Existence of Different Intermediate Hamiltonians in Type A $\mathcal{N}$-fold Supersymmetry II. The $\mathcal{N}=3$ Case

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Abstract

We continue the previous study on the existence of different intermediate Hamiltonians in type A $\mathcal{N}$-fold supersymmetric systems and carry out an exhaustive investigation on the $\mathcal{N}=3$ case. In contrast with the $\mathcal{N}=2$ case, we find various patterns in the existence of intermediate Hamiltonians due to the presence of two different intermediate positions in a factorized type A 3-fold supercharge. In addition, all the $\mathcal{N}=3$ models are strictly restricted to at most elliptic type, which enables us to make the complete classification of the systems which admit intermediate Hamiltonians. Finally, we show realizations of third-order parasupersymmetry and variant generalized 3-fold superalgebras by such systems.

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I. INTRODUCTION

Recently we reported [1] on the existence of different intermediate Hamiltonians in type A $\mathcal{N}$-fold supersymmetry (SUSY) [2]. Apart from the connectivity to a wide range of solvable and quasi-solvable potentials including not only the well-known $\mathfrak{sl}(2)$ Lie-algebraic models [3] but also the new extended class of completely solvable rational ones [4], type A $\mathcal{N}$-fold SUSY is of independent interest because of the rich mathematical structure it provides (for a review, see Ref. [5]).

The main motivation of our study came from the issue of non-uniqueness of factorizing operators as a consequence of the underlying $GL(2,\mathbb{C})$ symmetry. Type A $\mathcal{N}$-fold supercharge admits of a one-parameter family of factorizations that is expressible as a product of $\mathcal{N}$ first-order linear differential operators due to the aforementioned symmetry [6]. This in turn implies that a type A $\mathcal{N}$-fold SUSY can have different intermediate Hamiltonians corresponding to different factorizations. However the existence of intermediate Hamiltonians is not guaranteed in general [2]. In Ref. [1], we investigated under what conditions type A $\mathcal{N}$-fold SUSY systems can admit intermediate Hamiltonians and that how many sets of such Hamiltonians are plausible for the specific case of $\mathcal{N}=2$. We then concluded that the number of admissible intermediate Hamiltonians would be a more suitable index than the concept of *reducibility* introduced in Refs. [7, 8] to characterize higher-order intertwining operators. Furthermore, we found that it naturally leads to a realization of second-order parasupersymmetry (paraSUSY) [9] and generalized 2-fold superalgebra [10].

In this article, we pursue our investigations further and focus on the $\mathcal{N}=3$ case. In comparison with the previous $\mathcal{N}=2$ case, there appear mainly two novel features in the $\mathcal{N}=3$ case. The one is the fact that the functional types of type A $\mathcal{N}$-fold SUSY potentials for $\mathcal{N}\geq 3$ are strictly restricted to at most elliptic functions due to an additional constraint. But due to this constraint, all the type A $\mathcal{N}$-fold SUSY models for $\mathcal{N}\geq 3$ were completely classified in Ref. [6]. We find that we can also classify entirely all the type A 3-fold SUSY models which have intermediate Hamiltonians.

The other novel feature of $\mathcal{N}=3$ is concerned with the fact that there are two different intermediate positions in each factorized 3-fold supercharge. Due to the latter fact, one can consider different patterns in the existence of intermediate Hamiltonians. That is, in certain cases systems admit intermediate Hamiltonians at both the two intermediate positions while in other cases systems admit them at only one of the two positions. Explorations of the underlying conditions for each case reveal that for the former classes a system with intermediate Hamiltonians at the two positions turns out to be always solvable and even shape invariant. On the other hand, for the latter classes a system with an intermediate Hamiltonian at only one position is led to quasi-solvability only.

Thanks to the aforementioned different patterns in the $\mathcal{N}=3$ case, we further find intriguing and quite rich structure which does not exist in the $\mathcal{N}=2$ case when we consider the existence of more than one sets of intermediate Hamiltonians. For instance, we find that there are systems which have intermediate Hamiltonians at the two positions in one factorization but has only one at one of the positions in another factorization. Throughout the analyses of such systems, we realize that in contrast to the $\mathcal{N}=2$ case we must consider not only the number of admissible intermediate Hamiltonians but also the variety in the existence of them to characterize $\mathcal{N}$th-order intertwining operators for the $\mathcal{N}\geq 3$ cases.

We also investigate a parafermionic formulation of such systems and realizations of third-order paraSUSY [11, 12] and generalized 3-fold superalgebra [13]. We find not only that such realizations are indeed possible but also that variations of the latter superalgebra hold according to the different patterns in the existence of intermediate Hamiltonians.

The article is organized as follows. In Section II, we review type A 3-fold SUSY and discuss some of its salient features. Then, we introduce three classes according to three different patterns in the existence of intermediate Hamiltonians. In Section III, we work out explicitly the necessary and sufficient conditions for each of the three classes. In Section IV, we classify exhaustively different type A 3-fold SUSY potentials with one set of intermediate Hamiltonians. In particular, we find an intimate relation between the patterns and the degree of solvability. That is, all the systems which have intermediate Hamiltonians at the two positions consist of shape invariant potentials. In contrast, those which have one at only one position comprise sextic anharmonic oscillators, deformed Pöschl-Teller or Scarf potentials, and one-body elliptic Inozemtsev potentials, all of which are quasi-solvable. In Section V, we discuss and study in full detail the necessary and sufficient conditions for a system to have different sets of intermediate Hamiltonians in various patterns. In Section VI, we then give the complete classification of the type A 3-fold SUSY models which admit simultaneously more than one sets of intermediate Hamiltonians. In Section VII, we show that a system with intermediate Hamiltonians at the two positions always admits a realization of third-order paraSUSY. In addition, each system belonging to one of the three classes always admits a realization of variant generalized 3-fold superalgebras. In the final section, we summarize and discuss various aspects of the obtained results.

II. TYPE A 3-FOLD SUPERSYMMETRY

A type A 3-fold SUSY system is given by

$$H^{\pm} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}W(x)^2 - \frac{1}{3}\left(2E'(x) - E(x)^2\right) - R \pm \frac{3}{2}W'(x), \tag{2.1a}$$

$$P_3^- = P_{31}^- P_{32}^- P_{33}^-, \qquad P_3^+ = -(P_3^-)^{\mathrm{T}} = P_{33}^+ P_{32}^+ P_{31}^+, \tag{2.1b}$$

where the superscript T denotes the transposition in the x-space and $P_{3i}^{\pm}$ $(i = 1, 2, 3)$ are defined by

$$P_{31}^{\pm} = \mp\partial + W - E, \qquad P_{32}^{\pm} = \mp\partial + W, \qquad P_{33}^{\pm} = \mp\partial + W + E. \tag{2.2}$$

In the expanded form, the type A 3-fold supercharge component P_3^- reads

$$\begin{aligned} P_3^- &= \partial^3 + 3W\partial^2 + \left(3W' + 3W^2 + 2E' - E^2\right)\partial \\ &\quad + W'' + 3WW' + W^3 + \left(2E' - E^2\right)W + \frac{1}{2}\left(2E' - E^2\right)'. \end{aligned} \tag{2.3}$$

The functions $E(x)$ and $W(x)$ are not arbitrary but are connected with a fourth-degree polynomial $A(z)$ and a second-degree polynomial $Q(z)$

$$A(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0, \tag{2.4a}$$

$$Q(z) = b_2 z^2 + b_1 z + b_0, \tag{2.4b}$$

through the following relations

$$z''(x) = E(x)z'(x), \quad 2A(z) = z'(x)^2, \quad Q(z) = -z'(x)W(x). \tag{2.5}$$

From these relations, we obtain in particular

$$W'(x) + E(x)W(x) = -Q'(z), \qquad E'(x) + E(x)^2 = A''(z), \tag{2.6}$$

which will play important roles in later analyses. The restriction on the form of $A(z)$ arises in type A $\mathcal{N}$-fold SUSY for all $\mathcal{N} \geq 3$ and is absent for $\mathcal{N} = 2$. This strongly limits the possible forms of potentials for $\mathcal{N} \geq 3$.

Due to the underlying algebraic structure of type A $\mathcal{N}$-fold SUSY systems, they are invariant under the linear projective transformations of $GL(2, \mathbb{C})$ defined by

$$z = \frac{\alpha w + \beta}{\gamma w + \delta}, \quad (\alpha, \beta\gamma, \delta \in \mathbb{C},\ \Delta = \alpha\delta - \beta\gamma \neq 0). \tag{2.7}$$

The polynomials $A(z)$ and $Q(z)$ are then transformed under the $GL(2, \mathbb{C})$ transformations as

$$A(z) \mapsto \hat{A}(w) = \Delta^{-2}(\gamma w + \delta)^4 A(z)\Big|_{z=\frac{\alpha w+\beta}{\gamma w+\delta}} = \sum_{k=0}^{4} \hat{a}_k w^k, \tag{2.8}$$

$$Q(z) \mapsto \hat{Q}(w) = \Delta^{-1}(\gamma w + \delta)^2 Q(z)\Big|_{z=\frac{\alpha w+\beta}{\gamma w+\delta}} = \sum_{k=0}^{2} \hat{b}_k w^k, \tag{2.9}$$

where the new coefficients $\hat{a}_i$ $(i = 0, \dots, 4)$ and $\hat{b}_i$ $(i = 0, 1, 2)$ are respectively given by

$$\begin{pmatrix} \hat{a}_4 \\ \hat{a}_3 \\ \hat{a}_2 \\ \hat{a}_1 \\ \hat{a}_0 \end{pmatrix} = \Delta^{-2} \begin{pmatrix} \alpha^4 & \alpha^3\gamma & \alpha^2\gamma^2 & \alpha\gamma^3 & \gamma^4 \\ 4\alpha^3\beta & \alpha^2(\alpha\delta + 3\beta\gamma) & 2\alpha\gamma(\alpha\delta + \beta\gamma) & \gamma^2(3\alpha\delta + \beta\gamma) & 4\gamma^3\delta \\ 6\alpha^2\beta^2 & 3\alpha\beta(\alpha\delta + \beta\gamma) & \alpha^2\delta^2 + 4\alpha\beta\gamma\delta + \beta^2\gamma^2 & 3\gamma\delta(\alpha\delta + \beta\gamma) & 6\gamma^2\delta^2 \\ 4\alpha\beta^3 & \beta^2(3\alpha\delta + \beta\gamma) & 2\beta\delta(\alpha\delta + \beta\gamma) & \delta^2(\alpha\delta + 3\beta\gamma) & 4\gamma\delta^3 \\ \beta^4 & \beta^3\delta & \beta^2\delta^2 & \beta\delta^3 & \delta^4 \end{pmatrix} \begin{pmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix}, \tag{2.10}$$

and

$$\begin{pmatrix} \hat{b}_2 \\ \hat{b}_1 \\ \hat{b}_0 \end{pmatrix} = \Delta^{-1} \begin{pmatrix} \alpha^2 & \alpha\gamma & \gamma^2 \\ 2\alpha\beta & \alpha\delta + \beta\gamma & 2\gamma\delta \\ \beta^2 & \beta\delta & \delta^2 \end{pmatrix} \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}. \tag{2.11}$$

From the formulas (2.5), the induced transformations of functions $E(x)$ and $W(x)$ read

$$W(x) \mapsto \hat{W}(x) = W(x), \qquad E(x) \mapsto \hat{E}(x) = E(x) - \frac{2\gamma z'(x)}{\gamma z(x) - \alpha}. \tag{2.12}$$

The latter transformation rule in particular implies the invariance of the following function:

$$2\hat{E}'(x) - \hat{E}(x)^2 = 2E'(x) - E(x)^2. \tag{2.13}$$

The $GL(2,\mathbb{C})$ invariance of any type A 3-fold SUSY system is now manifest since both the pair of Hamiltonians $H^{\pm}$ in (2.1a) and the operator P_3^- in (2.3) only depend on W and $2E' - E^2$ which are both invariant under any $GL(2,\mathbb{C})$ transformations.

The superHamiltonian $\boldsymbol{H}_3$ and the type A 3-fold supercharges $\boldsymbol{Q}_3^{\pm}$ introduced with the ordinary fermionic variables $\psi^{\pm}$ as

$$\boldsymbol{H}_3 = H^- \psi^- \psi^+ + H^+ \psi^+ \psi^-, \qquad \boldsymbol{Q}_3^{\pm} = P_3^{\mp} \psi^{\pm}, \tag{2.14}$$

satisfy the type A 3-fold superalgebra [6]

$$\left[\boldsymbol{Q}_3^{\pm}, \boldsymbol{H}_3\right] = \left\{\boldsymbol{Q}_3^{-}, \boldsymbol{Q}_3^{+}\right\} = 0, \tag{2.15a}$$

$$\begin{aligned}\left\{\boldsymbol{Q}_3^{-}, \boldsymbol{Q}_3^{+}\right\} &= 8(\boldsymbol{H}_3 + R)^3 - \frac{8}{3}\left(i_2[A] - 3D_2[Q]\right)(\boldsymbol{H}_3 + R) \\ &\quad + \frac{16}{27}\left(j_3[A] + 9I_{1,2}[A,Q]\right),\end{aligned} \tag{2.15b}$$

where $i_2[A]$, $D_2[Q]$, $j_3[A]$, and $I_{1,2}[A,Q]$ are the absolute invariants composed of $A(z)$ and $Q(z)$ as the followings (cf., Ref. [6] and the references cited therein):

$$\begin{aligned} i_2[A] &= 12a_0a_4 - 3a_1a_3 + a_2^2, \qquad D_2[Q] = 4b_0b_2 - b_1^2, \\ 2j_3[A] &= 72a_0a_2a_4 - 27a_0a_3^2 - 27a_1^2a_4 + 9a_1a_2a_3 - 2a_2^3, \\ I_{1,2}[A,Q] &= 6a_4b_0^2 - 3a_3b_0b_1 + 2a_2b_0b_2 + a_2b_1^2 - 3a_1b_1b_2 + 6a_0b_2^2. \end{aligned} \tag{2.16}$$

One of the most important aspects of the type A 3-fold SUSY system (2.1) is that its components satisfy the third-order intertwining relations:

$$P_{31}^- P_{32}^- P_{33}^- H^- = H^+ P_{31}^- P_{32}^- P_{33}^-, \quad P_{33}^+ P_{32}^+ P_{31}^+ H^+ = H^- P_{33}^+ P_{32}^+ P_{31}^+, \tag{2.17}$$

which are responsible for the first commutation relation in (2.15a). Before considering the central issues on the existence of intermediate Hamiltonians, we shall refer to another transformation property of the system. That is, the type A 3-fold SUSY system (2.1) is transformed under $W(x) \to -W(x)$ as follows:

$$H^- \leftrightarrow H^+, \quad P_{31}^- \leftrightarrow -P_{33}^+, \quad P_{32}^- \leftrightarrow -P_{32}^+, \quad P_{33}^- \leftrightarrow -P_{31}^+, \quad P_3^- \leftrightarrow -P_3^+. \tag{2.18}$$

Hence, the system as a whole remains invariant under the transformation $W(x) \to -W(x)$. In particular, it leaves the pair of intertwining relations (2.17) invariant. This transformation however does not belong to the $GL(2,\mathbb{C})$ transformation since it changes the sign of $W(x)$ while any $GL(2,\mathbb{C})$ transformation does not change $W(x)$ at all as shown in (2.12).

