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CP violation due to compactification

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We address the challenging issue of how *CP* violation is realized in higher dimensional gauge theories without higher dimensional elementary scalar fields. In such theories interactions are basically governed by a gauge principle and therefore to get *CP* violating phases is a nontrivial task. It is demonstrated that *CP* violation is achieved as the result of compactification of extra dimensions, which is incompatible with the 4-dimensional *CP* transformation. As a simple example we adopt a 6-dimensional U(1) model compactified on a 2-dimensional orbifold T^2/Z_4. We argue that the 4-dimensional *CP* transformation is related to the complex structure of the extra space and show how the Z_4 orbifolding leads to *CP* violation. We confirm by explicit calculation of the interaction vertices that *CP* violating phases remain even after the rephasing of relevant fields. For completeness, we derive a rephasing invariant *CP* violating quantity, following a similar argument in the Kobayashi-Maskawa model which led to the Jarlskog parameter. As an example of a *CP* violating observable we briefly comment on the electric dipole moment of the electron.

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I. INTRODUCTION

In spite of the great success of the Kobayashi-Maskawa model [1], the fundamental origin of *CP* violation still seems to be elusive. Once space-time is enlarged such that it contains extra spatial dimensions, some new types of mechanisms of *CP* violation may be possible. In this paper we address the question as to whether *CP* violation is realized as the result of compactification of the extra spatial dimensions.

In order to extract the new type of *CP* violating mechanism due to the compactification, we will work in the framework of higher dimensional gauge theories without (higher dimensional) elementary scalar fields. Namely we exclude, e.g. the models of universal extra dimension, where the Higgs scalar is introduced and the same mechanism of *CP* violation as that in the Kobayashi-Maskawa model is operative.

A typical example of such theories is 10-dimensional supersymmetric Yang-Mills theory, which is the low-energy point particle limit of the open string sector of superstring theory. An interesting and nontrivial question is how to get *CP* violation in this type of higher dimensional gauge theory. Let us note that in such theories all interactions including possible four-dimensional Yukawa couplings are basically controlled by the gauge principle and therefore, the theory to start with is expected to be *CP* invariant, since all gauge couplings are of course real. Thus to realize *CP* violation is a challenging issue.

Since the original theory is *CP* invariant, a possible way to break *CP* would-be "spontaneous violation." More precisely, one of the few possibilities to break *CP* symmetry in such theories is to invoke the manner of compactification [2,3], which determines the vacuum state of the theory. (See also Ref. [4] for a discussion of *CP* symmetry in orbifold superstring theories.) An important observation in the argument is that although *C* and *P* transformations in higher dimensions can be easily found such that $\psi^c = C\bar{\psi}^t$, $C^\dagger \Gamma^M C = -(\Gamma^M)^t$; for instance, they do not simply reduce to ordinary 4-dimensional transformations and should be modified in order to recover the 4-dimensional ones. Interestingly, such a modified *CP* transformation was demonstrated to be equivalent (for even space-time dimensions) to the complex conjugation of the complex homogeneous coordinates z^a describing the extra space [3],

$$CP\colon\ z^a \to z^{a*}. \tag{1}$$

For illustrative purpose, let us consider the four generation model in Type-I superstring theory with six-dimensional Calabi-Yau manifold defined by a quintic polynomial

$$\sum_{a=1}^{5}(z^a)^5 - C(z^1 z^2 \cdots z^5) = 0. \tag{2}$$

CP is broken only when the coefficient *C* is complex, since otherwise the above defining equation is clearly invariant under $z^a \to z^{a*}$. Another possibility of spontaneous *CP* violation in this type of theory is due to the vacuum expectation value of an effective four-dimensional scalar field, which is originally the extra space component of the gauge field and may have an odd *CP* eigenvalue [5,6].

Unfortunately, the Calabi-Yau manifold is not easy to handle and to derive the resultant 4-dimensional couplings is very challenging. In this paper, we focus on a much simpler compactification, where interaction vertices are easily obtained. Namely, we discuss *CP* violation in the six-dimensional U(1) model due to the compactification on a 2-dimensional orbifold T^2/Z_4. We note that the six-dimensional model is the simplest possibility for incorporating a complex structure for the extra space.

The Z_4 orbifolding turns out to lead to CP violation. Without explicit calculations of interaction vertices we can easily understand the reason for CP violation from the following geometrical argument. Let the extra space coordinates be (y, z) and combine the pair of coordinates to form a complex coordinate $\omega = \frac{y+iz}{\sqrt{2}}$. The orbifold is obtained by identifying the points related by the action of Z_4, the rotation on the $y-z$ plane by an angle $\frac{\pi}{2}$, $(-z, y) \sim (y, z)$ (see Fig. 1). Or, by use of the complex coordinate,

$$i\omega \sim \omega. \tag{3}$$

As was discussed above and is explicitly shown below [see Eq. (27)], in terms of the complex coordinate the CP transformation is known to be equivalent to a complex conjugation [3]

$$CP\colon\ \omega \to \omega^*. \tag{4}$$

Therefore, as a result of the CP transformation $i\omega$ and ω in (3) are transformed into $(i\omega)^* = -i\omega^*$ and ω^*, respectively, and after the CP transformation the orbifold condition becomes

$$(-i)\omega^* \sim \omega^*. \tag{5}$$

Namely CP acts as an orientation-changing operator; the rotation by an angle $\frac{\pi}{2}$ has been changed into a rotation by an angle $-\frac{\pi}{2}$. This feature is illustrated in Fig. 2. Hence the orbifold condition is not compatible with the CP transformation and therefore the CP symmetry is broken as the consequence of the orbifold compactification.

This argument implies that Z_2 orbifolding does not lead to CP violation, since the identification $-\omega \sim \omega$ is equivalent to $-\omega^* \sim \omega^*$, or in other words, a rotation by an angle π is equivalent to a rotation by an angle $-\pi$.

In the language of the KK (Kaluza-Klein) mode function, CP violation in Z_4 orbifolding may be understood as follows. A generic mode function ϕ of (y, z) or equivalently of $\omega = \frac{y+iz}{\sqrt{2}}$ should have an eigenvalue t under the action of Z_4, as is seen in Eq. (10) in the next section,

$$\phi(i\omega) = t\phi(\omega) \qquad (t^4 = 1). \tag{6}$$

What we obtain from (6) by taking its complex conjugation is

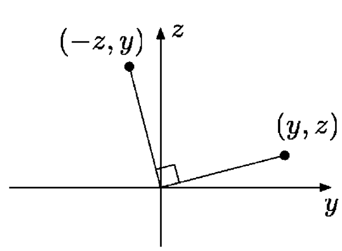

FIG. 1. The identification of points in the orbifold T^2/Z_4.

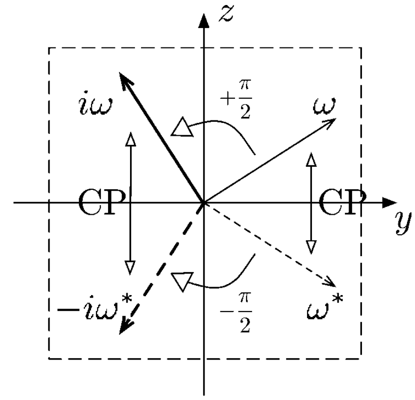

FIG. 2. The CP transformation acting on the orbifold T^2/Z_4.

$$\phi^*(-i\omega^*) = t^*\phi^*(\omega^*). \tag{7}$$

This means after the CP transformation, $\phi(\omega) \to \phi^*(\omega^*)$, etc., the mode function has eigenvalue t^* under the action of Z_4. Thus for mode functions with complex eigenvalues $t = \pm i$ the orbifold condition is not compatible with the CP transformation, leading to the CP violation originating from these mode functions. In fact, we will see in this article that the presence of such mode functions results in CP violating interaction vertices.

