of $r(x)=kx$ and $q(x)=\lambda x+\frac{k^2}{9}x^3$ (which are odd functions of x) one obtains

$$\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+\lambda x=0, \quad \lambda>0\,. \tag{3}$$

This equation represents a cubic oscillator subject to a damped nonlinear force as indicated by the product $kx\dot{x}$. Naturally, the presence of the damping is a challenge whenever one tries to contemplate a quantization of the model.

2.2 Hamiltonians

The classically damped systems leading to non-Hermitian quantum Hamiltonians need not necessarily imply a quantum damping [20]. Moreover, a backward classical-optics reinterpretation of quantum Hamiltonians of this type may exist in photonics playing the role of so called lossy amplifiers [21]. Last but not least, the similar classical-physics contexts (related, e.g., to the study of Bose-Einstein condensates [22, 23]) re-open the phenomenological appeal of classical nonlinearities as sampled by Eq. (1). For all of these reasons it is worth checking the conversion of the Lagrangians of Eq. (3) into Hamiltonians.

A brief account of some background and references is in order. First of all, let us recollect [24] that the toy-model Lagrangians of the form

$$L = \frac{27\lambda^3}{2k^2}(k\dot{x} + \frac{k^2x^2}{3} + 3\lambda)^{-1} + \frac{3\lambda\dot{x}}{2k} - \frac{9\lambda^2}{2k^2} \tag{4}$$

admit the evaluation of the canonically conjugate momentum,

$$p = -\frac{27\lambda^3}{2k}(k\dot{x} + \frac{k^2x^2}{3} + 3\lambda)^2 + \frac{3\lambda}{2k}\,. \tag{5}$$

Thus, the Hamiltonian can be cast in compact form

$$H_{(x,p)} = \frac{9\lambda^2}{2k^2}\left[2 - 2\left(1 - \frac{2kp}{3\lambda}\right)^{\frac{1}{2}} + \frac{k^2x^2}{9\lambda} - \frac{2kp}{3\lambda} - \frac{2k^3x^2p}{27\lambda^2}\right]\,. \tag{6}$$

Such a Hamiltonian is of non-standard type. The co-ordinates and potentials are mixed so that the expression cannot be split into individual kinetic and potential energy terms. However, we can write

$$H_{(x,p)} = \frac{1}{2}f(p)x^2 + U(p) \tag{7}$$

where

$$f(p) = \omega^2(1 - \frac{2kp}{3\omega^2}), \quad U(p) = \frac{9\omega^4}{2k^2}(\sqrt{1 - \frac{2kp}{3\omega^2}} - 1)^2 \tag{8}$$

with $\omega = \sqrt{\lambda}$. The roles of coordinate and momentum have been transposed implying a momentum-dependent system at play. The first term in (8) represents a mixed function of both position and momentum variables while the second term is a function of momentum alone. The classical Hamiltonian $H_{(x,p)}$ is invariant under a joint action of coordinate reflection and time reversal transformation.

Exact quantization of the Hamiltonian can be carried out by going over to the momentum space with $\hat{x} = i\hbar\frac{\partial}{\partial p}$. The Hamiltonian turns out to be of momentum-dependent mass type. Adopting a von Roos strategy of quantizing the problem by considering a general symmetric ordering [25], the underlying Schrodinger equation can be formulated and solved, in principle at least.

3 Two illustrative models of dynamics

In practice, the price to pay for having the model quantized lies in the clarification of the role of singularities in the eigenfunction. In this sense the road to quantization is, expectedly, not free of difficulties. One can, however, feel encouraged by the work by Chithiika Ruby et al [24] who showed that the energy spectrum and normalized solutions could still be obtained for a class of Hamiltonians that are nonsymmetric and non-Hermitian.

Needless to add that the non-Hermitian nature of quantum Hamiltonian may bring about a number of unpleasant consequences as, for example, the emergence of the exceptional

points [26] or the breakdown of the adiabatic theorem [27, 28]. At the same time, the acceptance of the anomaly may prove innovative, e.g., by giving a new physical interpretation to wave packets [29] or to pseudospectra [30, 31].

3.1 An amendment of the Lagrangian

In a way inspired by Eq. (1) let us consider a higher-power Lagrangian

$$L = C(v + f(x))^{\frac{2m+1}{2m-1}} - \delta, \quad C = \left(\frac{1-2m}{1+2m}\right)(\delta)^{\frac{2}{1-2m}}, \quad \delta > 0. \tag{9}$$

The main difference from (1) lies in our choice of a general function $f(x)$ in place of $f(x) = -1$ in our initial L. The other point is that the inverse exponent with respect to the model of Curtright and Zachos [14] is done for convenience of calculus. Further, we have omitted an explicit presence of the potential function assuming that the interaction re-appears in a more natural manner via a suitable choice of an auxiliary free parameter δ and of a nontrivial function $f(x)$. As long as our Lagrangian L is of a nonstandard type, we will not feel disturbed by the absence of the explicit potential $V(x)$.