For type A 3-fold SUSY systems, there are essentially three different patterns in the existence of intermediate Hamiltonians according to which we shall classify them as follows:

Class $(1,1)$:

$$P_{33}^- H^- = H^{\mathrm{i}1} P_{33}^-, \qquad P_{33}^+ H^{\mathrm{i}1} = H^- P_{33}^+, \tag{2.19a}$$

$$P_{32}^- H^{\mathrm{i}1} = H^{\mathrm{j}1} P_{32}^-, \qquad P_{32}^+ H^{\mathrm{j}1} = H^{\mathrm{i}1} P_{32}^+, \tag{2.19b}$$

$$P_{31}^- H^{\mathrm{j}1} = H^+ P_{31}^-, \qquad P_{31}^+ H^+ = H^{\mathrm{j}1} P_{31}^+. \tag{2.19c}$$

Class $(0,1)$:

$$P_{32}^{-}P_{33}^{-}H^{-} = H^{\mathrm{j}1}P_{32}^{-}P_{33}^{-}, \qquad P_{33}^{+}P_{32}^{+}H^{\mathrm{j}1} = H^{-}P_{33}^{+}P_{32}^{+}, \tag{2.20a}$$
$$P_{31}^{-}H^{\mathrm{j}1} = H^{+}P_{31}^{-}, \qquad P_{31}^{+}H^{+} = H^{\mathrm{j}1}P_{31}^{+}. \tag{2.20b}$$

Class $(1,0)$:

$$P_{33}^{-}H^{-} = H^{\mathrm{i}1}P_{33}^{-}, \qquad P_{33}^{+}H^{\mathrm{i}1} = H^{-}P_{33}^{+}, \tag{2.21a}$$
$$P_{31}^{-}P_{32}^{-}H^{\mathrm{i}1} = H^{+}P_{31}^{-}P_{32}^{-}, \qquad P_{32}^{+}P_{31}^{+}H^{+} = H^{\mathrm{i}1}P_{32}^{+}P_{31}^{+}. \tag{2.21b}$$

That is, in Class $(1,1)$ we have considered a set of intermediate Hamiltonians $H^{\mathrm{i}1}$ and $H^{\mathrm{j}1}$ both at the place between $P_{31}^{\pm}$ and $P_{32}^{\pm}$ and at the place between $P_{32}^{\pm}$ and $P_{33}^{\pm}$ simultaneously while in Class $(0,1)$ and Class $(1,0)$ we have considered an intermediate Hamiltonian at only one intermediate place, at the place between $P_{31}^{\pm}$ and $P_{32}^{\pm}$ in the former and at the place between $P_{32}^{\pm}$ and $P_{33}^{\pm}$ in the latter.

But we easily find that the set of the intertwining relations (2.21) in Class $(1,0)$ is transformed to the one (2.20) in Class $(0,1)$ by the transformations (2.18) which are caused by $W(x) \to -W(x)$ provided that $H^{\mathrm{i}1}$ is transformed to $H^{\mathrm{j}1}$ simultaneously under the same transformation. We shall hereafter call the set of the transformations (2.18) accompanied with the interchange $H^{\mathrm{j}1} \leftrightarrow H^{\mathrm{i}1}$ the *reflective* transformation of a type A 3-fold SUSY system. Hence, as long as only one set of intermediate Hamiltonians is concerned, we only need to consider Class $(0,1)$ without any loss of generality. When we shall examine more than one sets of intermediate Hamiltonians, however, we must consider simultaneously both Class $(0,1)$ and Class $(1,0)$, as we shall discuss in Sections V and VI.

III. EXISTENCE OF INTERMEDIATE HAMILTONIANS

In this section, we shall derive the necessary and sufficient conditions for the existence of (at least) one set of intermediate Hamiltonians in each class classified in the last section. We note that Class $(1,1)$ can be regarded as a special case of either Class $(0,1)$ or Class $(1,0)$. Thus, we only consider models in Class $(0,1)$ and Class $(1,0)$ which do not belong to Class $(1,1)$ without any loss of generality. In the subsequent sections, we shall investigate each case in details.

A. Conditions for Class $(1,1)$

The necessary and sufficient condition for satisfying the first formula (2.19a) is that there exists a constant C_{33} such that H^{-} and $H^{\mathrm{i}1}$ are expressed as

$$2H^{-} = P_{33}^{+}P_{33}^{-} + 2C_{33} = -\partial^2 - W' - E' + W^2 + 2EW + E^2 + 2C_{33}, \tag{3.1a}$$
$$2H^{\mathrm{i}1} = P_{33}^{-}P_{33}^{+} + 2C_{33} = -\partial^2 + W' + E' + W^2 + 2EW + E^2 + 2C_{33}. \tag{3.1b}$$

Similarly, the necessary and sufficient condition for satisfying the second formula in (2.19b) is that there exists another constant C_{32} such that $H^{\mathrm{i}1}$ and $H^{\mathrm{j}1}$ are expressed as

$$2H^{\mathrm{i}1} = P_{32}^{+}P_{32}^{-} + 2C_{32} = -\partial^2 - W' + W^2 + 2C_{32}, \tag{3.2a}$$
$$2H^{\mathrm{j}1} = P_{32}^{-}P_{32}^{+} + 2C_{32} = -\partial^2 + W' + W^2 + 2C_{32}, \tag{3.2b}$$

and the necessary and sufficient condition for satisfying the third formula in (2.19c) is that there exists another constant C_{31} such that $H^{\mathrm{j}1}$ and H^{+} are expressed as

$$2H^{\mathrm{j}1} = P_{31}^{+}P_{31}^{-} + 2C_{31} = -\partial^2 - W' + E' + W^2 - 2EW + E^2 + 2C_{31}, \tag{3.3a}$$

$$2H^{+} = P_{31}^{-}P_{31}^{+} + 2C_{31} = -\partial^2 + W' - E' + W^2 - 2EW + E^2 + 2C_{31}. \tag{3.3b}$$

Comparing (2.1a), (3.1), (3.2), and (3.3) each other, we obtain

$$6(W' + EW) + E' + E^2 = -6(R + C_{33}), \tag{3.4a}$$

$$2(W' + EW) + E' + E^2 = 2(C_{32} - C_{33}), \tag{3.4b}$$

$$2(W' + EW) - E' - E^2 = 2(C_{31} - C_{32}), \tag{3.4c}$$

$$6(W' + EW) - E' - E^2 = 6(C_{31} + R). \tag{3.4d}$$

Arranging the set of formulas (3.4) and using the relations (2.6), we have

$$2Q'(z) = C_{33} - C_{31}, \quad A''(z) = -3(2R + C_{33} + C_{31}) = 2C_{32} - C_{33} - C_{31}. \tag{3.5}$$

In terms of the coefficients of polynomials $A(z)$ and $Q(z)$ in (2.4), the necessary and sufficient conditions (3.5) are written as

$$\begin{aligned} a_4 = a_3 = b_2 = 0, &\qquad 2a_2 = 2C_{32} - C_{33} - C_{31}, \\ 2b_1 = C_{33} - C_{31}, &\qquad -3R = C_{33} + C_{32} + C_{31}. \end{aligned} \tag{3.6}$$

The last three equalities in (3.6) just determine the parameters C_{31}, C_{32}, and C_{33} in terms of the parameters of the original type A 3-fold SUSY system as

$$C_{31} = -\frac{a_2}{3} - b_1 - R, \qquad C_{32} = \frac{2a_2}{3} - R, \qquad C_{33} = -\frac{a_2}{3} + b_1 - R. \tag{3.7}$$

We note that the first equality in (3.6) is identical to the solvability condition of type A $\mathcal{N}$-fold SUSY systems (cf., Ref. [14], Eq. (6.13)), namely, the necessary and sufficient condition for the pair of any type A $\mathcal{N}$-fold SUSY Hamiltonians $H^{\pm}$ to be not only quasi-solvable but also solvable. Hence, a type A 3-fold SUSY system with a set of two intermediate Hamiltonians $H^{\mathrm{i}1}$ and $H^{\mathrm{i}2}$ is always solvable, and conversely a solvable type A 3-fold SUSY system always admits a set of two intermediate Hamiltonians.

B. Conditions for Class $(0, 1)$

To consider the first formula in (2.20a), we first note that the second-order operator $P_{32}^{-}P_{33}^{-}$ belongs to a type A 2-fold supercharge

$$P_{32}^{-}P_{33}^{-} = \left(\partial + W_1 - \frac{E}{2}\right)\left(\partial + W_1 + \frac{E}{2}\right), \tag{3.8}$$

with

$$W_1(x) = W(x) + \frac{E(x)}{2}. \tag{3.9}$$

Hence, the necessary and sufficient conditions for satisfying the first formula (2.20a) are that there exists a second-degree polynomial $Q_1(z)$ in z such that (cf., Ref. [6])

$$Q_1(z) = -z'(x)W_1(x), \qquad \frac{\mathrm{d}^3}{\mathrm{d}z^3}Q_1(z) = 0, \tag{3.10}$$

and that H^- and $H^{\mathrm{j}1}$ are expressed as

$$2H^- = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_1^2 - \frac{E'}{2} + \frac{E^2}{4} - 2R_1 - 2W_1', \tag{3.11a}$$

$$2H^{\mathrm{j}1} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_1^2 - \frac{E'}{2} + \frac{E^2}{4} - 2R_1 + 2W_1', \tag{3.11b}$$

where R_1 is a constant. The necessary and sufficient condition for satisfying the second formula (2.20b) is that there exists another constant C_{31} such that $H^{\mathrm{j}1}$ and H^+ are expressed as

$$2H^{\mathrm{j}1} = P_{31}^+P_{31}^- + 2C_{31} = -\partial^2 - W' + E' + W^2 - 2EW + E^2 + 2C_{31}, \tag{3.12a}$$

$$2H^+ = P_{31}^-P_{31}^+ + 2C_{31} = -\partial^2 + W' - E' + W^2 - 2EW + E^2 + 2C_{31}. \tag{3.12b}$$

Comparing (2.1a), (3.11), and (3.12) each other, and using (3.9), we obtain

$$6(W' + EW) - E' - E^2 = -12(R - R_1) = 4(C_{31} + R_1) = 6(C_{31} + R), \tag{3.13}$$

which is equivalent, in view of the relations (2.6), to

$$-6Q'(z) - A''(z) = -12(R - R_1) = 4(C_{31} + R_1) = 6(C_{31} + R). \tag{3.14}$$

The condition (3.10), if combined with (2.5), leads to

$$A'(z) = 2\left[Q(z) - Q_1(z)\right], \tag{3.15}$$

which means that $A'(z)$ is also a polynomial of at most second degree. In terms of the coefficients of polynomials $A(z)$ and $Q(z)$ in (2.4), the necessary and sufficient conditions (3.14) and (3.15) are written as

$$a_4 = a_3 + 2b_2 = 0, \quad a_2 + 3b_1 = -2(C_{31} + R_1), \quad 3R = 2R_1 - C_{31}. \tag{3.16}$$

The last two equalities in (3.16) just determine the constants C_{31} and R_1 in terms of the parameters of the original type A 3-fold SUSY system as

$$C_{31} = -\frac{a_2}{3} - b_1 - R, \qquad R_1 = -\frac{a_2}{6} - \frac{b_1}{2} + R. \tag{3.17}$$

We now find that a model in Class $(0, 1)$ which satisfies (3.16) can also satisfy the condition (3.6) and thus belong to Class $(1, 1)$ if and only if $a_3 = b_2 = 0$. Hence, we assume $a_3b_2 \neq 0$ without any loss of generality in the subsequent analyses of Class $(0, 1)$. With the latter assumption, however, the solvability condition cannot be satisfied inevitably. Hence, all the models in Class $(0, 1)$ are only quasi-solvable but may not be completely solvable.

Finally, we note that the superHamiltonian and type A 2-fold supercharges constructed from H^-, $H^{\mathrm{j}1}$, and $P_{32}^-P_{33}^-$ form a type A 2-fold superalgebra since H^- and $H^{\mathrm{j}1}$ are a type

A 2-fold SUSY pair with respect to the operator $P_{32}^{-}P_{33}^{-}$. In particular, the anti-commutator of the type A 2-fold supercharges constructed in this way reads in components

$$P_{33}^{+}P_{32}^{+}P_{32}^{-}P_{33}^{-} = 4S_2^{(0,1)}(H^{-} + R), \qquad P_{32}^{-}P_{33}^{-}P_{33}^{+}P_{32}^{+} = 4S_2^{(0,1)}(H^{\mathrm{j1}} + R), \tag{3.18}$$

where $S_2^{(0,1)}$ is a monic polynomial of second-degree given by

$$S_2^{(0,1)}(t) = t^2 - \left(\frac{a_2}{3} + b_1\right) t - 2(a_1 - 2b_0)b_2 - \frac{2}{9}(a_2 - 3b_1)a_2. \tag{3.19}$$

C. Conditions for Class $(1, 0)$

The analysis for Class $(1, 0)$ is almost the same as for Class $(0, 1)$ in the previous section. That is, the first formula in (2.21) requires that H^{-} and H^{i1} must form an ordinary SUSY pair with respect to $P_{33}^{\pm}$:

$$2H^{-} = P_{33}^{+}P_{33}^{-} + 2C_{33} = -\partial^2 - W' - E' + W^2 + 2EW + E^2 + 2C_{33}, \tag{3.20a}$$

$$2H^{\mathrm{i1}} = P_{33}^{-}P_{33}^{+} + 2C_{33} = -\partial^2 + W' + E' + W^2 + 2EW + E^2 + 2C_{33}, \tag{3.20b}$$

while the second formula in (2.21) requires that H^{i1} and H^{+} must form a type A 2-fold SUSY pair with respect to $P_{31}^{-}P_{32}^{-}$ and its conjugate:

$$2H^{\mathrm{i1}} = -\partial^2 + W_2^{\,2} - \frac{E'}{2} + \frac{E^2}{4} - 2R_2 - 2W_2', \tag{3.21a}$$

$$2H^{+} = -\partial^2 + W_2^{\,2} - \frac{E'}{2} + \frac{E^2}{4} - 2R_2 + 2W_2', \tag{3.21b}$$

where $W_2(x) = W(x) - E(x)/2$. By following a similar route as in the previous section, it is straightforward to show that the above requirements are satisfied if and only if

$$a_4 = a_3 - 2b_2 = 0, \quad C_{33} = -\frac{a_2}{3} + b_1 - R, \quad R_2 = -\frac{a_2}{6} + \frac{b_1}{2} + R. \tag{3.22}$$