We will discuss how the CP violating phases emerge in the vertices of four-dimensional gauge and Yukawa interactions including nonzero KK modes. The phases are confirmed to remain even after the rephasing of relevant fields by showing a concrete example and also constructing rephasing invariant quantities. As the typical example of CP violating observable, which may be relevant for a model without generation structure, we briefly comment on the electric dipole moment (EDM) of electron.

This paper is organized as follows. In the next section, the mode functions on the T^2/Z_4 orbifold are constructed and their orthonormality conditions are derived. In Sec. III, our model is introduced and after fixing the Z_4 eigenvalue of each field the fields are expanded as the sum of KK modes by use of the mode functions. In Sec. IV, four-dimensional mass eigenstates and corresponding mass eigenvalues are obtained from the free Lagrangian, where a R_ξ-type gauge-fixing term for the sector of gauge-Higgs bosons is added. In Sec. V, we derive gauge and Yukawa interaction vertices with respect to the mass eigenstates. In Sec. VI, we demonstrate, as a simple example, that CP violating phases appear in the interaction vertices of KK photons and argue that the phases remain even after possible rephasing of the fields. In Sec. VII, we derive a rephasing invariant CP violating quantity, following a similar argument in the Kobayashi-Maskawa model [1], which led to the Jarlskog parameter [7]. In Sec. VIII, we briefly comment on the EDM of electron in our model. Section IX is devoted to the summary.

II. MODE FUNCTIONS ON T^2/Z_4

The orthonormal set of mode functions on T^2 is given as

$$\varphi^{(m,n)}(y,z) = \frac{1}{2\pi R} e^{i((my+nz)/R)} \qquad (m, n\text{: integers}). \quad (8)$$

Then the eigenfunctions of Z_4 with eigenvalues $t = \pm 1$, $\pm i$ are constructed by a superposition

$$\tilde{\Phi}_t^{(m,n)}(y,z) = \tfrac{1}{2}[\varphi^{(m,n)}(y,z) + t^3\varphi^{(m,n)}(-z,y) + t^2\varphi^{(m,n)}(-y,-z) + t\varphi^{(m,n)}(z,-y)], \quad (9)$$

which satisfies (with $t^4 = 1$) [8]

$$\tilde{\Phi}_t^{(m,n)}(-z,y) = t\tilde{\Phi}_t^{(m,n)}(y,z). \quad (10)$$

Namely the eigenfunctions are obtained by the successive action of Z_4, the rotation on the $y - z$ plane by an angle $\frac{\pi}{2}$, on the mode functions on T^2.

Let us note $\tilde{\Phi}_t^{(m,n)}$ are also obtained by the rotation in the momentum space $(\frac{m}{R}, \frac{n}{R}) \to (\frac{n}{R}, -\frac{m}{R})$:

$$\tilde{\Phi}_t^{(m,n)}(y,z) = \frac{1}{2}[\varphi^{(m,n)}(y,z) + t^3\varphi^{(n,-m)}(y,z) + t^2\varphi^{(-m,-n)}(y,z) + t\varphi^{(-n,m)}(y,z)]. \quad (11)$$

This means the extra-dimensional momenta can be restricted to a "fundamental domain" shown in Fig. 3: ($m \geq 1$, $n \geq 0$ or $m = n = 0$). By use of the orthonormality of $\varphi^{(m,n)}$,

$$\int_{-\pi R}^{\pi R} dy \int_{-\pi R}^{\pi R} dz \varphi^{(m,n)}(y,z)^* \varphi^{(m',n')}(y,z) = \delta_{m,m'}\delta_{n,n'} \qquad (m, n, m', n'\text{: integers}), \quad (12)$$

and (11), we easily get

$$\int_{-\pi R}^{\pi R} dy \int_{-\pi R}^{\pi R} dz \tilde{\Phi}_t^{(m,n)}(y,z)^* \tilde{\Phi}_{t'}^{(m',n')}(y,z) = \delta_{m,m'}\delta_{n,n'} \times \begin{cases} \delta_{t,t'} & (m \geq 1, n \geq 0) \\ 4\delta_{t,1}\delta_{t',1} & (m = n = 0) \end{cases}. \quad (13)$$

Thus the orthogonal set of mode functions are known to be [$m \geq 1$, $n \geq 0$ or $m = n = 0$ (only for $t = 1$)]

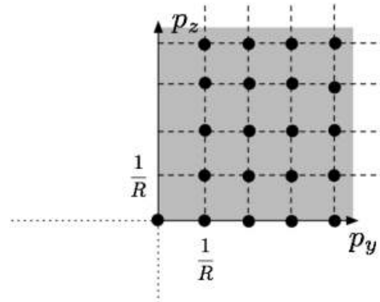

FIG. 3. The fundamental domain in the plane of extra space components of momentum.

$$\begin{aligned}
\Phi_{t=1}^{(m,n)}(y,z) &= \frac{1}{\sqrt{1 + 3\delta_{m,0}\delta_{n,0}}}\tilde{\Phi}_{t=1}^{(m,n)}(y,z) = \frac{1}{2\pi R}\frac{1}{\sqrt{1 + 3\delta_{m,0}\delta_{n,0}}}\left[\cos\left(\frac{my+nz}{R}\right) + \cos\left(\frac{ny-mz}{R}\right)\right], \\
\Phi_{t=-1}^{(m,n)}(y,z) &= \tilde{\Phi}_{t=-1}^{(m,n)}(y,z) = \frac{1}{2\pi R}\left[\cos\left(\frac{my+nz}{R}\right) - \cos\left(\frac{ny-mz}{R}\right)\right], \\
\Phi_{t=i}^{(m,n)}(y,z) &= -i\tilde{\Phi}_{t=i}^{(m,n)}(y,z) = \frac{1}{2\pi R}\left[\sin\left(\frac{my+nz}{R}\right) - i\sin\left(\frac{ny-mz}{R}\right)\right], \\
\Phi_{t=-i}^{(m,n)}(y,z) &= -i\tilde{\Phi}_{t=-i}^{(m,n)}(y,z) = \frac{1}{2\pi R}\left[\sin\left(\frac{my+nz}{R}\right) + i\sin\left(\frac{ny-mz}{R}\right)\right].
\end{aligned} \quad (14)$$

In terms of these mode functions, a generic bulk field $F(x,y,z)$ is KK-mode-expanded as follows depending on its Z_4-eigenvalue t;

$$F(x,y,z) = \begin{cases}
\frac{1}{2\pi R}F^{(0)}(x) + \frac{1}{2\pi R}\sum_{m=1}^\infty\sum_{n=0}^\infty F^{(m,n)}(x)[\cos(\frac{my+nz}{R}) + \cos(\frac{ny-mz}{R})] & (\text{for } t = 1), \\
\frac{1}{2\pi R}\sum_{m=1}^\infty\sum_{n=0}^\infty F^{(m,n)}(x)[\cos(\frac{my+nz}{R}) - \cos(\frac{ny-mz}{R})] & (\text{for } t = -1), \\
\frac{1}{2\pi R}\sum_{m=1}^\infty\sum_{n=0}^\infty F^{(m,n)}(x)[\sin(\frac{my+nz}{R}) - i\sin(\frac{ny-mz}{R})] & (\text{for } t = i), \\
\frac{1}{2\pi R}\sum_{m=1}^\infty\sum_{n=0}^\infty F^{(m,n)}(x)[\sin(\frac{my+nz}{R}) + i\sin(\frac{ny-mz}{R})] & (\text{for } t = -i).
\end{cases} \quad (15)$$

The presence of the factor i in the fields with $t = \pm i$ signals *CP* violation.