Parameter C is non-negative for $0 \leq m < \frac{1}{2}$ and the canonical momentum is given by formula

$$p = \frac{\partial L}{\partial v} = -(\delta)^{\frac{2}{1-2m}}(v + f(x))^{\frac{2}{2m-1}} \tag{10}$$

which can easily be inverted to yield

$$v = -f(x) + \delta(\pm\sqrt{-p})^{2m-1}\,. \tag{11}$$

The Hamiltonian has the structure

$$H_{\pm}(x,p) = (-p)f(x) - \frac{2\delta}{2m+1}(\pm\sqrt{-p})^{2m+1} \quad + \delta\,. \tag{12}$$

For the present purposes it is worthwhile to inquire into the specific case with $m = 0$.

3.2 Type I Hamiltonian at $m = 0$

At $m = 0$ we easily derive the double-valued

$$v = v_{\pm} = -f(x) \pm \frac{\delta}{\sqrt{-p}}\,. \tag{13}$$

The Hamiltonian branches out to the components

$$H_{\pm}(x,p) = (-p)f(x) \mp 2\delta\sqrt{-p} + \delta\,. \tag{14}$$

It is readily noticed that the real or the complex character of $H_\pm$ depends on the sign of the momentum p. Once we specify

$$f(x) = \frac{\lambda}{2}x^2 + \frac{9\lambda^2}{2k^2}, \quad \delta = \frac{9\lambda^2}{2k^2}, \quad \lambda > 0 \tag{15}$$

then under a shift $p \to \frac{2k}{3\lambda}p - 1$, $H_\pm$ move over to the corresponding forms

$$H_\pm(x,p) = \frac{9\lambda^2}{2k^2}\left[2 \mp 2\left(1 - \frac{2kp}{3\lambda}\right)^{\frac{1}{2}} + \frac{k^2x^2}{9\lambda} - \frac{2kp}{3\lambda} - \frac{2k^3x^2p}{27\lambda^2}\right]. \tag{16}$$

These represent a set of plausible Hamiltonians for the nonlinear system (3) possessing coordinate-momentum coupling terms. While H_+ coincides with the form of $H_{(x,p)}$ as furnished in (6), in both $H_\pm$ the presence of a linear harmonic oscillator potential is revealed in the limit $k \to 0$. The specific choice of the parameter δ as given above allows equation (3) to be converted to a harmonic oscillator form under the nonlocal transformation $U = xe^{\frac{k}{3}\int x(\tau)d\tau}$ [32].

In the classical scenario one needs to restrict p to $-\infty < p \leq \frac{3\omega^2}{2k}$ in order to address the physical properties of the system in the real space. However the presence of a branch point singularity at $p = \frac{3\lambda}{2k}$ makes the study of $H_\pm$ quite complicated. At the boundary $p = \frac{3\lambda}{2k}$, the two Hamiltonians $H_\pm$ coincide.

3.3 Type II Hamiltonian

We next focus on a class of models under the influence of the Lagrangian

$$L = \frac{1}{s}\left(\frac{1}{3}sx^2 + \frac{3}{s}\lambda - v\right)^{-1} \tag{17}$$

where s is a real parameter. This implies, for the canonical momentum, the result

$$p = \frac{\partial L}{\partial v} = \frac{1}{s}\left(\frac{1}{3}sx^2 + \frac{3}{s}\lambda - v\right)^{-2}. \tag{18}$$

Inversion of (18) leads to

$$v = \frac{1}{3}sx^2 + \frac{3}{s}\lambda \pm \frac{1}{\sqrt{sp}}. \tag{19}$$

Finally, employing the Legendre transform, the accompanying Hamiltonian to the above Lagrangian can be projected, for the two-v branches, as

$$H_\pm(x,p) = pv_\pm(p) - L = \frac{s}{3}x^2p + \frac{3}{s}\lambda p \mp 2\sqrt{\frac{p}{s}}. \tag{20}$$

Eq. (20) describes a set of alternative Hamiltonians for (3). The component H_+ for the case $s < 0$ has earlier been derived in [32] and found to possess an interesting $\lambda \to 0$ limit.