The last two equalities in the above just determine the constants C_{33} and R_2 and thus only the first formula stands essentially as the necessary and sufficient condition for a system to belong to Class $(1, 0)$. As in the case of Class $(0, 1)$, we assume $a_3 b_2 \neq 0$ without any loss of generality to prevent a model in Class $(1, 0)$ from belonging also to Class $(1, 1)$. Hence, by the assumption, any system in Class $(1, 0)$ is only quasi-solvable but may not be completely solvable. In addition, the superHamiltonian and type A 2-fold supercharges constructed from H^{i1}, H^{+}, and $P_{31}^{-}P_{32}^{-}$ form another type A 2-fold superalgebra since H^{i1} and H^{+} are a type A 2-fold SUSY pair with respect to the operator $P_{31}^{-}P_{32}^{-}$. In particular, the anti-commutator of the type A 2-fold supercharges constructed in this way reads in components

$$P_{32}^{+}P_{31}^{+}P_{31}^{-}P_{32}^{-} = 4S_2^{(1,0)}(H^{\mathrm{i1}} + R), \qquad P_{31}^{-}P_{32}^{-}P_{32}^{+}P_{31}^{+} = 4S_2^{(0,1)}(H^{+} + R), \tag{3.23}$$

where $S_2^{(1,0)}$ is a monic polynomial of second-degree given by

$$S_2^{(1,0)}(t) = t^2 - \left(\frac{a_2}{3} - b_1\right) t + 2(a_1 + 2b_0)b_2 - \frac{2}{9}(a_2 + 3b_1)a_2. \tag{3.24}$$

IV. CLASSIFICATION: CASES OF ONLY ONE SET

We are now in a position to classify completely all the possible type A 3-fold SUSY potentials which admit one and only one set of intermediate Hamiltonians. In the previous section, we showed that there are three different classes, Class $(1,1)$, Class $(0,1)$, and Class $(1,0)$, and that the necessary and sufficient conditions for the existence of intermediate Hamiltonians are

$$\text{Class } (1,1): \quad a_4 = a_3 = b_2 = 0, \tag{4.1}$$

$$\text{Class } (0,1): \quad a_4 = a_3 + 2b_2 = 0 \quad (a_3 b_2 \neq 0), \tag{4.2}$$

$$\text{Class } (1,0): \quad a_4 = a_3 - 2b_2 = 0 \quad (a_3 b_2 \neq 0), \tag{4.3}$$

where we have neglected the other irrelevant constraints on the parameter relations. It is evident that these conditions explicitly break the $GL(2,\mathbb{C})$ covariance of the original type A 3-fold systems. Hence, we cannot apply the same classification scheme as the one in Ref. [6] which employs the full $GL(2,\mathbb{C})$ covariance of the systems. However, we note that there exists a residual symmetry remained intact in the present cases. In fact, the transformation formulas (2.12) for the functions $E(x)$ and $W(x)$ tell us that for the set of linear projective transformations with $\gamma = 0$ the functions $E(x)$ and $W(x)$ are both invariant. The latter fact means in particular that each factor of the type A 3-fold supercharges $P_{3i}^{\pm}$ $(i = 1, 2, 3)$ in (2.2) and thus the intermediate Hamiltonians $H^{\mathrm{i}1}$ and $H^{\mathrm{j}1}$ introduced through the intertwining relations (2.19)–(2.21) as well are all invariant under an arbitrary $GL(2,\mathbb{C})$ transformation with $\gamma = 0$. The set of linear projective transformations with $\gamma = 0$ consists of the set of (complex) inhomogeneous linear transformations

$$z = \alpha w + \beta \quad (\alpha, \beta \in \mathbb{C}, \alpha \neq 0), \tag{4.4}$$

where we have set $\delta = 1$ without any loss of generality. For an arbitrary inhomogeneous linear transformation (4.4), the transformation matrices of the parameters a_i $(i = 0, \dots, 4)$ and b_i $(i = 0, 1, 2)$ given by (2.10) and (2.11) respectively become triangle, and all of the conditions (4.1)–(4.3) are covariant, that is,

$$a_4 = a_3 = b_2 = 0 \quad \Rightarrow \quad \hat{a}_4 = \hat{a}_3 = \hat{b}_2 = 0. \tag{4.5}$$

$$a_4 = a_3 \pm 2b_2 = 0 \quad \Rightarrow \quad \hat{a}_4 = 0,\ \hat{a}_3 \pm 2\hat{b}_2 = \alpha(a_3 \pm 2b_2) = 0. \tag{4.6}$$

Therefore, all type A 3-fold SUSY systems with one set of intermediate Hamiltonians are classified by considering the equivalence class under the set of inhomogeneous linear transformations (4.4). We now easily show that the representatives of $A(z)$ under this equivalence class in each case can be chosen, by noting the constraint $a_3 \neq 0$ in Class $(0,1)$ and Class $(1,0)$, as listed in Table I. We note that Case II and Case II' (Case IV and Case IV', respectively) are transformed to each other by a projective transformation of $GL(2,\mathbb{C})$ but not by any inhomogeneous linear transformation (4.4). That is the reason why we must treat them as separate cases though they are classified as one equivalent case in the classification of general type A $\mathcal{N}$-fold SUSY models [6]. The still arbitrarily determinable constant a in Table I can be fixed by considering the scaling relations under the scale transformations of the parameter space spanned by $\{a_i, b_i, R\}$:

$$z(x; \nu a_i, \nu b_i, \nu R) = z(\sqrt{\nu} x; a_i, b_i, R), \tag{4.7a}$$

Case	Class $(1,1)$	Class $(0,1)$, $(1,0)$
I	$a/2$	
II	$2z$	
II'		$2z^3$
III	$z^2/2$	
IV	$2a(z^2-1)$	
IV'		$2az^2(z+1)$
V		$2z^3 - g_2 z/2 - g_3/2$

TABLE I: The complex classification scheme of type A 3-fold SUSY models with one set of intermediate Hamiltonians. In the above, $g_2, g_3 \in \mathbb{C}$ satisfy $g_2^3 \neq 27g_3^2$ and $a \in \mathbb{C}$ is an arbitrary constant.

$$E(x;\nu a_i,\nu b_i,\nu R) = \sqrt{\nu}E(\sqrt{\nu}x;a_i,b_i,R), \tag{4.7b}$$
$$W(x;\nu a_i,\nu b_i,\nu R) = \sqrt{\nu}W(\sqrt{\nu}x;a_i,b_i,R), \tag{4.7c}$$
$$V(x;\nu a_i,\nu b_i,\nu R) = \nu V(\sqrt{\nu}x;a_i,b_i,R). \tag{4.7d}$$

Hence, in what follows we set $a=1$ in all the cases without any loss of generality. For the classification, we recall the fact that Class $(0,1)$ and Class $(1,0)$ are connected with each other by the reflective transformation. Hence, we shall present only models belonging to Class $(0,1)$. We shall show in each case the change of variable $z=z(x)$ and the two functions $E(x)$ and $W(x)$ determined by (2.5) as well as the potential parts of the 3-fold SUSY pair Hamiltonians $H^{\pm}$ and of the intermediate Hamiltonian(s) $H^{\mathrm{i}1}$ and/or $H^{\mathrm{j}1}$ determined by (3.1)–(3.3) in Class $(1,1)$ and by (3.11)–(3.12) in Class $(0,1)$.

A. Classification of Class $(1,1)$

I) $A(z) = 1/2$.

Functions:

$$z(x) = x, \qquad E(x) = 0, \qquad W(x) = -b_1 x - b_0. \tag{4.8}$$

Potentials:

$$2V^{-}(x) = (b_1 x + b_0)^2 + 3b_1 - 2R, \tag{4.9a}$$
$$2V^{\mathrm{i}1}(x) = (b_1 x + b_0)^2 + b_1 - 2R, \tag{4.9b}$$
$$2V^{\mathrm{j}1}(x) = (b_1 x + b_0)^2 - b_1 - 2R, \tag{4.9c}$$
$$2V^{+}(x) = (b_1 x + b_0)^2 - 3b_1 - 2R. \tag{4.9d}$$

In this case, all the potentials (4.9) are harmonic oscillators.

II) $A(z) = 2z$.

Functions:

$$z(x) = x^2, \qquad E(x) = \frac{1}{x}, \qquad W(x) = -\frac{b_1}{2}x - \frac{b_0}{2x}. \tag{4.10}$$

Potentials:

$$V^-(x) = \frac{b_1^2}{8}x^2 + \frac{(b_0-2)(b_0-4)}{8x^2} + \frac{b_1(b_0+3)}{4} - R, \tag{4.11a}$$
$$V^{\mathrm{i}1}(x) = \frac{b_1^2}{8}x^2 + \frac{b_0(b_0-2)}{8x^2} + \frac{b_1(b_0+1)}{4} - R, \tag{4.11b}$$
$$V^{\mathrm{j}1}(x) = \frac{b_1^2}{8}x^2 + \frac{b_0(b_0+2)}{8x^2} + \frac{b_1(b_0-1)}{4} - R, \tag{4.11c}$$
$$V^+(x) = \frac{b_1^2}{8}x^2 + \frac{(b_0+2)(b_0+4)}{8x^2} + \frac{b_1(b_0-3)}{4} - R. \tag{4.11d}$$

In this case, all the potentials (4.11) are radial harmonic oscillators with centrifugal potentials.

III) $A(z) = z^2/2$.

Functions:

$$z(x) = \mathrm{e}^x, \qquad E(x) = 1, \qquad W(x) = -b_1 - b_0\mathrm{e}^{-x}. \tag{4.12}$$

Potentials:

$$2V^-(x) = b_0(2b_1-3)\mathrm{e}^{-x} + b_0^2\mathrm{e}^{-2x} - 2\bar{R}, \tag{4.13a}$$
$$2V^{\mathrm{i}1}(x) = b_0(2b_1-1)\mathrm{e}^{-x} + b_0^2\mathrm{e}^{-2x} - 2\bar{R}, \tag{4.13b}$$
$$2V^{\mathrm{j}1}(x) = b_0(2b_1+1)\mathrm{e}^{-x} + b_0^2\mathrm{e}^{-2x} - 2\bar{R}, \tag{4.13c}$$
$$2V^+(x) = b_0(2b_1+3)\mathrm{e}^{-x} + b_0^2\mathrm{e}^{-2x} - 2\bar{R}, \tag{4.13d}$$

where $\bar{R} = R - b_1^2/2 - 1/3$ is a constant. In this case, all the potentials (4.13) are Morse potentials.

IV) $A(z) = 2(z^2-1)$.

Functions:

$$z(x) = \cosh 2x, \qquad E(x) = \frac{2\cosh 2x}{\sinh 2x}, \qquad W(x) = -\frac{b_1\cosh 2x + b_0}{2\sinh 2x}. \tag{4.14}$$

Potentials:

$$V^-(x) = \frac{2b_0(b_1-6)\cosh 2x + b_0^2 + (b_1-4)(b_1-8)}{8\sinh^2 2x} - \bar{R}, \tag{4.15a}$$
$$V^{\mathrm{i}1}(x) = \frac{2b_0(b_1-2)\cosh 2x + b_0^2 + b_1(b_1-4)}{8\sinh^2 2x} - \bar{R}, \tag{4.15b}$$
$$V^{\mathrm{j}1}(x) = \frac{2b_0(b_1+2)\cosh 2x + b_0^2 + b_1(b_1+4)}{8\sinh^2 2x} - \bar{R}, \tag{4.15c}$$

$$V^{+}(x) = \frac{2b_0(b_1+6)\cosh 2x + b_0^2 + (b_1+4)(b_1+8)}{8\sinh^2 2x} - \bar{R}, \tag{4.15d}$$

where $\bar{R} = R - b_1^2/8 - 4/3$ is a constant. In this case, all the potentials (4.15) are Pöschl–Teller potentials. We note that a system of Scarf potentials can be obtained by the choice $A(z) = 2(z^2+1)$ which is connected with the representative $A(z) = 2(z^2-1)$ by the combination of a complex linear transformation (4.4) and a scale transformation (4.7).

It is interesting to observe that all the potentials given by (4.9), (4.11), (4.13), and (4.15) are shape invariant as can be easily checked by some obvious scaling of parameters [15]. It bears mention that the schemes of SUSY quantum mechanics [15–18] are consistent with the factorization method [19], intertwining relationships [20], and the shape invariance condition [21]. The latter was first utilized by Gendenshtein [21] to track down solvable potentials. To be shape invariant the partner potentials while sharing a similar coordinate dependence can at most differ in the presence of some parameters as precisely has happened in the potentials above.

B. Classification of Class $(0,1)$

II') $A(z) = 2z^3$.

Functions:

$$z(x) = \frac{1}{x^2}, \qquad E(x) = -\frac{3}{x}, \qquad W(x) = -\frac{1}{2x} + \frac{b_1}{2}x + \frac{b_0}{2}x^3. \tag{4.16}$$

Potentials:

$$V^{-}(x) = \frac{b_0^2}{8}x^6 + \frac{b_1 b_0}{4}x^4 + \frac{b_1^2 - 20b_0}{8}x^2 + \frac{3}{8x^2} - b_1 - R, \tag{4.17a}$$

$$V^{\mathrm{j}1}(x) = \frac{b_0^2}{8}x^6 + \frac{b_1 b_0}{4}x^4 + \frac{b_1^2 + 8b_0}{8}x^2 + \frac{35}{8x^2} - R, \tag{4.17b}$$

$$V^{+}(x) = \frac{b_0^2}{8}x^6 + \frac{b_1 b_0}{4}x^4 + \frac{b_1^2 + 16b_0}{8}x^2 + \frac{15}{8x^2} + \frac{b_1}{2} - R. \tag{4.17c}$$

In this case, all the potentials (4.17) are well-known quasi-solvable sextic anharmonic oscillators [22].

IV') $A(z) = 2z^2(z+1)$.

Functions:

$$z(x) = \frac{1}{\sinh^2 x}, \quad E(x) = -\frac{3 + 2\sinh^2 x}{\sinh x \cosh x}, \quad W(x) = \frac{b_0 \sinh^4 x + b_1 \sinh^2 x - 1}{2\sinh x \cosh x}. \tag{4.18}$$

Potentials:

$$\begin{aligned} V^{-}(x) = {} & \frac{b_0^2}{8}\cosh^4 x + \frac{(2b_1 - 3b_0 - 12)b_0}{8}\cosh^2 x \\ & - \frac{(b_1 - b_0 + 3)(b_1 - b_0 + 5)}{8\cosh^2 x} + \frac{3}{8\sinh^2 x} + \frac{b_0}{2} - \bar{R}, \end{aligned} \tag{4.19a}$$

$$
\begin{aligned}
V^{\mathrm{j1}}(x) &= \frac{b_0^2}{8}\cosh^4 x + \frac{(2b_1 - 3b_0 + 4)b_0}{8}\cosh^2 x \\
&\quad - \frac{(b_1 - b_0 - 1)(b_1 - b_0 + 1)}{8\cosh^2 x} + \frac{35}{8\sinh^2 x} - \frac{b_0}{2} - \bar{R}, \qquad (4.19\mathrm{b}) \\
V^{+}(x) &= \frac{b_0^2}{8}\cosh^4 x + \frac{(2b_1 - 3b_0 + 12)b_0}{8}\cosh^2 x \\
&\quad - \frac{(b_1 - b_0 - 3)(b_1 - b_0 - 1)}{8\cosh^2 x} + \frac{15}{8\sinh^2 x} - b_0 - \bar{R}, \qquad (4.19\mathrm{c})
\end{aligned}
$$

where $\bar{R} = R - (b_1 - b_0)(b_1 - 3b_0)/8 - 4/3$ is a constant. In this case, all the potentials (4.19) are quasi-solvable deformed Pöschl–Teller or Scarf potentials [23].