III. THE MODEL AND Z_4-EIGENVALUE ASSIGNMENT

As the simplest realization of *CP* violation, we consider six-dimensional QED compactified on T^2/Z_4, whose Lagrangian is given as

$$\mathcal{L}_{\rm QED} = \bar{\Psi}_6\{\Gamma^M(i\partial_M + gA_M) - m_B\}\Psi_6 - \tfrac{1}{4}(\partial_M A_N - \partial_N A_M)(\partial^M A^N - \partial^N A^M) \qquad (M, N = 0-3, y, z), \quad (16)$$

where gauge fixing and F-P ghost terms have not been shown explicitly. Let us note that in contrast to the case of a five-dimensional model with S^1/Z_2 orbifold, the bulk mass term $-m_B\bar{\Psi}_6\Psi_6$ is allowed, since Z_4 is a rotation in the y-z plane, under which $\bar{\Psi}_6\Psi_6$ is obviously invariant. The electron described by the zero mode of (the half of) Ψ_6 thus has a mass m_B. Note also that A_y and A_z have nontrivial Z_4-eigenvalues, as is discussed later, and therefore, have neither zero modes nor VEV's.

The Z_4 symmetry implies that the extra space components of A_M, i.e. A_y and A_z, and Ψ_6 should properly transform under the action of Z_4. First, defining a complexified coordinate and vector potential

$$\omega \equiv \frac{y+iz}{\sqrt{2}}, \qquad A_\omega \equiv \frac{A_y - iA_z}{\sqrt{2}}, \tag{17}$$

the transformation properties of A_y and A_z are equivalent to

$$A_\omega(x, i\omega) = (-i)A_\omega(x, \omega). \tag{18}$$

Namely, A_ω is an eigenfunction under Z_4 with eigenvalue $-i$ and is mode-expanded as

$$A_\omega(x,y,z) = \frac{1}{2\pi R}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_\omega^{(m,n)}(x)\left[\sin\left(\frac{my+nz}{R}\right) + i\sin\left(\frac{ny-mz}{R}\right)\right], \tag{19}$$

where $A_\omega^{(m,n)}$ are complex functions, whose real and imaginary parts are denoted by $A_y^{(m,n)}$ and $A_z^{(m,n)}$, respectively: $A_\omega^{(m,n)} \equiv \frac{A_y^{(m,n)} - iA_z^{(m,n)}}{\sqrt{2}}$.

Since Z_4 is a rotation of an angle $\frac{\pi}{2}$, the 6-dimensional Dirac fermion transforms as

$$\frac{\mathbf{I} - \Gamma^y\Gamma^z}{\sqrt{2}}\Psi_6(x, i\omega) = (-i)^{1/2}\Psi_6(x,\omega), \tag{20}$$

where the phase-factor has an arbitrariness and is chosen such that Ψ_6 has a zero mode. We decompose Ψ_6 into two four-dimensional Dirac spinors:

$$\Psi_6 \equiv \begin{pmatrix}\psi \\ \Psi\end{pmatrix}. \tag{21}$$

In this base,

$$\begin{aligned}
\Gamma^\mu &= \gamma^\mu \otimes I_2 = \begin{bmatrix}\gamma^\mu & \\ & \gamma^\mu\end{bmatrix}, \\
\Gamma^y &= \gamma^5 \otimes i\sigma_1 = \begin{bmatrix} & i\gamma^5 \\ i\gamma^5 & \end{bmatrix}, \\
\Gamma^z &= \gamma^5 \otimes i\sigma_2 = \begin{bmatrix} & \gamma^5 \\ -\gamma^5 & \end{bmatrix}.
\end{aligned} \tag{22}$$

Then from (20) and (22) we find

$$\psi(x, i\omega) = (-i)\psi(x,\omega), \qquad \Psi(x, i\omega) = \Psi(x,\omega). \tag{23}$$

Let us note that only Ψ is allowed to have a zero mode. Accordingly, each field is mode-expanded as

$$\begin{aligned}
\psi(x,y,z) &= \frac{1}{2\pi R}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\psi^{(m,n)}(x)\left[\sin\left(\frac{my+nz}{R}\right) + i\sin\left(\frac{ny-mz}{R}\right)\right], \\
\Psi(x,y,z) &= \frac{1}{2\pi R}\Psi^{(0)}(x) + \frac{1}{2\pi R}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\Psi^{(m,n)}(x) \times\left[\cos\left(\frac{my+nz}{R}\right) + \cos\left(\frac{ny-mz}{R}\right)\right].
\end{aligned} \tag{24}$$

We will show below that the factor i in front of $\sin(\frac{ny-mz}{R})$ in the mode expansion of $\psi(x,y,z)$ results in CP violating phases in the interaction vertices of A_μ with ψ.

One may wonder if the requirement of anomaly cancellation affects the CP violation by enforcing the introduction of additional fields. Fortunately, our model is easily shown to be free from both four-dimensional and six-dimensional anomalies, and there is no need for additional fields. First, the four-dimensional anomaly due to the zero mode $\Psi^{(0)}(x)$ trivially vanishes, since $\Psi^{(0)}(x)$ is a four-dimensional Dirac spinor and its coupling to the photon is vectorlike. Concerning the six-dimensional anomaly [9], we note that each of ψ and Ψ is "nonchiral" in a six-dimensional sense. Namely, each fermion has both eigenvalues ± 1 of $\Gamma_7 \equiv \Gamma^0\Gamma^1\cdots\Gamma^y\Gamma^z$. This is easily seen, since in the base (22) $\Gamma_7 = \gamma^5 \otimes \sigma_3$ and the eigenvalue of Γ_7 is the product of the eigenvalues of the four-dimensional chirality and extra-dimensional chirality, namely, the eigenvalues of γ^5 and σ_3, respectively. Each of ψ and Ψ has eigenvalues 1 and -1 of σ_3, respectively, while each spinor is a four-dimensional Dirac spinor and has both eigenvalues, ± 1 of γ^5. Thus, each of ψ and Ψ has both eigenvalues ± 1 of Γ_7. These properties come essentially from the fact that we have started with a 6-dimensional Dirac spinor. Hence, each of ψ and Ψ, being "nonchiral" in six-dimensional sense, does not yield any six-dimensional anomalies.

In the base of gamma matrix (22) the "modified" P and C transformations, corresponding to ordinary four-dimensional ones, are explicitly given as

$$\begin{aligned}
&P\colon\ \Psi_6 \to (\gamma^0 \otimes \sigma_3)\Psi_6, \\
&C\colon\ \Psi_6 \to (c_4 \otimes \sigma_3)\bar{\Psi}_6^t \qquad (c_4 = i\gamma^2\gamma^0).
\end{aligned} \tag{25}$$

We can easily check that under the C transformation defined by (25), a pair of bilinears of Ψ_6, namely $(V^y, V^z) = (\bar{\Psi}_6\Gamma^y\Psi_6, \bar{\Psi}_6\Gamma^z\Psi_6)$ transforms into $(-V^y, V^z)$, while (V^y, V^z) is invariant under the P transformation. Accordingly, a pair of extra space coordinates (y,z) should transform as

$$P\colon\ (y,z) \to (y,z), \qquad C, CP\colon\ (y,z) \to (y,-z). \tag{26}$$

We thus explicitly confirm the transformation property of the complex coordinate discussed in [3]:

$$CP\colon\ \omega \to \omega^*. \tag{27}$$

IV. THE MASS EIGENSTATES AND MASS EIGENVALUES

Substituting the mode expansions (19) and (24), together with

$$A_\mu(x,y,z) = \frac{1}{2\pi R}A_\mu^{(0)}(x) + \frac{1}{2\pi R}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}A_\mu^{(m,n)}(x) \times\left[\cos\left(\frac{my+nz}{R}\right) + \cos\left(\frac{ny-mz}{R}\right)\right], \tag{28}$$

in the Lagrangian (16) and integrating over the extra space coordinates y and z we get the effective theory from four-dimensional perspective.

We first focus on the free Lagrangian to get the mass matrices for various fields of a fixed KK mode. Let us note that there should be a mixing between ψ and Ψ for fermions and a mixing between A_μ and a certain linear combination of A_y and A_z through a Higgs-like mechanism, both only for nonzero KK modes. To get mass eigenstates and their mass eigenvalues, we need to diagonalize the mass matrix in the base of ψ and Ψ for fermions and put a suitable gauge-fixing term to eliminate mixing for the gauge-Higgs sector.