4 Quantization

4.1 Schrödinger equation

Quantization of the branched system (16) can be handled in a typical way as detailed in [24], i.e., by adopting a suitable ordering procedure and implementing appropriate boundary conditions. In order to carry out quantization for the Type II Hamiltonian we will consider here the case of $s > 0$ and subject it to a perturbative treatment. In this regard we will study the spectrum of Hamiltonians entering Schrödinger equation

$$H_{\pm}(x,p)\,\psi^{(\pm)}(p) = \eta^{(\pm)}\,\psi^{(\pm)}(p) \tag{21}$$

where, in units $\hbar = 1$,

$$\frac{3}{s}H_{\pm}(x,p)/p = -\frac{d^2}{dp^2} \mp 6\sqrt{\frac{1}{s^3 p}} + \frac{9}{s^2}\lambda\,. \tag{22}$$

After the change of variables $p = r^{\varrho}$ (with special $\varrho = 2$) and having set $p\psi^{(\pm)}(p) = r^{\xi}\chi(r)$ (with special $\xi = \varrho + \frac{1}{2}$) one ends up with the following transformed Schrödinger equation

$$-\chi''(r) + \frac{\ell(\ell+1)}{r^2}\,\chi(r) + \frac{36}{s^2}\,r^2\,\chi(r) \mp \frac{24}{s^{3/2}}\,r\,\chi(r) = \frac{12}{s}\,\eta^{(\pm)}\,\chi(r) \tag{23}$$

where the emergence of the centrifugal term is a byproduct of the change of variables [33] and where the quantity resembling the angular momentum is equal to $\ell = 1/2$ for our choice of exponent $\varrho = 2$.

4.2 Perturbation expansions

Eigenvalue problem (23) is well suited for application of Rayleigh-Schrödinger perturbation theory [34]. Towards its implementation we abbreviate $\mp\frac{24}{s^{3/2}}\,r\, = g\,V$. Assuming here that the value of s is sufficiently large, the auxiliary variable g may be treated as a small parameter. Hamiltonians $H = H^{(\pm)}$ as well as the related infinite sets of wave-function kets $|\chi\rangle = |\chi^{(\pm)}_{(n)}\rangle$ and of the bound-state energies $E = E^{(\pm)}_{(n)}$ may be then formally Taylor-expanded,

$$H = H^{(HO)} + g\,V\,, \qquad |\chi\rangle = \sum_{m=0}^{\infty} g^m\,|\chi_m\rangle\,, \qquad E = \sum_{m=0}^{\infty} g^m\,E_m\,. \tag{24}$$

Recalling also that the unperturbed Hamiltonian $H^{(HO)}$ stands for the $\ell-$wave harmonic oscillator living on half-line of $r \in (0,\infty)$ we arrive at the well known formulae for the zero-order perturbation energies $E_0 = \omega\,(4n + 2\ell + 3)$ where $\omega = 6s^{-1}$. The unperturbed eigenfunctions may also be expressed in the compact form

$$\langle r|\chi_0\rangle = \mathcal{N}\,r^{\ell+1}\exp(-\,\omega\,r^2/2)\,L^{(\alpha)}_n(\omega\,r) \tag{25}$$

containing a suitable normalization constant $\mathcal{N}$ and associated Laguerre polynomials $L_n^{(\alpha)}(\omega\, r)$ with, in our present special case, $\alpha = \ell + 1/2 = 1$.

At this moment one would have to return to our hypothesis of smallness of g and to prove that the infinite-series ansatz of eq. (24) are all convergent whenever $|g| \leq R$, i.e., at some non-vanishing radius of convergence $R > 0$. Fortunately the proof is an easy consequence of the fact that our operator of perturbation V is relatively bounded with respect to $H^{(HO)}$.

In the final step of the construction of bound states we only have to insert formulae (24) in Schrödinger equation, deduce the recurrent Rayleigh-Schrödinger perturbation recipe and write down the corresponding general formulae that yield, typically,

$$\eta^{(\pm)} = \eta_{(n)}^{(\pm)} = 2n + 2 \mp \eta_{(n)} \qquad n = 0, 1, \ldots \tag{26}$$

where the quantities $\eta_{(n)}$ must be evaluated numerically.

5 Summary

Although virtually all of the available quantization recipes are ambiguous, one can still restrict one's attention to certain sufficiently elementary classical scenarios and succeed in the construction of a mathematically consistent quantum model the practical acceptability of which is to be verified via its appropriate experimental realization.

In such a setting the key motivation of our present study emerged after we imagined that the classical double-valued Hamiltonians could offer a unique possibility of moving over to the quantum sector thus opening up branched partners in such a setting too. We decided to follow this idea and to perform the first steps in this direction.

Our present results should be read as encouraging, confirming the purely technical feasibility of the general strategy we described. During its application we revealed that there exist several nontrivial examples of the branched Hamiltonians for which the quantization appeared not only feasible but also comparatively friendly.

Beyond this framework, our present approach also seems to open an entirely new direction of research in which one might be able to combine the generic numerical techniques with the non-numerical insight provided by the mathematics of multivalued functions in combination with the analytic tools of perturbation theory.

In the future research we expect that our present approach could also open a way towards the emergence of new qualitative features of branched Hamiltonians. In particular, we expect that the new classes of doublets of supersymmetric-partner Hamiltonians could emerge here in some exotic manner and in a way which has not yet been followed in the literature (cf., e.g., [35, 36, 37, 38]).