V) $A(z) = 2z^3 - g_2 z/2 - g_3/2$.

Functions:

$$
z(x) = \wp(x), \qquad E(x) = \frac{\wp''(x)}{\wp'(x)}, \qquad W(x) = \frac{\wp(x)^2 - b_1\wp(x) - b_0}{\wp'(x)}. \qquad (4.20)
$$

Potentials:

$$
\begin{aligned}
V^{-}(x) &= \frac{b_1\wp(x) + \bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x) + \bar{b}_0 - \frac{10}{3}\wp''(x)\right] - \frac{8}{9}\wp(x) + \frac{91}{18}\wp(2x) + \frac{3b_1}{2} - R, \qquad (4.21\mathrm{a}) \\
V^{\mathrm{j1}}(x) &= \frac{b_1\wp(x) + \bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x) + \bar{b}_0 + \frac{2}{3}\wp''(x)\right] + \frac{40}{9}\wp(x) - \frac{5}{18}\wp(2x) - \frac{b_1}{2} - R, \qquad (4.21\mathrm{b}) \\
V^{+}(x) &= \frac{b_1\wp(x) + \bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x) + \bar{b}_0 + \frac{8}{3}\wp''(x)\right] + \frac{10}{9}\wp(x) + \frac{55}{18}\wp(2x) - \frac{3b_1}{2} - R. \qquad (4.21\mathrm{c})
\end{aligned}
$$

In the above, $\wp(x)$ is the Weierstrass elliptic function and $\bar{b}_0 = b_0 - g_2/12$ is another parameter. The first term of each the potential is a rational function of $\wp(x)$ by the formulas $\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3$ and $\wp''(x) = 6\wp(x)^2 - g_2/2$. In this case, all the potentials (4.21) are quasi-solvable one-body elliptic BC type Inozemtsev potentials (cf., Ref. [24] and the references cited therein).

V. DIFFERENT SETS OF INTERMEDIATE HAMILTONIANS

In this section, we shall study under what conditions a type A 3-fold SUSY system can have more than one sets of intermediate Hamiltonians. The latter possibility originates from the fact that each factor of a type A $\mathcal{N}$-fold supercharge in a factorized form is not invariant under a subset of the $GL(2,\mathbb{C})$ transformations although any type A $\mathcal{N}$-fold supercharge as a whole is invariant under all the $GL(2,\mathbb{C})$ transformations [6]. For the $\mathcal{N} = 3$ case, we easily check the latter fact directly from the definition (2.2) and the transformation formulas (2.12):

$$
P_{31}^{\pm}[\hat{W},\hat{E}] = P_{31}^{\pm}[W,E] + \frac{2\gamma z'}{\gamma z - \alpha}, \quad P_{33}^{\pm}[\hat{W},\hat{E}] = P_{33}^{\pm}[W,E] - \frac{2\gamma z'}{\gamma z - \alpha}. \qquad (5.1)
$$

Hence, the factors $P_{31}^{\pm}$ and $P_{33}^{\pm}$ in $P_3^{\pm}$ are in fact not invariant under any $GL(2,\mathbb{C})$ transformation with $\gamma \neq 0$. It means in particular that intermediate Hamiltonians H^{i1} and/or

$H^{\mathrm{j}1}$ defined by the relations (2.19), (2.20), or (2.21) could not be invariant under any transformation with $\gamma \neq 0$, even if they exist after the transformation. In fact, the existence of intermediate Hamiltonians after a transformation is not guaranteed automatically since the conditions for the existence (4.1) or (4.2) are not preserved under any $GL(2,\mathbb{C})$ transformation with $\gamma \neq 0$ as was already discussed in Section IV. Therefore, another different set of intermediate Hamiltonians exists if there is a $GL(2,\mathbb{C})$ transformation with $\gamma \neq 0$ for which the transformed parameters $\hat{a}_i$ and $\hat{b}_i$ given by (2.10) and (2.11) also satisfy the existence conditions (4.1) or (4.2). Furthermore, if there exist simultaneously such $n(>1)$ $GL(2,\mathbb{C})$ transformations characterized by the sets $\{\alpha_i,\beta_i,\gamma_i,\delta_i\}_{i=1}^{n}$ satisfying $\gamma_i \neq 0$ and $\alpha_i/\gamma_i \neq \alpha_j/\gamma_j$ for all $i \neq j$, then the corresponding n sets of intermediate Hamiltonians are different from each other since the formula (2.12) tells us that the deformation is characterized by the one parameter α/γ. We shall hereafter say that two $GL(2,\mathbb{C})$ transformations with $\gamma_i,\gamma_j \neq 0$ are *inequivalent* if $\alpha_i/\gamma_i \neq \alpha_j/\gamma_j$.

In the classification of the systems which admit one set of intermediate Hamiltonians in Section IV, we considered the two different classes, namely, Class $(1,1)$ where an intermediate Hamiltonian exists at each of the two intermediate positions in the factorized type A 3-fold supercharge and Class $(0,1)$ where an intermediate Hamiltonian exists only at one intermediate position. And any system belonging to Class $(1,0)$ can be obtained from the corresponding system belonging to Class $(0,1)$ by the reflective transformation $W(x) \to -W(x)$. Accordingly, we can consider a type A 3-fold SUSY system which admits simultaneously m different sets of intermediate Hamiltonians $\{H^{\mathrm{i}k},H^{\mathrm{j}k}\}_{k=1}^{m}$ of Class $(1,1)$ and n different sets of an intermediate Hamiltonian $\{H^{\mathrm{j}k}\}_{k=m+1}^{m+n}$ of Class $(0,1)$. We shall call such a class of systems Class $(m,m+n)$ with an obvious implication of the terminology. It is evident that any system which admits simultaneously m different sets of intermediate Hamiltonians $\{H^{\mathrm{i}k},H^{\mathrm{j}k}\}_{k=1}^{m}$ of Class $(1,1)$ and n different sets of an intermediate Hamiltonian $\{H^{\mathrm{i}k}\}_{k=m+1}^{m+n}$ of Class $(1,0)$, which would be called to belong Class $(m+n,m)$, can be obtained from the corresponding system belonging to Class $(m,m+n)$ by the reflective transformation (with the obvious accompaniment of the interchanges $H^{\mathrm{j}k} \leftrightarrow H^{\mathrm{i}k}$ for $k>1$). To investigate each class systematically, we first note that any system which belongs to Class $(m,m+n)$ with $m,n>0$ always belongs to Class $(1,1)$ since the conditions for Class $(1,1)$ is stricter than for Class $(0,1)$. In other words, an arbitrary system in Class $(m,m+n)$ with $m>0$ is a special case of one of the four systems in Class $(1,1)$ classified in Section IV A and thus is always solvable in particular. Hence, only Class $(0,n)$ with $n>1$, which is an abbreviation for Class $(0,0+n)$, can have a quasi-solvable system which must be a special case of one of the three systems in Class $(0,1)$ classified in Section IV B.

In addition to these classes, there could exist *hybrid* classes. That is, there is the possibility that a type A 3-fold SUSY system admits simultaneously different sets of intermediate Hamiltonians some of which belong to Class $(0,1)$ and the others of which belong to Class $(1,0)$. These systems, if exist, could belong to neither Class $(m,m+n)$ nor Class $(0,n)$ since Class $(0,1)$ and Class $(1,0)$ are, as already mentioned, connected by the reflective transformation but not by any $GL(2,\mathbb{C})$ transformation. We can thus consider a class of systems belonging to Class $(m,m+n)$ which admit simultaneously l additional different sets of intermediate Hamiltonians $\{H^{\mathrm{i}k}\}_{k=m+1}^{m+l}$ of Class $(1,0)$. We shall call such a class of systems Class $(m+l,m+n)$. We can assume $n \geq l$ without any loss of generality since the reflective transformation interchanges Class $(m+n,m+l)$ and Class $(m+l,m+n)$. By following an argument similar to in the last paragraph, we shall separate the hybrid classes in two, the one is Class $(m+l,m+n)$ with $m,l,n>0$ which is a special case of Class

$(m, m+n)$ with $m, n > 0$ and the other is Class $(l; n)$, which is an abbreviation for Class $(0+l, 0+n)$, with $n \geq l \geq 1$ and is a special case of Class $(0, n)$ with $n \geq 1$.

In the subsequent sections, we shall study Class $(m, m+n)$ with $m, n > 0$ and Class $(0, n)$ with $n \geq 1$ separately.

A. Conditions for Class $(m, m+n)$ with $m, n > 0$

The necessary and sufficient conditions for the existence of a set of intermediate Hamiltonians of Class $(1,1)$ are given by (4.1). A system in Class $(1,1)$ also belongs to Class $(m, m+n)$ if there exist simultaneously n inequivalent $GL(2,\mathbb{C})$ transformations for which the conditions (4.2) for Class $(0,1)$ are satisfied by the transformed coefficients $\hat{a}_i$ $(i = 0, \ldots, 4)$ and $\hat{b}_i$ $(i = 0, 1, 2)$, that is,

$$\hat{a}_4 = \hat{a}_3 + 2\hat{b}_2 = 0 \quad (\hat{a}_3 \hat{b}_2 \neq 0), \tag{5.2}$$

and in addition if there exist simultaneously $m-1$ additional inequivalent $GL(2,\mathbb{C})$ transformations for which the conditions (4.1) for Class $(1,1)$ are satisfied by $\hat{a}_i$ and $\hat{b}_i$, that is,

$$\hat{a}_4 = \hat{a}_3 = \hat{b}_2 = 0. \tag{5.3}$$

From the transformation formulas (2.10) and (2.11), we see that when the condition (4.1) is satisfied, the transformed coefficients $\hat{a}_4$, $\hat{a}_3$, and $\hat{b}_2$ respectively read

$$\Delta^2 \hat{a}_4 = \alpha^2\gamma^2 a_2 + \alpha\gamma^3 a_1 + \gamma^4 a_0, \tag{5.4a}$$
$$\Delta^2 \hat{a}_3 = 2\alpha\gamma(\alpha\delta + \beta\gamma) a_2 + \gamma^2(3\alpha\delta + \beta\gamma) a_1 + 4\gamma^3\delta a_0, \tag{5.4b}$$
$$\Delta \hat{b}_2 = \alpha\gamma b_1 + \gamma^2 b_0. \tag{5.4c}$$

Hence, except for the trivial case $\gamma = 0$, the conditions (5.2) are satisfied if and only if

$$\alpha^2 a_2 + \alpha\gamma a_1 + \gamma^2 a_0 = 0, \qquad 2\alpha a_2 + \gamma a_1 - 2\alpha b_1 - 2\gamma b_0 = 0, \tag{5.5}$$

with $\hat{a}_3 \hat{b}_2 \neq 0$, and the conditions (5.3) are satisfied if and only if

$$\alpha^2 a_2 + \alpha\gamma a_1 + \gamma^2 a_0 = 0, \qquad 2\alpha a_2 + \gamma a_1 = 0, \qquad \alpha b_1 + \gamma b_0 = 0. \tag{5.6}$$

In addition, a system of Class $(m, m+n)$ also belongs to Class $(m+l, m+n)$ if there exist simultaneously additional l inequivalent $GL(2,\mathbb{C})$ transformations for which the transformed coefficients $\hat{a}_i$ and $\hat{b}_i$ satisfy the conditions (4.3) for Class $(1,0)$, namely,

$$\hat{a}_4 = \hat{a}_3 - 2\hat{b}_2 = 0 \quad (\hat{a}_3 \hat{b}_2 \neq 0). \tag{5.7}$$

By the transformation formulas (5.4), they are satisfied if and only if (with $\hat{a}_3 \hat{b}_2 \neq 0$)

$$\alpha^2 a_2 + \alpha\gamma a_1 + \gamma^2 a_0 = 0, \qquad 2\alpha a_2 + \gamma a_1 + 2\alpha b_1 + 2\gamma b_0 = 0. \tag{5.8}$$

In Section VI A, we shall investigate the conditions (5.5), (5.6), and (5.8) in each case separately.

B. Conditions for Class $(0, n)$ with $n > 1$

The necessary and sufficient conditions for the existence of an intermediate Hamiltonian of Class $(0, 1)$ are given by (4.2). A system in Class $(0, 1)$ also belongs to Class $(0, n)$ if there exist simultaneously $n-1$ inequivalent $GL(2, \mathbb{C})$ transformations for which the transformed coefficients $\hat{a}_i$ and $\hat{b}_i$ satisfy the condition for Class $(0, 1)$, namely, Eq. (5.2). From the transformation formulas (2.10) and (2.11), we see that when the condition (4.2) is satisfied, the transformed coefficients $\hat{a}_4$, $\hat{a}_3$, and $\hat{b}_2$ respectively read

$$\Delta^2 \hat{a}_4 = \alpha^3\gamma a_3 + \alpha^2\gamma^2 a_2 + \alpha\gamma^3 a_1 + \gamma^4 a_0, \tag{5.9a}$$

$$\Delta^2 \hat{a}_3 = \alpha^2(\alpha\delta + 3\beta\gamma)a_3 + 2\alpha\gamma(\alpha\delta + \beta\gamma)a_2 + \gamma^2(3\alpha\delta + \beta\gamma)a_1 + 4\gamma^3\delta a_0, \tag{5.9b}$$

$$\Delta \hat{b}_2 = -\alpha^2 a_3/2 + \alpha\gamma b_1 + \gamma^2 b_0. \tag{5.9c}$$

Hence, we obtain

$$\begin{aligned}\Delta^2(\hat{a}_3 + 2\hat{b}_2) = {} & 4\alpha^2\beta\gamma a_3 + 2\alpha\gamma(\alpha\delta + \beta\gamma)a_2 + \gamma^2(3\alpha\delta + \beta\gamma)a_1 + 4\gamma^3\delta a_0 \\ & + 2\Delta\alpha\gamma b_1 + 2\Delta\gamma^2 b_0.\end{aligned} \tag{5.10}$$

Therefore, the condition (5.2) is satisfied, except for the trivial case $\gamma = 0$, if and only if

$$\alpha^3 a_3 + \alpha^2\gamma a_2 + \alpha\gamma^2 a_1 + \gamma^3 a_0 = 0, \tag{5.11a}$$

$$4\alpha^2 a_3 + 2\alpha\gamma a_2 + \gamma^2 a_1 - 2\alpha\gamma b_1 - 2\gamma^2 b_0 = 0, \tag{5.11b}$$

with $\hat{a}_3\hat{b}_2 \neq 0$. The second condition is derived by the elimination of a_0. On the other hand, a system in Class $(0, n)$ also belongs to Class $(l; n)$ if there exist simultaneously l additional inequivalent $GL(2, \mathbb{C})$ transformations for which the transformed coefficients $\hat{a}_i$ and $\hat{b}_i$ satisfy the conditions for Class $(1, 0)$, namely, Eq. (5.7). By the transformation formulas (5.9) they are satisfied if and only if (with $\hat{a}_3\hat{b}_2\gamma \neq 0$)

$$\alpha^3 a_3 + \alpha^2\gamma a_2 + \alpha\gamma^2 a_1 + \gamma^3 a_0 = 0, \tag{5.12a}$$

$$2\alpha^2 a_3 + 2\alpha\gamma a_2 + \gamma^2 a_1 + 2\alpha\gamma b_1 + 2\gamma^2 b_0 = 0. \tag{5.12b}$$

In Section VI B, we shall investigate the conditions (5.11) and (5.12) in each case separately.