The mass matrix for the fermion in the base of $\psi^{(m,n)}$ and $\Psi^{(m,n)}$ for a given nonzero mode (m,n) $(m \geq 1, n \geq 0)$ can be read off from the part $\bar{\Psi}_6\{i(\Gamma^y\partial_y + \Gamma^z\partial_z) - m_B\}\Psi_6$. After the y, z integrations this part yields the mass term

$$\left(\frac{m-in}{R}\bar{\psi}^{(m,n)}\gamma^5\Psi^{(m,n)} + \text{H.c.}\right) - m_B(\bar{\psi}^{(m,n)}\psi^{(m,n)} + \bar{\Psi}^{(m,n)}\Psi^{(m,n)}). \tag{29}$$

In order to eliminate γ^5 in the first parenthesis, we perform a chiral rotation,

$$\Psi^{(m,n)} \to \tilde{\Psi}^{(m,n)} \equiv \gamma^5\Psi^{(m,n)}. \tag{30}$$

Then in terms of $\psi^{(m,n)}$ and $\tilde{\Psi}^{(m,n)}$ the mass term is written as

$$\left(\frac{m-in}{R}\bar{\psi}^{(m,n)}\tilde{\Psi}^{(m,n)} + \text{H.c.}\right) - m_B\bar{\psi}^{(m,n)}\psi^{(m,n)} + m_B\bar{\tilde{\Psi}}^{(m,n)}\tilde{\Psi}^{(m,n)}, \tag{31}$$

whose mass matrix is now Hermitian, i.e.

$$M_f^{(m,n)} = \begin{pmatrix} m_B & -\frac{m-in}{R} \\ -\frac{m+in}{R} & -m_B \end{pmatrix}. \tag{32}$$

The matrix $M_f^{(m,n)}$ is diagonalized by an unitary matrix $U^{(m,n)}$,

$$\begin{aligned} U^{(m,n)\dagger}M_f^{(m,n)}U^{(m,n)} &= \begin{pmatrix} m_f^{(m,n)} & 0 \\ 0 & -m_f^{(m,n)} \end{pmatrix}, \\ U^{(m,n)} &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi^{(m,n)}} \end{pmatrix}\begin{pmatrix} \cos\theta^{(m,n)} & \sin\theta^{(m,n)} \\ -\sin\theta^{(m,n)} & \cos\theta^{(m,n)} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta^{(m,n)} & \sin\theta^{(m,n)} \\ -\sin\theta^{(m,n)}e^{i\varphi^{(m,n)}} & \cos\theta^{(m,n)}e^{i\varphi^{(m,n)}} \end{pmatrix} \end{aligned} \tag{33}$$

where

$$m_f^{(m,n)} \equiv \sqrt{m_B^2 + \frac{m^2+n^2}{R^2}}, \qquad \tan\varphi^{(m,n)} \equiv \frac{n}{m}, \qquad \tan 2\theta^{(m,n)} \equiv \frac{\frac{\sqrt{m^2+n^2}}{R}}{m_B}. \tag{34}$$

Let us note that by a chiral transformation to change the sign of the eigenvalue $-m_f^{(m,n)}$ we have degenerate mass eigenvalues and then a further unitary transformation by an arbitrary unitary matrix $V^{(m,n)}$ becomes possible. Thus, the mass eigenstates $\psi'^{(m,n)}$ and $\Psi'^{(m,n)}$ are related to the original fields as

$$\begin{pmatrix} \psi^{(m,n)} \\ \tilde{\Psi}^{(m,n)} \end{pmatrix} = U^{(m,n)}\begin{pmatrix} 1 & 0 \\ 0 & \gamma_5 \end{pmatrix}V^{(m,n)}\begin{pmatrix} \psi'^{(m,n)} \\ \Psi'^{(m,n)} \end{pmatrix}, \tag{35}$$

or in terms of Weyl fermions[1] as

$$\begin{aligned} \begin{pmatrix} \psi_R^{(m,n)} \\ \tilde{\Psi}_R^{(m,n)} \end{pmatrix} &= U^{(m,n)}V^{(m,n)}\begin{pmatrix} \psi_R'^{(m,n)} \\ \Psi_R'^{(m,n)} \end{pmatrix}, \\ \begin{pmatrix} \psi_L^{(m,n)} \\ \tilde{\Psi}_L^{(m,n)} \end{pmatrix} &= U^{(m,n)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}V^{(m,n)}\begin{pmatrix} \psi_L'^{(m,n)} \\ \Psi_L'^{(m,n)} \end{pmatrix}. \end{aligned} \tag{36}$$

The freedom of $V^{(m,n)}$ may signal the internal symmetry between two Dirac fermions obtained from a massive six-dimensional Dirac fermion by dimensional reduction. Any physical observables should be invariant under the unitary transformation due to $V^{(m,n)}$. So, without loss of generality we can choose a base where $V^{(m,n)} = U^{(m,n)\dagger}$, to get

$$\begin{aligned} \begin{pmatrix} \psi_R^{(m,n)} \\ \tilde{\Psi}_R^{(m,n)} \end{pmatrix} &= \begin{pmatrix} \psi_R'^{(m,n)} \\ \Psi_R'^{(m,n)} \end{pmatrix}, \\ \begin{pmatrix} \psi_L^{(m,n)} \\ \tilde{\Psi}_L^{(m,n)} \end{pmatrix} &= \hat{U}^{(m,n)}\begin{pmatrix} \psi_L'^{(m,n)} \\ \Psi_L'^{(m,n)} \end{pmatrix}, \end{aligned} \tag{37}$$

where the unitary and Hermitian matrix $\hat{U}^{(m,n)}$ $(\hat{U}^{(m,n)\dagger} = \hat{U}^{(m,n)}, (\hat{U}^{(m,n)})^2 = I_2)$ is given as [as is easily derived from (33)]

[1]We define four-dimensional Weyl fermions as $\psi_{R,L} \equiv \frac{1\pm\gamma^5}{2}\psi$.

$$\hat{U}^{(m,n)} = U^{(m,n)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}U^{(m,n)\dagger} = \frac{1}{m_f^{(m,n)}}M_f^{(m,n)} = \frac{1}{m_f^{(m,n)}}\begin{pmatrix} m_B & -\frac{m-in}{R} \\ -\frac{m+in}{R} & -m_B \end{pmatrix}. \tag{38}$$

$\hat{U}^{(m,n)}$ denotes the asymmetry in the unitary transformations (36) between the right-handed and left-handed fermions, which is of real physical interest. The mass term for nonzero KK modes is thus written with degenerate mass $m_f^{(m,n)}$ as

$$-m_f^{(m,n)}(\bar{\psi}'^{(m,n)}\psi'^{(m,n)} + \bar{\Psi}'^{(m,n)}\Psi'^{(m,n)}), \tag{39}$$

where

$$\psi'^{(m,n)} \equiv \psi_R'^{(m,n)} + \psi_L'^{(m,n)}, \qquad \Psi'^{(m,n)} \equiv \Psi_R'^{(m,n)} + \Psi_L'^{(m,n)}. \tag{40}$$

Concerning the mass term for the zero mode, there is no mixing between ψ and Ψ, since only the state Ψ exists: $\Psi'^{(0)} = \Psi^{(0)}$. The mass term takes a simple form

$$-m_B\bar{\Psi}'^{(0)}\Psi'^{(0)}. \tag{41}$$

Let us note that $\Psi'^{(0)} = \Psi^{(0)}$ and (41) are naturally obtained formally setting $m = n = 0$ in (37)–(39).