6 Acknowledgements

One of us (BB) thanks Anindya Ghose Choudhury and Partha Guha for discussions. We also thank the referee for constructive suggestions.

References

[1] N Bohr *Philos. Mag.* **26** (1913) 476

[2] F Bagarello, J-P Gazeau, F H Szafraniec and M Znojil, editors, *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects*, John Wiley & Sons, 2015

[3] M Znojil *Ann. Phys.* (NY) **361** (2015) 226; *J. Phys. A: Math. Theor.* **48** (2015) 195303

[4] M Znojil and G Levai *Phys. Lett.* **A 376** (2012) 3000

[5] C Quesne and V M Tkachuk *J. Phys. A: Math. Gen.* **37** (2004) 4267

[6] B Bagchi, A Banerjee, C Quesne and V M Tkachuk*J. Phys. A: Math. Gen.* **38** (2005) 2929

[7] B Bagchi and T Tanaka *Phys. Lett.* **A 372** (2008) 5390

[8] A Shapere and F Wilczek *Phys. Rev. Lett.*, **109** (2012) 160402, arXiv: 1202.2537

[9] A Shapere and F Wilczek *Phys. Rev. Lett.*, **109** (2012) 200402, arXiv: 1207.2677

[10] T L Curtright and C K Zachos *J. Phys. A: Math. Gen.* **43** (2010) 445101, arXiv: 1002.0104

[11] T L Curtright *SIGMA* **7** (2011) 042, arXiv: 1011.6056

[12] T L Curtright and A Veitia *Phys. Lett. A* **375** (2011) 276, arXiv: 1005.5030

[13] H C Rosu, S C Mancas and P Chen *Phys.Scr.* **90** (2015) 055208, arXiv: 1409.5365

[14] T L Curtright and C K Zachos *J. Phys. A: Math. Gen.* **47** (2014) 145201

[15] A Liénard *Revue g?en?erale de l ??electricit?e* **23** (1928) 901

[16] A Liénard *Revue g?en?erale de l ??electricit?e* **23** (1928) 946

[17] N J Doran and D Wood *Optics Lett.* **13** (1988) 56

[18] J Zhou, S Sun and Z Liu *App. Maths. and Comp.* **161** (2005) 655

[19] B Bagchi, A Ghose Choudhury and P Guha *J. Math. Phys.* **56** (2014) 012105, arXiv: 1305.4566

[20] J B Gong and Q H Wang, arXiv:1412.3549 (2014)

[21] K G Makris, L Ge and H Türeci, *Phys. Rev.* **X 4** (2014) 041044

[22] D Dizdarevic, D Dast, D Haag, J Main, H Cartarius and G. Wunner, *Phys. Rev.* **A 91** (2015) 033636

[23] T Dohnal and P Siegl, arXiv:1504.00054

[24] V Chithiika Ruby, M Senthivelan and M Lakshmanan *J. Phys. A: Math. Gen.* **45** (2012) 382002

[25] O von Roos *Phys. Rev.* **B 27** (1981) 7547, ibid. **B 31** (1985) 2294

[26] F Bagarello and F Gargano, *Phys. Rev.* **A 89** (2014) 0321132

[27] K Hashimoto, K. Kanki, H Hayakawa and T Petrosky, *Prog. Theor. Exp. Phys.* **2015** (2015) 023A02;

[28] T J Milburn, J Doppler, C A Holmes, S Portolan, S Rotter and P Rabl, arXiv:1410.1882 (2014)

[29] E M Graefe, H J Korsch, A Rush and R Schubert, *J. Phys. A: Math. Gen.* **48** (2015) 055301

[30] R Novak, *Int. J. Theor. Phys.* **doi:** 10.1007/s10773-015-2530-5 (in print)

[31] F Ruzicka, *Int. J. Theor. Phys.* **doi:** 10.1007/s10773-015-2531-4 (in print), arXiv:1501.04601

[32] V K Chandrasekar, M Senthivelan and M Lakshmanan *Phys. Rev. E***72** (2005) 066203, arXiv: 0408054 (nlin.SI)

[33] M Znojil *J. Phys. A: Math. Gen.* **27** (1994) 4945

[34] B H Bransden and C J Joachain *Introduction to Quantum Mechanics*, ELBS, 1989

[35] E Witten *Nucl. Phys.* **B 188** (1981) 513

[36] F Cooper, A Khare and U Sukhatme *Phys. Rep.***251** (1995) 267

[37] B Bagchi *Supersymmetry in Quantum and Classical Mechanics*, Chapman and Hall/CRC, 2000

[38] M Znojil, *J. Phys. A: Math. Gen.* **35** (2002) 2341