VI. CLASSIFICATION: CASES OF MORE THAN ONE SETS

Now that we have derived the existence conditions for another different set of intermediate Hamiltonians in a general form, we are in a position to proceed a detailed analysis for each case classified in Section IV. In what follows, we first investigate the systems which belong to Class $(m, m+n)$ with $m, n > 0$ and next the ones which belong to Class $(0, n)$ with $n > 1$. All the former models are not only quasi-solvable but also solvable since they are special cases of Class $(1, 1)$. On the other hand, all the latter models are merely quasi-solvable since we have excluded either cases of $a_3 = b_2 = 0$ or of $\hat{a}_3 = \hat{b}_2 = 0$ in Class $(0, n)$.

A. Classification of Class $(m, m+n)$ with $m, n > 0$

I) $A(z) = 1/2$:

In this case, both the conditions (5.5) and (5.6) only have a trivial solution $\gamma = 0$. Thus, the system admits no different sets of intermediate Hamiltonians.

II) $A(z) = 2z$:

In this case, the conditions (5.6) only have a trivial solution but the conditions (5.5) have one set of non-trivial solutions

$$\alpha = 0, \quad b_0 = 1 \quad \text{with} \quad \hat{a}_3 = -2\hat{b}_2 = 2\gamma/\beta(\neq 0). \tag{6.1}$$

Thus, the system admits another set of an intermediate Hamiltonian of Class $(0, 1)$ and belongs to Class $(1, 2)$, which is an abbreviation for Class $(1, 1+1)$. On the other hand, the conditions (5.8) also have one set of non-trivial solutions

$$\alpha = 0, \quad b_0 = -1 \quad \text{with} \quad \hat{a}_3 = 2\hat{b}_2 = 2\gamma/\beta(\neq 0). \tag{6.2}$$

However, the latter $GL(2, \mathbb{C})$ transformation $\alpha = 0$ is equivalent to the one in (6.1), and the latter solution $b_0 = -1$ is not compatible with $b_0 = 1$ in (6.1). Hence, the system does not admit a realization of Class $(1+1, 1+1)$.

Functions:

$$z(x) = x^2, \quad E(x) = \frac{1}{x}, \quad \hat{E}(x) = -\frac{3}{x}, \quad W(x) = \hat{W}(x) = -\frac{b_1}{2}x - \frac{1}{2x}. \tag{6.3}$$

Potentials:

$$V^-(x) = \frac{b_1^2}{8}x^2 + \frac{3}{8x^2} + b_1 - R, \tag{6.4a}$$

$$V^{\mathrm{i}1}(x) = \frac{b_1^2}{8}x^2 - \frac{1}{8x^2} + \frac{b_1}{2} - R, \qquad V^{\mathrm{j}1}(x) = \frac{b_1^2}{8}x^2 + \frac{3}{8x^2} - R, \tag{6.4b}$$

$$V^{\mathrm{j}2}(x) = \frac{b_1^2}{8}x^2 + \frac{35}{8x^2} - R, \tag{6.4c}$$

$$V^+(x) = \frac{b_1^2}{8}x^2 + \frac{15}{8x^2} - \frac{b_1}{2} - R. \tag{6.4d}$$

It is interesting to note that the formulas (3.1) and (3.2) are not valid with $\hat{E}(x)$ and $\hat{W}(x)$ given in (6.3); the potential term $V^{\mathrm{i}2}(x)$, for instance, calculated by (3.1) and calculated by (3.2) do not coincide with each other. Hence, the system admits only $H^{\mathrm{j}2}$ but not $H^{\mathrm{i}2}$ as the second set of intermediate Hamiltonians. All the potentials (6.4) including the newly appeared intermediate one $V^{\mathrm{j}2}(x)$ in (6.4c) belong to the class of radial harmonic oscillators with a *particular* angular momentum.

III) $A(z) = z^2/2$:

In this case, only the conditions (5.6) have one set of non-trivial solutions

$$\alpha = 0, \quad b_0 = 0 \quad \text{with} \quad \hat{a}_3 = \hat{b}_2 = 0. \tag{6.5}$$

Thus, the system admits another set of intermediate Hamiltonians of Class $(1,1)$ and belongs to Class $(2,2)$. However, when the conditions (6.5) are satisfied, then $E(x) = -\hat{E}(x) = 1$ and $W(x) = \hat{W}(x) = -b_1$ so that all the potentials are just an identical constant. Hence, this case is nothing more than a trivial system.

IV) $A(z) = 2(z^2 - 1)$:

In this case, the conditions (5.6) only have a trivial solution but the conditions (5.5) have two sets of non-trivial solutions

$$\alpha = \pm\gamma, \quad b_0 = \mp b_1 \pm 2 \quad \text{with} \quad \hat{a}_3 = -2\hat{b}_2 = -4\gamma/(\delta \mp \beta)(\neq 0). \tag{6.6}$$

Thus, the system admits another set of intermediate Hamiltonians of Class $(0,1)$ corresponding to each of the solutions and belongs to Class $(1,2)$. In addition, the system with the specific values of parameters $b_1 = 2$ and $b_0 = 0$ admits the two different solutions *simultaneously* and thus can have additional two different sets of intermediate Hamiltonians of Class $(0,1)$ corresponding to the two solutions. In the latter case, the system belongs to Class $(1,3)$. On the other hand, the conditions (5.8) also have two sets of non-trivial solutions

$$\alpha = \pm\gamma, \quad b_0 = \mp b_1 \mp 2 \quad \text{with} \quad \hat{a}_3 = 2\hat{b}_2 = -4\gamma/(\delta \mp \beta)(\neq 0). \tag{6.7}$$

They are compatible with (6.6) if and only if $b_1 = 0$ and $b_0 \pm 2$ with the inequivalent transformations, namely, $\alpha = \pm\gamma$ for the former and $\alpha = \mp\gamma$ for the latter. In these particular two cases, the system belongs to Class $(1+1, 1+1)$.

IV-1) Class $(1,2)$

Parameters:

$$b_0 = \mp(b_1 - 2), \quad \hat{a}_3 = -2\hat{b}_2 = -\frac{4\gamma}{\delta \mp \beta}, \quad \hat{a}_2 = \frac{2(\beta \pm 5\delta)}{\beta \mp \delta}, \quad \hat{b}_1 = -b_1 + \frac{4\delta}{\delta \mp \beta}. \tag{6.8}$$

Functions:

$$\begin{aligned} z(x) &= \cosh 2x, & W(x) &= \hat{W}(x) = -\frac{b_1 \cosh 2x + b_0}{2\sinh 2x}, \\ E(x) &= \frac{2\cosh 2x}{\sinh 2x}, & \hat{E}(x) &= \frac{2\cosh 2x}{\sinh 2x} - \frac{4\sinh 2x}{\cosh 2x \mp 1}. \end{aligned} \tag{6.9}$$

Potentials:

$$V^-(x) = \frac{\mp(b_1 - 2)(b_1 - 6)\cosh 2x + b_1^2 - 8b_1 + 18}{4\sinh^2 2x} - \bar{R}, \tag{6.10a}$$

$$V^{\mathrm{i}1}(x) = \frac{\mp(b_1 - 2)^2\cosh 2x + b_1^2 - 4b_1 + 2}{4\sinh^2 2x} - \bar{R}, \tag{6.10b}$$

$$V^{\mathrm{j}1}(x) = \frac{\mp(b_1^2 - 4)\cosh 2x + b_1^2 + 2}{4\sinh^2 2x} - \bar{R}, \tag{6.10c}$$

$$V^{\mathrm{j}2}(x) = \frac{\mp(b_1^2 - 4)\cosh 2x + b_1^2 + 2}{4\sinh^2 2x} \pm \frac{8}{\cosh 2x \mp 1} - \bar{R}, \tag{6.10d}$$

$$V^+(x) = \frac{\mp(b_1 - 2)(b_1 + 6)\cosh 2x + b_1^2 + 4b_1 + 18}{4\sinh^2 2x} - \bar{R}, \tag{6.10e}$$

where $\bar{R} = R - b_1^2/8 - 4/3$ is a constant. The formulas (3.1) and (3.2) are again not valid with $\hat{E}(x)$ and $\hat{W}(x)$ given in (6.9) and thus the system admits only $H^{\mathrm{j}2}$ but not $H^{\mathrm{i}2}$ as the second set of intermediate Hamiltonians. All the potentials (6.10) but the newly appeared intermediate one $V^{\mathrm{j}2}(x)$ in (6.10d) belong to the class of Pöschl–Teller potential with only one free parameter. The deformation term in (6.10d) is reminiscent of the generalized Pöschl–Teller potentials constructed in Ref. [4]. It is worth noting that the deformed potential (6.10d) is connected with the other shape-invariant potentials by the intertwining relations (2.20) and thus is almost isospectral to them and is in particular solvable.

IV-2) Class $(1,3)$

Parameters:

$$\begin{aligned} &b_1 = 2, \quad \hat{a}_3 = -2\hat{b}_2 = -\frac{4\gamma}{\delta-\beta}, \quad \hat{a}_2 = \frac{2(\beta+5\delta)}{\beta-\delta}, \quad \hat{b}_1 = \frac{2(\delta+\beta)}{\delta-\beta}, \\ &b_0 = 0, \quad \hat{\hat{a}}_3 = -2\hat{\hat{b}}_2 = -\frac{4\gamma}{\delta+\beta}, \quad \hat{\hat{a}}_2 = \frac{2(\beta-5\delta)}{\beta+\delta}, \quad \hat{\hat{b}}_1 = \frac{2(\delta-\beta)}{\delta+\beta}. \end{aligned} \tag{6.11}$$

Functions:

$$\begin{aligned} &z(x) = \cosh 2x, \quad E(x) = \frac{2\cosh 2x}{\sinh 2x}, \quad W(x) = \hat{W}(x) = \hat{\hat{W}}(x) = -\frac{\cosh 2x}{\sinh 2x}, \\ &\hat{E}(x) = \frac{2\cosh 2x}{\sinh 2x} - \frac{4\sinh 2x}{\cosh 2x - 1}, \qquad \hat{\hat{E}}(x) = \frac{2\cosh 2x}{\sinh 2x} - \frac{4\sinh 2x}{\cosh 2x + 1}. \end{aligned} \tag{6.12}$$

Potentials:

$$V^-(x) = \frac{3}{2\sinh^2 2x} - \bar{R}, \tag{6.13a}$$

$$V^{\mathrm{i}1}(x) = -\frac{1}{2\sinh^2 2x} - \bar{R}, \qquad V^{\mathrm{j}1}(x) = \frac{3}{2\sinh^2 2x} - \bar{R}, \tag{6.13b}$$

$$V^{\mathrm{j}2}(x) = \frac{3}{2\sinh^2 2x} + \frac{8}{\cosh 2x - 1} - \bar{R}, \tag{6.13c}$$

$$V^{\mathrm{j}3}(x) = \frac{3}{2\sinh^2 2x} - \frac{8}{\cosh 2x + 1} - \bar{R}, \tag{6.13d}$$

$$V^+(x) = \frac{15}{2\sinh^2 2x} - \bar{R}, \tag{6.13e}$$

where $\bar{R} = R - 11/6$ is a constant. There are essentially no free parameters in this case.

IV-3) Class $(1+1, 1+1)$

Parameters:

$$\begin{aligned} &b_1 = 0, \quad \hat{a}_3 = -2\hat{b}_2 = -\frac{4\gamma}{\delta\mp\beta}, \quad \hat{a}_2 = \frac{2(\beta\pm 5\delta)}{\beta\mp\delta}, \quad \hat{b}_1 = \frac{4\delta}{\delta\mp\beta}, \\ &b_0 = \pm 2, \quad \hat{\hat{a}}_3 = 2\hat{\hat{b}}_2 = -\frac{4\gamma}{\delta\pm\beta}, \quad \hat{\hat{a}}_2 = \frac{2(\beta\mp 5\delta)}{\beta\pm\delta}, \quad \hat{\hat{b}}_1 = \frac{4\delta}{\delta\pm\beta}. \end{aligned} \tag{6.14}$$

Functions:

$$\begin{aligned} &z(x) = \cosh 2x, \quad E(x) = \frac{2\cosh 2x}{\sinh 2x}, \quad W(x) = \hat{W}(x) = \hat{\hat{W}}(x) = \mp\frac{1}{\sinh 2x}, \\ &\hat{E}(x) = \frac{2\cosh 2x}{\sinh 2x} - \frac{4\sinh 2x}{\cosh 2x \mp 1}, \qquad \hat{\hat{E}}(x) = \frac{2\cosh 2x}{\sinh 2x} - \frac{4\sinh 2x}{\cosh 2x \pm 1}. \end{aligned} \tag{6.15}$$

Potentials:

$$V^-(x) = \frac{\mp 6\cosh 2x + 9}{2\sinh^2 2x} - \bar{R}, \tag{6.16a}$$

$$V^{\mathrm{i}1}(x) = -\frac{\mp 2\cosh 2x + 1}{2\sinh^2 2x} - \bar{R}, \qquad V^{\mathrm{j}1}(x) = \frac{\pm 2\cosh 2x + 1}{2\sinh^2 2x} - \bar{R}, \tag{6.16b}$$

$$V^{\mathrm{i}2}(x) = \frac{\mp 2\cosh 2x + 1}{2\sinh^2 2x} \mp \frac{8}{\cosh 2x \pm 1} - \bar{R}, \tag{6.16c}$$

$$V^{\mathrm{j}2}(x) = \frac{\pm 2\cosh 2x + 1}{2\sinh^2 2x} \pm \frac{8}{\cosh 2x \mp 1} - \bar{R}, \tag{6.16d}$$

$$V^+(x) = \frac{\pm 6\cosh 2x + 9}{2\sinh^2 2x} - \bar{R}, \tag{6.16e}$$

where $\bar{R} = R - 4/3$ is a constant. As in the previous case, there are essentially no free parameters in this case.