We now move to the part relevant for the mass-squared of gauge-Higgs bosons:

$$-\tfrac{1}{2}(-F_{\mu y}F^{\mu}{}_{y} - F_{\mu z}F^{\mu}{}_{z} + F_{yz}^2) \qquad (F_{\mu y} = \partial_\mu A_y - \partial_y A_\mu, \text{etc.}), \tag{42}$$

which may be written in terms of A_ω as

$$(\partial_\mu A_\omega)(\partial^\mu A_{\bar{\omega}}) + (\partial_\omega A_\mu)(\partial_{\bar{\omega}}A^\mu) - \{(\partial_\mu A_\omega)(\partial_{\bar{\omega}}A^\mu) + (\partial_\mu A_{\bar{\omega}})(\partial_\omega A^\mu)\} + \frac{1}{2}(\partial_{\bar{\omega}}A_\omega - \partial_\omega A_{\bar{\omega}})^2 \quad \left(A_{\bar{\omega}} = A_\omega^*, \partial_\omega = \frac{\partial_y - i\partial_z}{\sqrt{2}}, \text{etc.}\right). \tag{43}$$

Since the sector of nonzero KK modes possess a Higgs-like mechanism, in order to form four-dimensional massive gauge bosons, we now introduce the gauge-fixing term à la R_ξ gauge [10]:

$$-\frac{1}{2\xi}\{\partial_\mu A^\mu - \xi(\partial_\omega A_{\bar{\omega}} + \partial_{\bar{\omega}}A_\omega)\}^2. \tag{44}$$

The aim is to eliminate the term in (43), which may be rewritten, after partial integrals, as $-(\partial_\omega A_{\bar{\omega}} + \partial_{\bar{\omega}}A_\omega)\partial_\mu A^\mu$.

Combining (43) and (44) we get

$$(\partial_\omega A_\mu)(\partial_{\bar{\omega}}A^\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (\partial_\mu A_\omega)(\partial^\mu A_{\bar{\omega}}) - \frac{\xi}{2}(\partial_\omega A_{\bar{\omega}} + \partial_{\bar{\omega}}A_\omega)^2 + \frac{1}{2}(\partial_{\bar{\omega}}A_\omega - \partial_\omega A_{\bar{\omega}})^2. \tag{45}$$

We thus realize that $\mathrm{Re}(\partial_{\bar{\omega}}A_\omega)$ and $\mathrm{Im}(\partial_{\bar{\omega}}A_\omega)$, i.e. $\frac{1}{\sqrt{m^2+n^2}}(mA_y^{(m,n)} + nA_z^{(m,n)})$ and $\frac{1}{\sqrt{m^2+n^2}}(-nA_y^{(m,n)} + mA_z^{(m,n)})$ behave as a would-be N-G boson and physical Higgs boson, respectively, for nonzero KK modes.

In fact, substituting (19) and (28) in (45) and integrating over y and z, we get corresponding four-dimensional effective Lagrangian,

$$\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{m^2+n^2}{R^2}A_\mu^{(m,n)}A^{\mu(m,n)} - \frac{1}{2\xi}\left\{(\partial_\mu A^{(0)\mu})^2 + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(\partial_\mu A^{(m,n)\mu})^2\right\} + \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{(\partial_\mu G^{(m,n)}) \times (\partial^\mu G^{(m,n)}) + (\partial_\mu H^{(m,n)})(\partial^\mu H^{(m,n)}) - \xi\frac{m^2+n^2}{R^2}G^{(m,n)2} - \frac{m^2+n^2}{R^2}H^{(m,n)2}\right\}, \tag{46}$$

where $G^{(m,n)}$ and $H^{(m,n)}$ denote the would-be N-G boson and the physical Higgs boson:

$$G^{(m,n)}(x) = \frac{1}{\sqrt{m^2+n^2}}(mA_y^{(m,n)}(x) + nA_z^{(m,n)}(x)), \qquad H^{(m,n)}(x) = \frac{1}{\sqrt{m^2+n^2}}(-nA_y^{(m,n)}(x) + mA_z^{(m,n)}(x)). \tag{47}$$

It is now clear that we get a massless photon $A_\mu^{(0)}$, along with a massive photon $A_\mu^{(m,n)}$ and massive Higgs boson $H^{(m,n)}$, both having masses $M_V^{(m,n)} \equiv \frac{\sqrt{m^2+n^2}}{R}$, in the four-dimensional spectrum.

V. THE INTERACTION VERTICES

Having KK-mode expansions for each field, we are now ready to calculate the interaction vertices in terms of four-dimensional fields. First, we focus on the interaction vertices of four-dimensional gauge fields $A_\mu^{(m,n)}$ (and $A_\mu^{(0)}$). Since the interaction preserves the chirality of fermions and the right-handed fermions are not associated with unitary transformation when described by mass eigenstates [see (37)] we initially restrict ourselves to the interaction vertices for the right-handed fermions. The relevant part of the Lagrangian is ($\tilde{\Psi} \equiv \gamma^5\Psi$)

$$g(\bar{\psi}\gamma^\mu R\psi + \bar{\tilde{\Psi}}\gamma^\mu R\tilde{\Psi})A_\mu. \tag{48}$$

Substituting (24) and (28) in (48), with the mode expansion for Ψ being modified as

$$\tilde{\Psi}(x,y,z) = \frac{1}{2\pi R}\gamma^5\Psi^{(0)}(x) + \frac{1}{2\pi R}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\tilde{\Psi}^{(m,n)}(x) \times\left[\cos\left(\frac{my+nz}{R}\right) + \cos\left(\frac{mz-ny}{R}\right)\right], \tag{49}$$

we get after y and z integrations the interaction vertices with respect to nonzero KK modes,

$$\sum_{m,m',m''=1}^{\infty}\sum_{n,n',n''=0}^{\infty}\frac{g_4}{2}\big(\bar{\psi}'^{(m,n)}(x)\quad \bar{\Psi}'^{(m,n)}(x)\big)U^{(m',n')}_{R(m,n;m'',n'')}\times\gamma^\mu R\begin{pmatrix}\psi'^{(m'',n'')}(x)\\ \Psi'^{(m'',n'')}(x)\end{pmatrix}\times A^{(m',n')}_\mu(x),$$

$$U^{(m',n')}_{R(m,n;m'',n'')}=\begin{pmatrix}v^{(m',n')}_{(m,n;m'',n'')} & 0\\ 0 & V^{(m',n')}_{(m,n;m'',n'')}\end{pmatrix},\tag{50}$$

where $g_4\equiv\frac{g}{2\pi R}$ is the four-dimensional gauge coupling and the vertex functions are given as[2]

$$\begin{aligned}V^{(m',n')}_{(m,n;m'',n'')}=&\ \delta_{m+m'-m''}\delta_{n+n'-n''}+\delta_{m+n'-m''}\delta_{n-m'-n''}\\&+\delta_{m+n'-n''}\delta_{n-m'+m''}+\delta_{m-m'+m''}\delta_{n-n'+n''}\\&+\delta_{m-m'+n''}\delta_{n-n'-m''}+\delta_{m-m'-m''}\delta_{n-n'-n''}\\&+\delta_{m-m'-n''}\delta_{n-n'+m''}+\delta_{m-n'+n''}\delta_{n+m'-m''}\\&+\delta_{m-n'-m''}\delta_{n+m'-n''},\end{aligned}\tag{51}$$

$$\begin{aligned}v^{(m',n')}_{(m,n;m'',n'')}=&\ \delta_{m+m'-m''}\delta_{n+n'-n''}+\delta_{m+n'-m''}\delta_{n-m'-n''}\\&+i\delta_{m+n'-n''}\delta_{n-m'+m''}-\delta_{m-m'+m''}\delta_{n-n'+n''}\\&-i\delta_{m-m'+n''}\delta_{n-n'-m''}+\delta_{m-m'-m''}\delta_{n-n'-n''}\\&+i\delta_{m-m'-n''}\delta_{n-n'+m''}-i\delta_{m-n'+n''}\delta_{n+m'-m''}\\&+\delta_{m-n'-m''}\delta_{n+m'-n''}.\end{aligned}\tag{52}$$

The interaction vertices including at least one zero mode are also given by the same form as (50), but the vertex functions are different by factor 2 from what we obtain by formally generalizing (51) and (52) to the case of zero mode:

$$\begin{aligned}V^{(0,0)}_{(m,n;m'',n'')}&=2\delta_{m,m''}\delta_{n,n''}\qquad(m,m'',n,n''\geq 0),\\ v^{(0,0)}_{(m,n;m'',n'')}&=2\delta_{m,m''}\delta_{n,n''}\qquad(m,m''\geq 1,n,n''\geq 0),\\ V^{(m',n')}_{(0,0;m'',n'')}&=2\delta_{m',m''}\delta_{n',n''}\qquad(m'\geq 1,m''\geq 0,n',n''\geq 0),\\ V^{(m',n')}_{(m,n;0,0)}&=2\delta_{m,m'}\delta_{n,n'}\qquad(m\geq 0,m'\geq 1,n,n'\geq 0).\end{aligned}\tag{53}$$

In the case of the left-handed current, the A_μ interaction is written as

$$\sum_{m,m',m''=1}^{\infty}\sum_{n,n',n''=0}^{\infty}\frac{g_4}{2}\big(\bar{\psi}'^{(m,n)}(x)\quad \bar{\Psi}'^{(m,n)}(x)\big)U^{(m',n')}_{L(m,n;m'',n'')}\times\gamma^\mu L\begin{pmatrix}\psi'^{(m'',n'')}(x)\\ \Psi'^{(m'',n'')}(x)\end{pmatrix}\times A^{(m',n')}_\mu(x),$$

$$U^{(m',n')}_{L(m,n;m'',n'')}=\hat{U}^{(m,n)\dagger}U^{(m',n')}_{R(m,n;m'',n'')}\hat{U}^{(m'',n'')}.\tag{54}$$

[2]Here we use the abbreviation for Kronecker's delta: $\delta_m\equiv\delta_{m,0}$.

Actually, in a process where only the left-handed current appears without any chirality flip, the unitary matrices $\hat{U}$ are irrelevant,

$$\begin{aligned}&\cdots\hat{U}^{(m,n)}\hat{U}^{(m,n)\dagger}U^{(m',n')}_{R(m,n;m'',n'')}\hat{U}^{(m'',n'')}\hat{U}^{(m'',n'')\dagger}\cdots\\&=\cdots U^{(m',n')}_{R(m,n;m'',n'')}\cdots.\end{aligned}\tag{55}$$

This reflects the fact that the freedom of $V^{(m,n)}$ in (36) enables us to choose $V^{(m,n)}=\sigma_3U^{(m,n)\dagger}$ if we wish, so that $\hat{U}^{(m,n)}$ appears in the right-handed current instead of the left-handed current. On the other hand, in processes with chirality flip, the matrices $\hat{U}^{(m,n)}$ can not be eliminated and describe the amplitudes.

From (38), (50), (53), and (54), we find that the interaction vertex of the ordinary photon $A^{(0)}_\mu$ takes the usual QED form:

$$g_4\Big(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\bar{\Psi}'^{(m,n)}\gamma^\mu\Psi'^{(m,n)}+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\bar{\psi}'^{(m,n)}\gamma^\mu\psi'^{(m,n)}\Big)A^{(0)}_\mu.\tag{56}$$

Next, we discuss the interaction vertices of A_ω and $A_{\bar{\omega}}$. The relevant part of the Lagrangian is

$$\sqrt{2}g\{i\bar{\psi}\,\tilde{\Psi}A_\omega+\text{H.c.}\}.\tag{57}$$

Here A_ω is mode expanded in terms of $G^{(m,n)}$ and $H^{(m,n)}$ as

$$\begin{aligned}A_\omega(x,y,z)=&\frac{1}{2\pi R}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{m-in}{\sqrt{2(m^2+n^2)}}\{G^{(m,n)}(x)\\&-iH^{(m,n)}(x)\}\times\Big[\sin\Big(\frac{my+nz}{R}\Big)\\&+i\sin\Big(\frac{ny-mz}{R}\Big)\Big].\end{aligned}\tag{58}$$

We thus realize that once we get the vertex function for $G^{(m',n')}(x)$, then the vertex function for $H^{(m',n')}(x)$ is readily obtained by multiplying by $-i$.

As A_ω has Z_4-eigenvalue $-i$, the vertex functions can be written in terms of $v^{(m'',n'')}_{(m,n;m',n')}$, obtained by an exchange of $(m',n')\leftrightarrow(m'',n'')$ in (52):

$$\begin{aligned}&\sum_{m,m',m''=1}^{\infty}\sum_{n,n',n''=0}^{\infty}\frac{g_4}{2}\frac{m'-in'}{\sqrt{m'^2+n'^2}}\bar{\psi}^{(m,n)}(x)iv^{(m'',n'')}_{(m,n;m',n')}\tilde{\Psi}^{(m'',n'')}\\&\times(x)\{G^{(m',n')}(x)-iH^{(m',n')}(x)\}+\text{H.c.}\end{aligned}\tag{59}$$

Rewriting in terms of mass eigenstates for the fermions, we get (for nonzero KK modes)

$$\sum_{m,m',m''=1}^{\infty}\sum_{n,n',n''=0}^{\infty}\frac{g_4}{2}\frac{1}{\sqrt{m'^2+n'^2}}\Big\{\big(\bar{\psi}'^{(m,n)}(x)\quad\bar{\Psi}'^{(m,n)}(x)\big)\begin{pmatrix}-\frac{m'-in'}{m_f^{(m'',n'')}}\frac{m''+in''}{R}iv^{(m'',n'')}_{(m,n;m',n')} & -\frac{m'-in'}{m_f^{(m'',n'')}}m_Biv^{(m'',n'')}_{(m,n;m',n')}\\ +\frac{m'+in'}{m_f^{(m'',n'')}}m_B(-i)v^{(m,n)*}_{(m'',n'';m',n')} & -\frac{m'+in'}{m_f^{(m'',n'')}}\frac{m''-in''}{R}(-i)v^{(m,n)*}_{(m'',n'';m',n')}\end{pmatrix}$$

$$\times L\begin{pmatrix}\psi'^{(m'',n'')}(x)\\ \Psi'^{(m'',n'')}(x)\end{pmatrix}\times G^{(m',n')}(x)-i\big(\bar{\psi}'^{(m,n)}(x)\quad\bar{\Psi}'^{(m,n)}(x)\big)$$

$$\times\begin{pmatrix}-\frac{m'-in'}{m_f^{(m'',n'')}}\frac{m''+in''}{R}iv^{(m'',n'')}_{(m,n;m',n')} & -\frac{m'-in'}{m_f^{(m'',n'')}}m_Biv^{(m'',n'')}_{(m,n;m',n')}\\ -\frac{m'+in'}{m_f^{(m'',n'')}}m_B(-i)v^{(m,n)*}_{(m'',n'';m',n')} & +\frac{m'+in'}{m_f^{(m'',n'')}}\frac{m''-in''}{R}(-i)v^{(m,n)*}_{(m'',n'';m',n')}\end{pmatrix}L\begin{pmatrix}\psi'^{(m'',n'')}(x)\\ \Psi'^{(m'',n'')}(x)\end{pmatrix}\times H^{(m',n')}(x)\Big\}$$

$$+\sum_{m,m',m''=1}^{\infty}\sum_{n,n',n''=0}^{\infty}\frac{g_4}{2}\frac{1}{\sqrt{m'^2+n'^2}}\Big\{\big(\bar{\psi}'^{(m,n)}(x)\quad\bar{\Psi}'^{(m,n)}(x)\big)\begin{pmatrix}-\frac{m'+in'}{m_f^{(m,n)}}\frac{m-in}{R}(-i)v^{(m,n)*}_{(m'',n'';m',n')} & +\frac{m'-in'}{m_f^{(m,n)}}m_Biv^{(m'',n'')}_{(m,n;m',n')}\\ -\frac{m'+in'}{m_f^{(m,n)}}m_B(-i)v^{(m,n)*}_{(m'',n'';m',n')} & -\frac{m'-in'}{m_f^{(m,n)}}\frac{m+in}{R}iv^{(m'',n'')}_{(m,n;m',n')}\end{pmatrix}$$

$$\times R\begin{pmatrix}\psi'^{(m'',n'')}(x)\\ \Psi'^{(m'',n'')}(x)\end{pmatrix}\times G^{(m',n')}(x)-i\big(\bar{\psi}'^{(m,n)}(x)\quad\bar{\Psi}'^{(m,n)}(x)\big)\begin{pmatrix}+\frac{m'+in'}{m_f^{(m,n)}}\frac{m-in}{R}(-i)v^{(m,n)*}_{(m'',n'';m',n')} & +\frac{m'-in'}{m_f^{(m,n)}}m_Biv^{(m'',n'')}_{(m,n;m',n')}\\ +\frac{m'+in'}{m_f^{(m,n)}}m_B(-i)v^{(m,n)*}_{(m'',n'';m',n')} & -\frac{m'-in'}{m_f^{(m,n)}}\frac{m+in}{R}iv^{(m'',n'')}_{(m,n;m',n')}\end{pmatrix}$$

$$\times R\begin{pmatrix}\psi'^{(m'',n'')}(x)\\ \Psi'^{(m'',n'')}(x)\end{pmatrix}\times H^{(m',n')}(x)\Big\}. \tag{60}$$