B. Classification of Class $(0, n)$ with $n > 1$

II') $A(z) = 2z^3$:

In this case, the conditions (5.11) have one set of non-trivial solutions $\alpha = b_0 = 0$ but with $\hat{a}_3 = \hat{b}_2 = 0$ which should be excluded. In fact, the system corresponding to the latter solutions is identical with the one in the case II, which belongs to Class $(1, 2)$, already found in the previous section, Eqs. (6.4).

IV') $A(z) = 2z^2(z + 1)$:

In this case, the conditions (5.11) have two sets of non-trivial solutions, the one is

$$\alpha = -\gamma, \quad b_1 = b_0 - 2 \quad \text{with} \quad \hat{a}_3 = -2\hat{b}_2 = 2\gamma/(\beta + \delta)(\neq 0), \tag{6.17}$$

but the other is

$$\alpha = b_0 = 0 \quad \text{with} \quad \hat{a}_3 = \hat{b}_2 = 0, \tag{6.18}$$

and thus should be discarded. Indeed, the choice of the latter solution simply leads to the system of Class $(1, 2)$, and together with the former solution (6.17), to the system of Class $(1, 3)$ which are identical with the systems (6.10) and (6.13) respectively in the case IV already found in the previous section. Hence, only the solution (6.17) leads to a new system which belongs to Class $(0, 2)$.

On the other hand, the conditions (5.12) also have two sets of non-trivial solutions, the one is identical with (6.18) to be discarded while the other is

$$\alpha = -\gamma, \quad b_1 = b_0 \quad \text{with} \quad \hat{a}_3 = 2\hat{b}_2 = 2\gamma/(\beta + \delta)(\neq 0). \tag{6.19}$$

Hence, the Class $(0, 1)$ system (4.19) admits an intermediate Hamiltonian of Class $(1, 0)$ and thus belongs to Class $(1; 1)$. However, the $GL(2, \mathbb{C})$ transformation $\alpha = -\gamma$ of the latter solution is equivalent to the one in (6.17). Hence, the system does not admit a realization of Class $(1; 2)$. We note that the choice of the two solutions (6.18) and (6.19) leads to the system of Class $(1{+}1, 1{+}1)$ which is identical with the system (6.16).

IV'-1) Class $(0, 2)$

Parameters:

$$b_1 = b_0 - 2, \quad \hat{a}_3 = -2\hat{b}_2 = \frac{2\gamma}{\beta+\delta}, \quad \hat{a}_2 = \frac{2(\delta-2\beta)}{\delta+\beta}, \quad \hat{b}_1 = -b_0 - \frac{2\delta}{\beta+\delta}. \tag{6.20}$$

Functions:

$$\begin{aligned} z(x) &= \frac{1}{\sinh^2 x}, \quad W(x) = \hat{W}(x) = \frac{b_0 \sinh^2 x \cosh^2 x - 2\sinh^2 x - 1}{2\sinh x \cosh x}, \\ E(x) &= -\frac{3 + 2\sinh^2 x}{\sinh x \cosh x}, \qquad \hat{E}(x) = \frac{1 - 2\sinh^2 x}{\sinh x \cosh x}. \end{aligned} \tag{6.21}$$

Potentials:

$$V^-(x) = \frac{b_0^2}{8}\cosh^4 x - \frac{(b_0+16)b_0}{8}\cosh^2 x - \frac{3}{8\cosh^2 x} + \frac{3}{8\sinh^2 x} + \frac{11}{6} + b_0 - R, \tag{6.22a}$$

$$V^{\mathrm{j}1}(x) = \frac{b_0^2}{8}\cosh^4 x - \frac{b_0^2}{8}\cosh^2 x - \frac{3}{8\cosh^2 x} + \frac{35}{8\sinh^2 x} + \frac{11}{6} - R, \tag{6.22b}$$

$$V^{\mathrm{j}2}(x) = \frac{b_0^2}{8}\cosh^4 x - \frac{b_0^2}{8}\cosh^2 x - \frac{35}{8\cosh^2 x} + \frac{3}{8\sinh^2 x} + \frac{11}{6} - R, \tag{6.22c}$$

$$V^+(x) = \frac{b_0^2}{8}\cosh^4 x - \frac{(b_0-8)b_0}{8}\cosh^2 x - \frac{15}{8\cosh^2 x} + \frac{15}{8\sinh^2 x} + \frac{11}{6} - \frac{b_0}{2} - R. \tag{6.22d}$$

IV'-2) Class $(1; 1)$

Parameters:

$$b_1 = b_0, \quad \hat{a}_3 = 2\hat{b}_2 = \frac{2\gamma}{\beta+\delta}, \quad \hat{a}_2 = \frac{2(\delta-2\beta)}{\delta+\beta}, \quad \hat{b}_1 = -b_0 - \frac{2\beta}{\beta+\delta}. \tag{6.23}$$

Functions:

$$\begin{aligned} z(x) &= \frac{1}{\sinh^2 x}, \quad W(x) = \hat{W}(x) = \frac{b_0 \sinh^2 x \cosh^2 x - 1}{2\sinh x \cosh x}, \\ E(x) &= -\frac{3 + 2\sinh^2 x}{\sinh x \cosh x}, \qquad \hat{E}(x) = \frac{1 - 2\sinh^2 x}{\sinh x \cosh x}. \end{aligned} \tag{6.24}$$

Potentials:

$$V^-(x) = \frac{b_0^2}{8}\cosh^4 x - \frac{(b_0+12)b_0}{8}\cosh^2 x - \frac{15}{8\cosh^2 x} + \frac{3}{8\sinh^2 x} + \frac{4}{3} + \frac{b_0}{2} - R, \tag{6.25a}$$

$$V^{\mathrm{j}1}(x) = \frac{b_0^2}{8}\cosh^4 x - \frac{(b_0-4)b_0}{8}\cosh^2 x + \frac{1}{8\cosh^2 x} + \frac{35}{8\sinh^2 x} + \frac{4}{3} - \frac{b_0}{2} - R, \tag{6.25b}$$

$$V^{\mathrm{i}1}(x) = \frac{b_0^2}{8}\cosh^4 x - \frac{(b_0+4)b_0}{8}\cosh^2 x - \frac{35}{8\cosh^2 x} - \frac{1}{8\sinh^2 x} + \frac{4}{3} - R, \tag{6.25c}$$

$$V^+(x) = \frac{b_0^2}{8}\cosh^4 x - \frac{(b_0-12)b_0}{8}\cosh^2 x - \frac{3}{8\cosh^2 x} + \frac{15}{8\sinh^2 x} + \frac{4}{3} - b_0 - R. \tag{6.25d}$$

V) $A(z) = 2z^3 - g_2 z/2 - g_3/2$:

In this case, the conditions (5.11) have three sets of non-trivial solutions

$$\alpha = e_i\gamma, \quad b_0 = -e_i b_1 + 4e_i^2 - \frac{g_2}{4} \quad \text{with} \quad \hat{a}_3 = -2\hat{b}_2 = \frac{\wp''(\omega_i)\gamma}{\beta - e_i\delta}(\neq 0), \tag{6.26}$$

where each $e_i = \wp(\omega_i)$ $(i = 1, 2, 3)$ is the value of the Weierstrass elliptic function at the half of the fundamental period $2\omega_i$ which satisfies the third-degree algebraic equation

$$4e_i^3 - g_2 e_i - g_3 = 0. \tag{6.27}$$

Thus, the system admits another set of intermediate Hamiltonians of Class $(0, 1)$ corresponding to each of the solutions and in those cases the system belongs to Class $(0, 2)$. In addition, the system with the specific values of parameters $b_1 = 4(e_i + e_j)$ and $b_0 = -(e_i + e_j)^2 - 3e_i e_j$ $(i \neq j)$ admits the two different solutions *simultaneously* and thus can have additional two different sets of intermediate Hamiltonians of Class $(0, 1)$ corresponding to the two solutions. In the latter case, the system belongs to Class $(0, 3)$. We note, however, that the three different solutions are incompatible simultaneously and hence any Class $(0, n)$ with $n > 3$ cannot be realized. On the other hand, the conditions (5.12) also have three sets of non-trivial solutions

$$\alpha = e_i\gamma, \quad b_0 = -e_i b_1 - 2e_i^2 + \frac{g_2}{4} \quad \text{with} \quad \hat{a}_3 = 2\hat{b}_2 = \frac{\wp''(\omega_i)\gamma}{\beta - e_i\delta}(\neq 0). \tag{6.28}$$

Hence, the Class $(0, 1)$ system (4.21) also admits an intermediate Hamiltonian of Class $(1, 0)$ and thus belongs to Class $(1; 1)$. In addition, a choice of one solution $\alpha = e_i\gamma$ in (6.26) and another $\alpha = e_j\gamma$ $(j \neq i)$ in (6.28) is compatible with $b_1 = 2e_i$ and $b_0 = e_i^2 - e_i e_j - e_j^2$. In the latter case, the system possesses two additional intermediate Hamiltonians the one belongs to Class $(0, 1)$ and the other to Class $(1, 0)$, and thus it is a member of Class $(1, 2)$. However, a choice of three solutions, e.g., two $\alpha = e_i\gamma, e_j\gamma$ in (6.26) and one $\alpha = e_k\gamma$ in (6.28) with $i \neq j \neq k \neq i$, conflict with the assumption of non-degeneracy $g_2^3 \neq 27g_3^2$ (for the latter example, $e_i = -2e_j$ must hold). Therefore, the hybrid classes such as Class $(1, 3)$ and Class $(2; 2)$ cannot be realized anymore.

V-1) Class $(0, 2)$

Parameters:

$$\begin{aligned} b_0 &= -e_i b_1 + 4e_i^2 - \frac{g_2}{4}, & \hat{b}_1 &= -b_1 + \frac{4e_i^2\beta - (12e_i^3 + g_3)\delta}{2e_i(\beta - e_i\delta)}, \\ \hat{a}_3 &= -2\hat{b}_2 = \frac{\wp''(\omega_i)\gamma}{\beta - e_i\delta}, & \hat{a}_2 &= 3\frac{4e_i^2\beta^2 + g_3\beta\delta - (4e_i^3 + g_3)e_i\delta^2}{2e_i(\beta - e_i\delta)^2}. \end{aligned} \tag{6.29}$$

Functions:

$$\begin{aligned} z(x) &= \wp(x), \quad W(x) = \hat{W}(x) = \frac{\wp(x)^2 - b_1\wp(x) - b_0}{\wp'(x)}, \\ E(x) &= \frac{\wp''(x)}{\wp'(x)}, \qquad \hat{E}(x) = \frac{\wp''(x)}{\wp'(x)} - \frac{2\wp'(x)}{\wp(x) - e_i}. \end{aligned} \tag{6.30}$$

Potentials:

$$V^{-}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0-\frac{10}{3}\wp''(x)\right]-\frac{8}{9}\wp(x)+\frac{91}{18}\wp(2x)+\frac{3b_1}{2}-R, \tag{6.31a}$$

$$V^{\mathrm{j}1}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{2}{3}\wp''(x)\right]+\frac{40}{9}\wp(x)-\frac{5}{18}\wp(2x)-\frac{b_1}{2}-R, \tag{6.31b}$$

$$\begin{aligned}V^{\mathrm{j}2}(x)=&\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{2}{3}\wp''(x)\right]+\frac{4}{9}\wp(x)-\frac{5}{18}\wp(2x)\\&+\frac{2\wp''(\omega_i)}{\wp(x)-e_i}+4e_i-\frac{b_1}{2}-R,\end{aligned} \tag{6.31c}$$

$$V^{+}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{8}{3}\wp''(x)\right]+\frac{10}{9}\wp(x)+\frac{55}{18}\wp(2x)-\frac{3b_1}{2}-R, \tag{6.31d}$$

where $\bar{b}_0=b_0-g_2/12=-e_ib_1+2\wp''(\omega_i)/3$. The first term of each the potential can be expressed solely in terms of $\wp(x)$. With the latter value of $\bar{b}_0$, we have

$$\begin{aligned}&\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+C\wp''(x)\right]\\&\qquad=\frac{[3b_1(\wp(x)-e_i)+2\wp''(\omega_i)]^2}{72\prod_{l=1}^{3}(\wp(x)-e_l)}+\frac{C}{12}\sum_{l=1}^{3}\frac{3b_1(\wp(x)-e_i)+2\wp''(\omega_i)}{\wp(x)-e_l},\end{aligned} \tag{6.32}$$

where and hereafter $i\neq j\neq k\neq i$ $(i,j,k=1,2,3)$.

V-2) Class $(0,3)$

Parameters: $\hat{a}_i$ and $\hat{b}_i$ are the same as (6.29).

$$\begin{aligned}&b_1=-4e_k,\quad b_0=-e_k^2-3e_ie_j,\quad \hat{\hat{b}}_1=-b_1+\frac{4e_j^2\beta-(12e_j^3+g_3)\delta}{2e_j(\beta-e_j\delta)},\\&\hat{\hat{a}}_3=-2\hat{\hat{b}}_2=\frac{\wp''(\omega_j)\gamma}{\beta-e_j\delta},\qquad \hat{\hat{a}}_2=3\frac{4e_j^2\beta^2+g_3\beta\delta-(4e_j^3+g_3)e_j\delta^2}{2e_j(\beta-e_j\delta)^2}.\end{aligned} \tag{6.33}$$

Functions:

$$\begin{aligned}&z(x)=\wp(x),\qquad W(x)=\hat{W}(x)=\hat{\hat{W}}(x)=\frac{\wp(x)^2+4e_k\wp(x)+e_k^2-3e_ie_j}{\wp'(x)},\\&E(x)=\frac{\wp''(x)}{\wp'(x)},\quad \hat{E}(x)=\frac{\wp''(x)}{\wp'(x)}-\frac{2\wp'(x)}{\wp(x)-e_i},\quad \hat{\hat{E}}(x)=\frac{\wp''(x)}{\wp'(x)}-\frac{2\wp'(x)}{\wp(x)-e_j}.\end{aligned} \tag{6.34}$$

Potentials:

$$V^{-}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0-\frac{10}{3}\wp''(x)\right]-\frac{8}{9}\wp(x)+\frac{91}{18}\wp(2x)+\frac{3b_1}{2}-R, \tag{6.35a}$$

$$V^{\mathrm{j}1}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{2}{3}\wp''(x)\right]+\frac{40}{9}\wp(x)-\frac{5}{18}\wp(2x)-\frac{b_1}{2}-R, \tag{6.35b}$$

$$V^{\mathrm{j}2}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{2}{3}\wp''(x)\right]+\frac{4}{9}\wp(x)-\frac{5}{18}\wp(2x)$$
$$+\frac{2\wp''(\omega_i)}{\wp(x)-e_i}+4e_i-\frac{b_1}{2}-R, \tag{6.35c}$$
$$V^{\mathrm{j}3}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{2}{3}\wp''(x)\right]+\frac{4}{9}\wp(x)-\frac{5}{18}\wp(2x)$$
$$+\frac{2\wp''(\omega_j)}{\wp(x)-e_j}+4e_j-\frac{b_1}{2}-R, \tag{6.35d}$$
$$V^{+}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{8}{3}\wp''(x)\right]+\frac{10}{9}\wp(x)+\frac{55}{18}\wp(2x)-\frac{3b_1}{2}-R. \tag{6.35e}$$