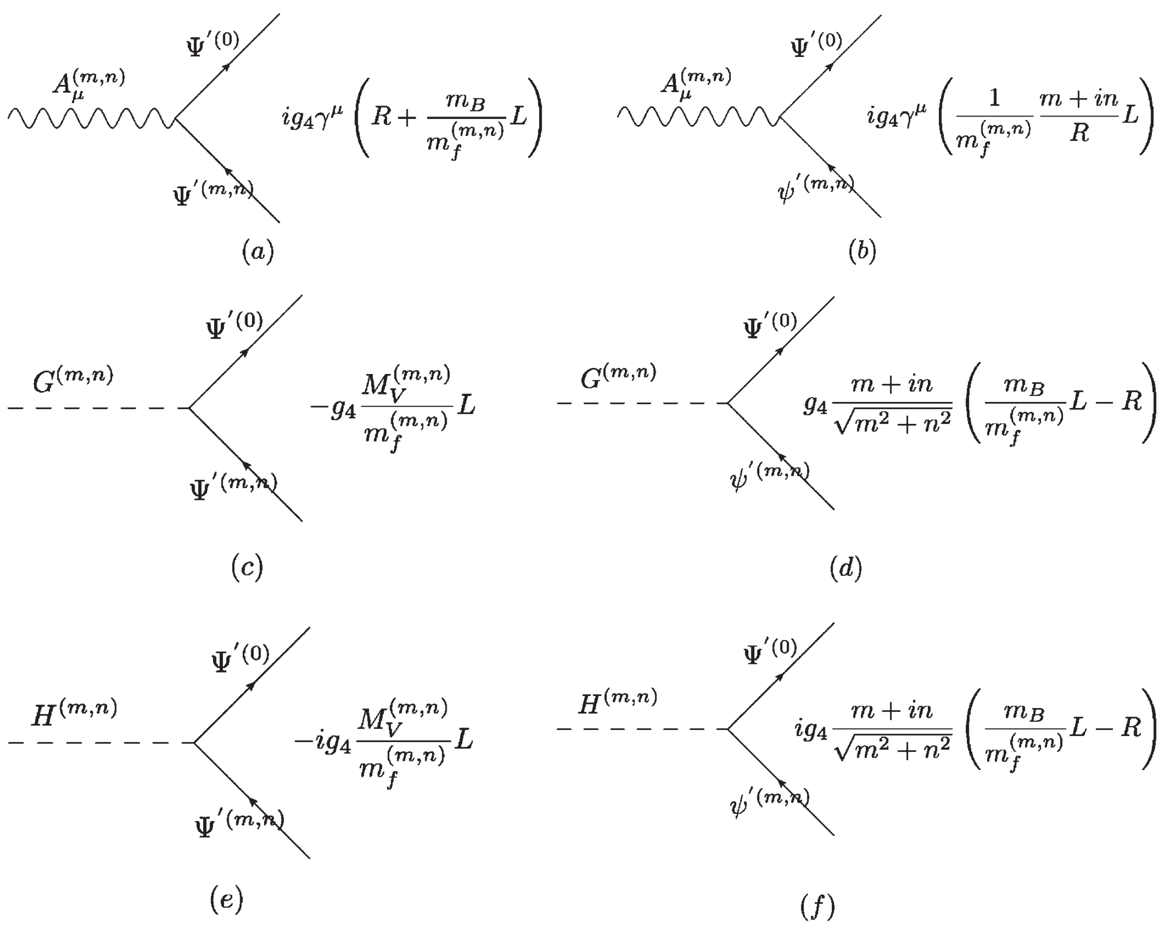

FIG. 4. The interaction vertices where one of the external fermion lines is the zero mode $\Psi'^{(0)}$.

The interaction vertices including at least one zero mode are also given by the same form as (60), by use of (53).

It is interesting to note that a sort of "equivalence theorem" holds concerning the interaction vertices of nonzero KK modes $A_\mu^{(m,n)}$ and $G^{(m,n)}$, which are expected to hold as the result of Higgs-like mechanism operative in the sector of massive gauge-Higgs bosons. For illustrative purpose, we focus on the interaction vertices where one of external fermion lines is the zero mode $\Psi'^{(0)}$ shown in Fig. 4, which are easily obtained from Eqs. (38), (50), (53), (54), and (60).

For instance, multiplying $\frac{k^\mu}{M_V^{(m,n)}}$ (k^μ: the 4-momentum of $A_\mu^{(m,n)}$) to the current coupled with the massive photon, we get a relation by use of equations of motions for fermions

$$\frac{k^\mu}{M_V^{(m,n)}} g_4 \bar{\Psi}^{(0)} \gamma_\mu \left(R + \frac{m_B}{m_f^{(m,n)}} L \right) \Psi'^{(m,n)} = -g_4 \frac{M_V^{(m,n)}}{m_f^{(m,n)}} \bar{\Psi}^{(0)} L \Psi'^{(m,n)}, \tag{61}$$

where the right-hand side just coincides with (i times) the current coupled with $G^{(m,n)}$.

VI. CP-VIOLATING PHOTON INTERACTION

Now we are ready to confirm that CP violation is realized in our theory by showing that the imaginary couplings in the interaction vertices remain even after the rephasing of fermions [1].

As we have seen in the previous section, the interaction vertices of KK modes of A_μ, G, and H are rather complicated. We thus restrict ourselves to the interaction vertices of A_μ, although the Yukawa couplings of KK modes of G and H also violate CP, as is suggested by the equivalence theorem. To make the analysis as transparent as possible, we consider only the right-handed current of ψ' coupled to A_μ, as shown in (50), since the corresponding current due to Ψ' described by V has no phases. If all of n, n' and n'' are set to 0 the interaction vertex becomes real, as is seen in (52) and (53). Thus we consider the case where $n = n' = 0$ but $n'' \geq 1$, as the simplest nontrivial possibility:

$$\sum_{m,m',m'',n''=1}^{\infty} \frac{g_4}{2} \bar{\psi}'^{(m)} v_{(m;m'',n'')}^{(m')} \gamma^\mu R \psi'^{(m'',n'')} A_\mu^{(m')}, \tag{62}$$

$$v_{(m;m'',n'')}^{(m')} \equiv v_{(m,0;m'',n'')}^{(m',0)} = i\delta_{m,n''}\delta_{m',m''} + \delta_{m,m''}\delta_{m',n''}, \tag{63}$$

where we use the notation $\psi'^{(m)} \equiv \psi'^{(m,0)}$, etc. Our task is to see whether the phase i in (63) can be eliminated by suitable rephasing of $\psi'^{(m)}$ and $\psi'^{(m'',n'')}$, or not. More explicitly (62) is written, by use of (63), as

$$\sum_{m,m'=1}^{\infty} \frac{g_4}{2} [\bar{\psi}'^{(m)} \gamma^\mu R \psi'^{(m,m')} A_\mu^{(m')} + i \bar{\psi}'^{(m')} \gamma^\mu R \psi'^{(m,m')} A_\mu^{(m)}]. \tag{64}$$