With the values of b_1 and b_0 in (6.33), the first term of each the potential can be expressed solely in terms of $\wp(x)$ as

$$\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+C\wp''(x)\right]$$
$$=\frac{2[3e_k\wp(x)+e_k^2+2e_ie_j]^2}{9\prod_{l=1}^{3}(\wp(x)-e_l)}-\frac{C}{3}\sum_{l=1}^{3}\frac{3e_k\wp(x)+e_k^2+2e_ie_j}{\wp(x)-e_l}. \tag{6.36}$$

V-3) Class $(1;1)$

Parameters:

$$b_0=-e_ib_1-2e_i^2+\frac{g_2}{4},\qquad \hat{b}_1=-b_1+\frac{4e_i^2\beta+(4e_i^3+g_3)\delta}{2e_i(\beta-e_i\delta)},$$
$$\hat{a}_3=2\hat{b}_2=\frac{\wp''(\omega_i)\gamma}{\beta-e_i\delta},\qquad \hat{a}_2=3\frac{4e_i^2\beta^2+g_3\beta\delta-(4e_i^3+g_3)e_i\delta^2}{2e_i(\beta-e_i\delta)^2}. \tag{6.37}$$

Functions:

$$z(x)=\wp(x),\quad W(x)=\hat{W}(x)=\frac{\wp(x)^2-b_1\wp(x)-b_0}{\wp'(x)},$$
$$E(x)=\frac{\wp''(x)}{\wp'(x)},\qquad \hat{E}(x)=\frac{\wp''(x)}{\wp'(x)}-\frac{2\wp'(x)}{\wp(x)-e_i}, \tag{6.38}$$

Potentials:

$$V^{-}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0-\frac{10}{3}\wp''(x)\right]-\frac{8}{9}\wp(x)+\frac{91}{18}\wp(2x)+\frac{3b_1}{2}-R, \tag{6.39a}$$
$$V^{\mathrm{i}1}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0-\frac{4}{3}\wp''(x)\right]-\frac{2}{9}\wp(x)+\frac{7}{18}\wp(2x)$$
$$+\frac{2\wp''(\omega_i)}{\wp(x)-e_i}+4e_i+\frac{b_1}{2}-R, \tag{6.39b}$$
$$V^{\mathrm{j}1}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{2}{3}\wp''(x)\right]+\frac{40}{9}\wp(x)-\frac{5}{18}\wp(2x)-\frac{b_1}{2}-R, \tag{6.39c}$$
$$V^{+}(x)=\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{8}{3}\wp''(x)\right]+\frac{10}{9}\wp(x)+\frac{55}{18}\wp(2x)-\frac{3b_1}{2}-R, \tag{6.39d}$$

where $\bar{b}_0 = b_0 - g_2/12 = -e_i b_1 - \wp''(\omega_i)/3$. The first term of each the potential can be expressed solely in terms of $\wp(x)$. With the latter value of $\bar{b}_0$, we have

$$\begin{aligned}
&\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+C\wp''(x)\right]\\
&\qquad=\frac{[3b_1(\wp(x)-e_i)-\wp''(\omega_i)]^2}{72\prod_{l=1}^3(\wp(x)-e_l)}+\frac{C}{12}\sum_{l=1}^3\frac{3b_1(\wp(x)-e_i)-\wp''(\omega_i)}{\wp(x)-e_l}. \qquad (6.40)
\end{aligned}$$

V-4) Class $(1;2)$

Parameters: $\hat{a}_i$ and $\hat{b}_i$ are the same as (6.29).

$$\begin{aligned}
&b_1 = 2e_i, \quad b_0 = e_i^2 + e_j e_k, \quad \hat{\hat{b}}_1 = -b_1 + \frac{4e_j^2\beta + (4e_j^3+g_3)\delta}{2e_j(\beta - e_j\delta)},\\
&\hat{\hat{a}}_3 = 2\hat{\hat{b}}_2 = \frac{\wp''(\omega_j)\gamma}{\beta - e_j\delta}, \qquad \hat{\hat{a}}_2 = 3\frac{4e_j^2\beta^2 + g_3\beta\delta - (4e_j^3+g_3)e_j\delta^2}{2e_j(\beta-e_j\delta)^2}.
\end{aligned} \qquad (6.41)$$

Functions:

$$\begin{aligned}
&z(x) = \wp(x), \qquad W(x) = \hat{W}(x) = \hat{\hat{W}}(x) = \frac{\wp(x)^2 - 2e_i\wp(x) - e_i^2 - e_j e_k}{\wp'(x)},\\
&E(x) = \frac{\wp''(x)}{\wp'(x)}, \quad \hat{E}(x) = \frac{\wp''(x)}{\wp'(x)} - \frac{2\wp'(x)}{\wp(x)-e_i}, \quad \hat{\hat{E}}(x) = \frac{\wp''(x)}{\wp'(x)} - \frac{2\wp'(x)}{\wp(x)-e_j},
\end{aligned} \qquad (6.42)$$

Potentials:

$$V^-(x) = \frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0-\frac{10}{3}\wp''(x)\right] - \frac{8}{9}\wp(x) + \frac{91}{18}\wp(2x) + \frac{3b_1}{2} - R, \qquad (6.43a)$$

$$\begin{aligned}
V^{\mathrm{i}1}(x) &= \frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0-\frac{4}{3}\wp''(x)\right] - \frac{2}{9}\wp(x) + \frac{7}{18}\wp(2x)\\
&\quad + \frac{2\wp''(\omega_j)}{\wp(x)-e_j} + 4e_j + \frac{b_1}{2} - R, \qquad (6.43b)
\end{aligned}$$

$$V^{\mathrm{j}1}(x) = \frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{2}{3}\wp''(x)\right] + \frac{40}{9}\wp(x) - \frac{5}{18}\wp(2x) - \frac{b_1}{2} - R, \qquad (6.43c)$$

$$\begin{aligned}
V^{\mathrm{j}2}(x) &= \frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{2}{3}\wp''(x)\right] + \frac{4}{9}\wp(x) - \frac{5}{18}\wp(2x)\\
&\quad + \frac{2\wp''(\omega_i)}{\wp(x)-e_i} + 4e_i - \frac{b_1}{2} - R, \qquad (6.43d)
\end{aligned}$$

$$V^+(x) = \frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+\frac{8}{3}\wp''(x)\right] + \frac{10}{9}\wp(x) + \frac{55}{18}\wp(2x) - \frac{3b_1}{2} - R. \qquad (6.43e)$$

With the values of b_1 and b_0 in (6.41), the first term of each the potential can be expressed solely in terms of $\wp(x)$ as

$$\begin{aligned}
&\frac{b_1\wp(x)+\bar{b}_0}{2\wp'(x)^2}\left[b_1\wp(x)+\bar{b}_0+C\wp''(x)\right]\\
&\qquad=\frac{[3e_i\wp(x)+e_i^2+2e_je_k]^2}{18\prod_{l=1}^3(\wp(x)-e_l)}+\frac{C}{6}\sum_{l=1}^3\frac{3e_i\wp(x)+e_i^2+2e_je_k}{\wp(x)-e_l}. \qquad (6.44)
\end{aligned}$$

VII. PARASUPERSYMMETRY AND GENERALIZED SUPERALGEBRAS

In the case of $\mathcal{N} = 2$, it was shown [1] that any type A 2-fold SUSY system which has an intermediate Hamiltonian admits a realization of second-order paraSUSY [9] and a generalized 2-fold superalgebra [10]. Thus, it is natural to ask whether an analogous realization is possible in the present $\mathcal{N} = 3$ case. In what follows, we show that it is indeed the case. More precisely, a type A 3-fold SUSY system of Class $(1,1)$ admits a realization of third-order paraSUSY [11, 12] and a generalized 3-fold superalgebra found in Ref. [13] while one of Class $(0,1)$ does only a restricted version of the latter algebra. We shall first discuss the former realization and then the latter.

A. Parasupersymmetry of Order 3 in Class $(1,1)$

Higher-order paraSUSY was introduced in Refs. [11, 12] as a generalization of second-order one [9]. In the case of third-order, it is characterized by the following algebraic relations:

$$(\boldsymbol{Q}_{\mathrm{P}}^{\pm})^3 \neq 0, \qquad (\boldsymbol{Q}_{\mathrm{P}}^{\pm})^4 = 0, \qquad \bigl[\boldsymbol{Q}_{\mathrm{P}}^{\pm}, \boldsymbol{H}_{\mathrm{P}}\bigr] = 0, \tag{7.1a}$$

$$(\boldsymbol{Q}_{\mathrm{P}}^{\pm})^3\boldsymbol{Q}_{\mathrm{P}}^{\mp} + (\boldsymbol{Q}_{\mathrm{P}}^{\pm})^2\boldsymbol{Q}_{\mathrm{P}}^{\mp}\boldsymbol{Q}_{\mathrm{P}}^{\pm} + \boldsymbol{Q}_{\mathrm{P}}^{\pm}\boldsymbol{Q}_{\mathrm{P}}^{\mp}(\boldsymbol{Q}_{\mathrm{P}}^{\pm})^2 + \boldsymbol{Q}_{\mathrm{P}}^{\mp}(\boldsymbol{Q}_{\mathrm{P}}^{\pm})^3 = 6(\boldsymbol{Q}_{\mathrm{P}}^{\pm})^2\boldsymbol{H}_{\mathrm{P}}. \tag{7.1b}$$

By the introduction of parafermionic coordinates $\psi_{\mathrm{P}}^{\pm}$ of order 3 satisfying [13]

$$(\psi_{\mathrm{P}}^{\pm})^4 = 0, \quad \bigl\{\psi_{\mathrm{P}}^{-}, \psi_{\mathrm{P}}^{+}\bigr\} + \bigl\{(\psi_{\mathrm{P}}^{-})^3, (\psi_{\mathrm{P}}^{+})^3\bigr\} = 2I, \quad \bigl\{(\psi_{\mathrm{P}}^{-})^2, (\psi_{\mathrm{P}}^{+})^2\bigr\} = I, \tag{7.2}$$

a quantum mechanical realization of paraSUSY of order 3 is achieved by defining the triple $(\boldsymbol{H}_{\mathrm{P}}, \boldsymbol{Q}_{\mathrm{P}}^{\pm})$ as

$$\begin{aligned} \boldsymbol{H}_{\mathrm{P}} &= H_0(\psi_{\mathrm{P}}^{-})^3(\psi_{\mathrm{P}}^{+})^3 + H_1(\psi_{\mathrm{P}}^{+}\psi_{\mathrm{P}}^{-} - (\psi_{\mathrm{P}}^{+})^2(\psi_{\mathrm{P}}^{-})^2) \\ &\quad + H_2((\psi_{\mathrm{P}}^{+})^2(\psi_{\mathrm{P}}^{-})^2 - (\psi_{\mathrm{P}}^{+})^3(\psi_{\mathrm{P}}^{-})^3) + H_3(\psi_{\mathrm{P}}^{+})^3(\psi_{\mathrm{P}}^{-})^3, \end{aligned} \tag{7.3a}$$

$$\boldsymbol{Q}_{\mathrm{P}}^{-} = Q_1^{-}(\psi_{\mathrm{P}}^{-})^3(\psi_{\mathrm{P}}^{+})^2 + Q_2^{-}(\psi_{\mathrm{P}}^{+}(\psi_{\mathrm{P}}^{-})^2 - (\psi_{\mathrm{P}}^{+})^2(\psi_{\mathrm{P}}^{-})^3) + Q_3^{-}(\psi_{\mathrm{P}}^{+})^2(\psi_{\mathrm{P}}^{-})^3, \tag{7.3b}$$

$$\boldsymbol{Q}_{\mathrm{P}}^{+} = Q_1^{+}(\psi_{\mathrm{P}}^{-})^2(\psi_{\mathrm{P}}^{+})^3 + Q_2^{+}(\psi_{\mathrm{P}}^{-}(\psi_{\mathrm{P}}^{+})^2 - (\psi_{\mathrm{P}}^{-})^2(\psi_{\mathrm{P}}^{+})^3) + Q_3^{+}(\psi_{\mathrm{P}}^{+})^3(\psi_{\mathrm{P}}^{-})^2, \tag{7.3c}$$

where

$$H_k = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_k(x), \qquad Q_k^{\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}x} + W_k(x). \tag{7.4}$$

The linear space in which the system $(\boldsymbol{H}_{\mathrm{P}}, \boldsymbol{Q}_{\mathrm{P}}^{\pm})$ shall be considered is $\mathfrak{F} \times \mathsf{V}_3$ where $\mathfrak{F}$ is a linear space of complex functions such as $L^2(\mathbb{R})$ and $\mathsf{V}_3 = \sum_{k=0}^{3} \mathsf{V}_3^{(k)}$ is the parafermionic Fock space of order 3 composed of the k-parafermionic subspaces $\mathsf{V}_3^{(k)}$ $(k = 0, \dots, 3)$. Then, the latter system satisfies the third-order paraSUSY algebra (7.1) if and only if [12, 13, 25]

$$2H_0 = Q_1^{-}Q_1^{+} - 2R_1, \quad 2H_1 = Q_1^{+}Q_1^{-} - 2R_1 = Q_2^{-}Q_2^{+} - 2R_2, \tag{7.5a}$$

$$2H_2 = Q_2^{+}Q_2^{-} - 2R_2 = Q_3^{-}Q_3^{+} - 2R_3, \quad 2H_3 = Q_3^{+}Q_3^{-} - 2R_3, \tag{7.5b}$$

where R_k $(k = 1, 2, 3)$ are constants satisfying

$$R_1 + R_2 + R_3 = 0. \tag{7.6}$$

Comparing now the paraSUSY conditions (7.5) with the Class $(1,1)$ conditions (3.1)–(3.3), we immediately notice that any type A 3-fold SUSY system which belongs to Class $(1,1)$ admits a realization of paraSUSY of order 3 by the following identifications:

$$Q_k^{\pm} = P_{3\,4-k}^{\mp}, \quad R_k = -C_{3\,4-k}, \quad H_0 = H^-, \quad H_1 = H^{\mathrm{i}1}, \quad H_2 = H^{\mathrm{j}1}, \quad H_3 = H^+. \tag{7.7}$$

By the formulas (3.7) and the second relation in the above, the constraint (7.6) is expressed in terms of the type A $\mathcal{N}$-fold SUSY parameters as $R = 0$, which is identical to the constraint in the $\mathcal{N} = 2$ case (cf., Ref. [1], Section 4).

The realization of third-order paraSUSY via the formulas (7.5) admits another nonlinear relation, Ref. [13], Eq. (4.33). For the present system, it reads by the formulas (3.7) and the second relation in (7.7)

$$\begin{aligned}(\boldsymbol{Q}_{\mathrm{P}}^-)^3(\boldsymbol{Q}_{\mathrm{P}}^+)^3 + \left\{ \begin{array}{c} \boldsymbol{Q}_{\mathrm{P}}^+(\boldsymbol{Q}_{\mathrm{P}}^-)^3(\boldsymbol{Q}_{\mathrm{P}}^+)^2 \\ (\boldsymbol{Q}_{\mathrm{P}}^-)^2(\boldsymbol{Q}_{\mathrm{P}}^+)^3\boldsymbol{Q}_{\mathrm{P}}^- \end{array} \right\} + \left\{ \begin{array}{c} \boldsymbol{Q}_{\mathrm{P}}^-(\boldsymbol{Q}_{\mathrm{P}}^+)^3(\boldsymbol{Q}_{\mathrm{P}}^-)^2 \\ (\boldsymbol{Q}_{\mathrm{P}}^+)^2(\boldsymbol{Q}_{\mathrm{P}}^-)^3\boldsymbol{Q}_{\mathrm{P}}^+ \end{array} \right\} + (\boldsymbol{Q}_{\mathrm{P}}^+)^3(\boldsymbol{Q}_{\mathrm{P}}^-)^3 \\ = 8\left(\left(\boldsymbol{H}_{\mathrm{P}} + R + \frac{a_2}{3}\right)^2 - b_1^2\right)\left(\boldsymbol{H}_{\mathrm{P}} + R - \frac{2a_2}{3}\right).\end{aligned} \tag{7.8}$$

We note that this algebraic relation holds irrespective of the paraSUSY constraint (7.6). Thus, in this sense the latter algebra (7.8) is more general than the paraSUSY algebra (7.1b).

In the subsector with the parafermion number 0 and 3, the nonlinear algebra (7.8) reduces to

$$\left\{(\boldsymbol{Q}_{\mathrm{P}}^-)^3, (\boldsymbol{Q}_{\mathrm{P}}^+)^3\right\} = 8\left(\left(\boldsymbol{H}_{\mathrm{P}} + R + \frac{a_2}{3}\right)^2 - b_1^2\right)\left(\boldsymbol{H}_{\mathrm{P}} + R - \frac{2a_2}{3}\right)\bigg|_{\mathfrak{F}\times(\mathsf{V}_3^{(0)}+\mathsf{V}_3^{(3)})}. \tag{7.9}$$

This, together with the relations

$$\left[(\boldsymbol{Q}_{\mathrm{P}}^{\pm})^3, \boldsymbol{H}_{\mathrm{P}}\right] = \left\{(\boldsymbol{Q}_{\mathrm{P}}^{\pm})^3, (\boldsymbol{Q}_{\mathrm{P}}^{\pm})^3\right\} = 0, \tag{7.10}$$

which follow directly from the third-order paraSUSY relations in (7.1a) forms a 3-fold superalgebra. We can now easily check that the latter algebra exactly coincides with type A 3-fold superalgebra (2.15) in Class $(1,1)$ by the conditions (4.1). Hence, an arbitrary type A 3-fold SUSY system which belongs to Class $(1,1)$ admits a realization of paraSUSY of order 3 and the generalized type A 3-fold superalgebra (7.8).

B. Generalized 3-fold Superalgebra in Class $(0,1)$ and Class $(1,0)$

Contrary to the case of Class $(1,1)$, any system belonging to Class $(0,1)$ and $(1,0)$ admits neither paraSUSY of order 3 nor quasi-paraSUSY of order $(3,q)$ [13]. The reason is the lack of a 'shape-invariant' condition at the place where an intermediate Hamiltonian is absent. However, as we shall show shortly, a restricted version of the generalized type A 3-fold superalgebra (7.8) still holds in each of Class $(0,1)$ and Class $(1,0)$ with the same parafermionic setting as (7.3) and (7.7). Indeed, substituting the relations (7.7) into the formulas (4.21)–(4.26) in Ref. [13] and using the intertwining relations (2.20) and the formula (3.18) for Class $(0,1)$, and (2.21) and (3.23) for Class $(1,0)$, respectively, we see that in the case of Class $(0,1)$ the following algebraic relation holds in the subsector with the parafermion number 0, 2, and 3

$$(\boldsymbol{Q}_{\mathrm{P}}^{-})^{3}(\boldsymbol{Q}_{\mathrm{P}}^{+})^{3}+\left\{\begin{array}{c}\boldsymbol{Q}_{\mathrm{P}}^{-}(\boldsymbol{Q}_{\mathrm{P}}^{+})^{3}(\boldsymbol{Q}_{\mathrm{P}}^{-})^{2}\\(\boldsymbol{Q}_{\mathrm{P}}^{+})^{2}(\boldsymbol{Q}_{\mathrm{P}}^{-})^{3}\boldsymbol{Q}_{\mathrm{P}}^{+}\end{array}\right\}+(\boldsymbol{Q}_{\mathrm{P}}^{+})^{3}(\boldsymbol{Q}_{\mathrm{P}}^{-})^{3}$$
$$=8S_{2}^{(0,1)}(\boldsymbol{H}_{\mathrm{P}}+R)\left(\boldsymbol{H}_{\mathrm{P}}+R+\frac{a_{2}}{3}+b_{1}\right)\bigg|_{\mathfrak{F}\times(\mathsf{V}_{3}^{(0)}+\mathsf{V}_{3}^{(2)}+\mathsf{V}_{3}^{(3)})}. \quad (7.11)$$

and that in the case of Class $(1,0)$ the algebra which holds in the subsector with the parafermion number 0, 1, and 3 reads

$$(\boldsymbol{Q}_{\mathrm{P}}^{-})^{3}(\boldsymbol{Q}_{\mathrm{P}}^{+})^{3}+\left\{\begin{array}{c}\boldsymbol{Q}_{\mathrm{P}}^{+}(\boldsymbol{Q}_{\mathrm{P}}^{-})^{3}(\boldsymbol{Q}_{\mathrm{P}}^{+})^{2}\\(\boldsymbol{Q}_{\mathrm{P}}^{-})^{2}(\boldsymbol{Q}_{\mathrm{P}}^{+})^{3}\boldsymbol{Q}_{\mathrm{P}}^{-}\end{array}\right\}+(\boldsymbol{Q}_{\mathrm{P}}^{+})^{3}(\boldsymbol{Q}_{\mathrm{P}}^{-})^{3}$$
$$=8S_{2}^{(1,0)}(\boldsymbol{H}_{\mathrm{P}}+R)\left(\boldsymbol{H}_{\mathrm{P}}+R+\frac{a_{2}}{3}-b_{1}\right)\bigg|_{\mathfrak{F}\times(\mathsf{V}_{3}^{(0)}+\mathsf{V}_{3}^{(1)}+\mathsf{V}_{3}^{(3)})}. \quad (7.12)$$

In the whole parafermionic vector space $\mathfrak{F}\times\mathsf{V}_3$, however, no algebraic relations like (7.8) hold for Class $(0,1)$ and Class $(1,0)$. In fact, for Class $(0,1)$ each of the second term in the l.h.s. of (7.8) is calculated in the subsector with the parafermion number 1 as

$$\begin{aligned}\boldsymbol{Q}_{\mathrm{P}}^{+}(\boldsymbol{Q}_{\mathrm{P}}^{-})^{3}(\boldsymbol{Q}_{\mathrm{P}}^{+})^{2}&=2P_{33}^{-}P_{33}^{+}P_{32}^{+}(H^{\mathrm{j}1}-C_{31})P_{32}^{-}\big|_{\mathfrak{F}\times\mathsf{V}_{3}^{(1)}},\\(\boldsymbol{Q}_{\mathrm{P}}^{-})^{2}(\boldsymbol{Q}_{\mathrm{P}}^{+})^{3}\boldsymbol{Q}_{\mathrm{P}}^{-}&=2P_{32}^{+}(H^{\mathrm{j}1}-C_{31})P_{32}^{-}P_{33}^{-}P_{33}^{+}\big|_{\mathfrak{F}\times\mathsf{V}_{3}^{(1)}},\end{aligned} \quad (7.13)$$

and cannot be expressed as a polynomial of $H^{\mathrm{j}1}$. Similarly, for Class $(1,0)$ each of the third term in the l.h.s. of (7.8) is calculated in the subsector with the parafermion number 2 as

$$\begin{aligned}\boldsymbol{Q}_{\mathrm{P}}^{-}(\boldsymbol{Q}_{\mathrm{P}}^{+})^{3}(\boldsymbol{Q}_{\mathrm{P}}^{-})^{2}&=2P_{31}^{+}P_{31}^{-}P_{32}^{-}(H^{\mathrm{i}1}-C_{33})P_{32}^{+}\big|_{\mathfrak{F}\times\mathsf{V}_{3}^{(2)}},\\(\boldsymbol{Q}_{\mathrm{P}}^{+})^{2}(\boldsymbol{Q}_{\mathrm{P}}^{-})^{3}\boldsymbol{Q}_{\mathrm{P}}^{+}&=2P_{32}^{-}(H^{\mathrm{i}1}-C_{33})P_{32}^{+}P_{31}^{+}P_{31}^{-}\big|_{\mathfrak{F}\times\mathsf{V}_{3}^{(2)}},\end{aligned} \quad (7.14)$$

and again cannot be expressed as a polynomial of $H^{\mathrm{i}1}$.

We note that both the nonlinear algebras (7.11) and (7.12) are compatible with the type A 3-fold superalgebra (2.15). In fact, it is easy to check that both (7.11) and (7.12) reduce to the anti-commutator of the type A 3-fold superalgebra (2.15b) in the more restricted subsector with the parafermion number 0 and 3 by noting the condition (4.2) for the former Class $(0,1)$ and by (4.3) for the latter Class $(1,0)$.

VIII. DISCUSSION AND SUMMARY

In this article, we have fully investigated the necessary and sufficient conditions for a type A 3-fold SUSY system to have one or more sets of intermediate Hamiltonians and then made the complete classification of them by the property of the $GL(2,\mathbb{C})$ transformations. When only one set of intermediate Hamiltonians is concerned, there are three different patterns in the existence and called Class $(1,1)$, Class $(0,1)$, and Class $(1,0)$, respectively. We have found that all the models which belong to Class $(1,1)$ are not only solvable but also shape invariant while the ones which belong to Class $(0,1)$ or Class $(1,0)$ are just quasi-solvable.

Case	Possible Classes
I	$(1,1)$
II	$(1,1) \supset (1,2)$
II'	$(0,1)$
III	$(1,1)$
IV	$(1,1) \supset (1,2) \supset \begin{cases} (1,3) \\ (1+1,1+1) \end{cases}$
IV'	$(0,1) \supset \begin{cases} (0,2) \\ (1;1) \end{cases}$
V	$(0,1) \supset \begin{cases} (0,2) \supset (0,3) \\ (1;1) \supset (1;2) \end{cases}$

TABLE II: The possible classes of intermediate Hamiltonians for each case of type A 3-fold SUSY models.

When more than one sets of intermediate Hamiltonians are concerned, there emerge various patterns depending on the functional type of each model. In Table II, we summarize the possible classes for each case of type A 3-fold SUSY models.

It is now evident from Table II that the structure of higher-order intertwining operators is much richer than the degree that one can classify them solely by the notion of reducibility introduced in Refs. [7, 8]. It is not only because the requirement of reality is restrictive but also because there are various patterns in the existence of intermediate Hamiltonians. Needless to say, the number of possible patterns drastically increase as the order $\mathcal{N}$ of intertwining operators gets higher.

Although we have not assumed in this article the reality of Hamiltonians and thus have analyzed general complex Hamiltonians by employing the $GL(2,\mathbb{C})$ transformations, it is straightforward to examine and classify real Hamiltonians by the use of the real $GL(2,\mathbb{R})$ transformations instead of $GL(2,\mathbb{C})$. We only show in Table III an example of the real classification scheme for that purpose. Note, however, that some of the possible classes for Case IV and Case V in Table II which can exist in the complex case might be missing in the real case since the solutions to the conditions (5.5), (5.6), (5.8), (5.11) or (5.12) are not necessarily real.

The realization of the variant generalized 3-fold superalgebras in Section VII indicates that the parafermionic formulation like (7.3) could provide a more adequate and advantageous framework to formulate $\mathcal{N}$-fold SUSY than the conventional fermionic formulation like (2.14). In the conventional approach, the type A 3-fold superalgebra (2.15) cannot characterize nor detect the existence of intermediate Hamiltonians at all. In contrast to it, in the parafermionic approach on the one hand the type A 3-fold superalgebra is always realized in the subsector with the fermion number 0 and 3, and on the other hand the various patterns in the existence of intermediate Hamiltonians are characterized by considering the other subsector with the fermion number 1 and/or 2. In addition, generalized $\mathcal{N}$-fold superalgebra like (7.8) could provide an alternative for defining paraSUSY of order $\mathcal{N}$. In

Case	Class $(1,1)$	Class $(0,1)$, $(1,0)$
I	$a/2$	
II	$2z$	
II'		$2z^3$
III	$z^2/2$	
IV	$\pm 2a(z^2-1)$	
	$\pm 2a(z^2+1)$	
IV'		$2az^2(z+1)$
V		$2z(1-z)(1-mz)$
		$z(z^2+2(1-2m)z+1)/2$

TABLE III: A real classification scheme of type A 3-fold SUSY models with one set of intermediate Hamiltonians. In the above, $a \in \mathbb{R}$ is an arbitrary constant and $0 < m < 1$.

the present $\mathcal{N} = 3$ case, the conventional defining algebra (7.1b) only produces in essence the additional constraint (7.6) whose physical relevance is unclear. Hence, there seems, at least until now, no physical evidence to claim that which definition is appropriate.

In principle, one can continue to study the $\mathcal{N} > 3$ cases, but as already mentioned the number of different patterns in the existence of intermediate Hamiltonians drastically increases as $\mathcal{N}$ increases. However, the number of different sets of intermediate Hamiltonians would be limited to at most 4 regardless of the number of patterns due to the following reasons. First, the transformation formula (2.10) which is valid for all $\mathcal{N} \geq 3$ could produce algebraic equations of at most fourth-degree and thus at most four additional sets of intermediate Hamiltonians would be admissible. But type A $\mathcal{N}$-fold SUSY systems for $\mathcal{N} \geq 3$ have at most three independent free parameters b_i $(i = 0, 1, 2)$ and thus at most three different solutions among the four would be compatible simultaneously. Hence, at most three *additional* sets would be available, which means that the maximum number of different sets is four in total. By a similar argument we conclude that it would be at most 3 if one constraint on the parameters b_i are inevitable for the existence of one set of intermediate Hamiltonians, as in the present $\mathcal{N} = 3$ case (4.1)–(4.3), since in this case there are essentially at most two independent free parameters and thus at most two algebraic solutions would be compatible simultaneously. The latter fact is indeed the reason why in the $\mathcal{N} = 3$ case there are at most three different sets and thus are no classes such as Class $(m, m+n)$ with $m+n > 3$, Class $(m+l, m+n)$ with $m+l+n > 3$, Class $(0, n)$ with $n > 3$, and Class $(l; n)$ with $l+n > 3$.

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