Write the rephasing by use of phases ϕ_m and $\phi_{m,m'}$ as

$$\psi'^{(m)} \to e^{i\phi_m} \psi'^{(m)}, \qquad \psi'^{(m,m')} \to e^{i\phi_{m,m'}} \psi'^{(m,m')}. \tag{65}$$

For the case of $m = m'$, two interaction terms in (64) are actually identical and the resulting complex coupling $(1 + i)g_4$ can be made real by rephasing satisfying a condition

$$\phi_m - \phi_{m,m} = \frac{\pi}{4} \qquad (\mathrm{mod}\,\pi), \tag{66}$$

where modπ reflects the freedom to add an arbitrary multiple of π. For $m \neq m'$, the two terms in (64) are mutually independent, and we get two independent conditions in order to eliminate the CP phases from the interaction Lagrangian:

$$\phi_m - \phi_{m,m'} = 0 \qquad (\mathrm{mod}\,\pi), \tag{67}$$

$$\phi_{m'} - \phi_{m,m'} = \frac{\pi}{2} \qquad (\mathrm{mod}\,\pi). \tag{68}$$

The condition (66) can be trivially satisfied. Namely, for given ϕ_m, we can always find the solution of $\phi_{m,m}$. The combination of (67) and (68), however, give rise to nontrivial conditions for ϕ_m:

$$\phi_m - \phi_{m'} = \frac{\pi}{2} \qquad (\mathrm{mod}\,\pi). \tag{69}$$

Since this condition should be met for arbitrary m and m' ($m \neq m'$), we realize that all ϕ_m ($m \neq m'$) must be the same (modπ) for given m'. As the m', in turn, can be arbitrary we find that all ϕ_m should be the same (modπ):

$$\phi_1 = \phi_2 = \cdots \qquad (\mathrm{mod}\,\pi). \tag{70}$$

On the other hand (70) clearly contradicts (69). Thus we conclude that the conditions (67) and (68) are incompatible with each other and the CP-violating phases cannot be removed by the rephasing. Let us note that if some part of the interaction Lagrangian violates CP, so does the whole Lagrangian. Hence we have confirmed that CP is violated in our model.

VII. REPHASING INVARIANT QUANTITIES

Although we have shown that CP is violated in our model by considering a concrete example, for the completeness of the argument it would be desirable to identify rephasing invariant CP violating parameters, à la the Jarlskog parameter in the Kobayashi-Maskawa model [7]. It would be also helpful in understanding what are the physical invariants appearing in the amplitudes of CP violating processes. As a matter of fact, in our model the free Lagrangian of fermions has a larger symmetry than the rephasing: it is invariant under the unitary transformation described by $V^{(m,n)}$ in (36). Thus the CP violating observables should be invariant under the transformation and therefore they are written as the imaginary parts of the trace of the products of the matrices $U^{(m',n')}_{R,L(m,n;m'',n'')}$ (for the case of A_μ interaction).

First, let us focus on the observables due to $A^{(m',n')}_\mu$ interactions where each fermion propagator possesses no chirality flip. Let us note that as far as processes without chirality flip are considered there is no difference between the processes due to right- and left-handed currents, as we have already discussed. Thus, here we consider only the processes due to the right-handed current.

Our task is to find nonvanishing imaginary parts of the trace of the products of $U^{(m',n')}_{R,L(m,n;m'',n'')}$, appearing in the $A^{(m',n')}_\mu$ vertex [see (50)–(52)]:

$$\begin{aligned}&\mathrm{Im}\,\mathrm{Tr}(U^{(m',n')}_{R(m,n;m'',n'')}U^{(m''',n''')}_{R(m'',n'';m'''',n'''')}\cdots)\\&\quad=\mathrm{Im}(v^{(m',n')}_{(m,n;m'',n'')}v^{(m''',n''')}_{(m'',n'';m'''',n'''')}\cdots),\end{aligned}\tag{71}$$

corresponding to the Feynman diagram shown in Fig. 5. In Fig. 5, the fermions form a closed loop, thus making (71) invariant under the unitary transformation,

$$U^{(m',n')}_{R(m,n;m'',n'')}\to V^{(m,n)\dagger}U^{(m',n')}_{R(m,n;m'',n'')}V^{(m'',n'')},\tag{72}$$

corresponding to the freedom of $V^{(m,n)}$ in (36). In the case of Kobayashi-Maskawa model, the Jarlskog parameter arises only at the fourth order of the KM matrix elements $V^{\mathrm{KM}}_{i\alpha}$; $J=|\mathrm{Im}(V^{\mathrm{KM}}_{i\alpha}V^{\mathrm{KM}*}_{j\alpha}V^{\mathrm{KM}}_{j\beta}V^{\mathrm{KM}*}_{i\beta})|(i\neq j,\alpha\neq\beta)$. In our model, however, the imaginary part is found to arise already at the second order of $v^{(m',n')}_{R(m,n;m'',n'')}$,

$$\mathrm{Im}\,(v^{(m',n')}_{(m,n;m'',n'')}v^{(m''',n''')}_{(m'',n'';m,n)}),\tag{73}$$

because of the variety of the KK modes $A^{(m',n')}_\mu$. For instance, if we set

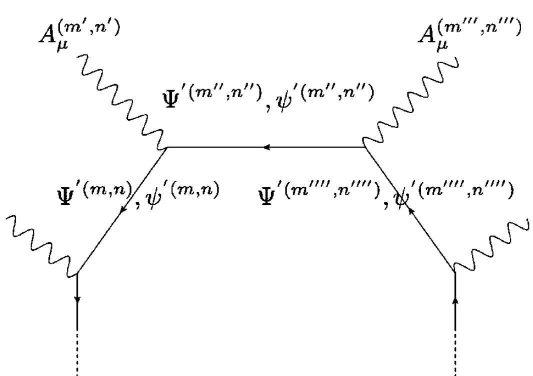

FIG. 5. An example of Feynman diagram describing rephasing invariant quantities.

$$\begin{aligned}&m=1,\qquad n=a,\qquad m'=2,\qquad n'=a+2,\\&m''=1,\qquad n''=2,\qquad m'''=a+1,\qquad n'''=1,\end{aligned}\tag{74}$$

with an arbitrary positive integer $a(\neq 2)$, we find

$$\mathrm{Im}\,(v^{(2,a+2)}_{(1,a;1,2)}v^{(a+1,1)}_{(1,2;1,a)})=1.\tag{75}$$

VIII. A BRIEF COMMENT ON THE EDM OF ELECTRON

Arguments given above have shown that the CP violating phases remain even after the rephasing of the fields and therefore CP is broken as the consequence of the compactification on T^2/Z_4. As a concrete example of a CP violating observable here we comment on the electric dipole moment (EDM) of electron, since the EDM does not need the fermion generation structure, which is ignored in our model.

We first focus on the possible 1-loop contributions to the EDM, where the intermediate states are the nonzero KK modes of electron and gauge-Higgs bosons. The relevant 1-loop diagrams are those in Fig. 6.

Since the Feynman diagrams are divided into two types, i.e. diagrams with the exchanges of four-dimensional vector and four-dimensional scalar, it may be useful to derive general formulas for the amplitudes of these two types of diagrams, due to generic interactions, $\bar{\tilde{\psi}}\gamma_\mu(aL+bR)\Psi'^{(0)}V^\mu+\mathrm{H.c.}$ and $\bar{\tilde{\psi}}(a'L+b'R)\Psi'^{(0)}S+\mathrm{H.c.}$, respectively, where $\tilde{\psi}$, V^μ, S denote intermediate states of fermion, four-dimensional vector and four-dimensional scalar. The general formulas for these two types of diagrams are known to be proportional to $2i\,\mathrm{Im}(ab^*)$ and $2i\,\mathrm{Im}(a'b'^*)$, respectively [6].

This observation immediately leads to an important conclusion that we do not get the EDM at least at the 1-loop level. Namely, all of $2i\,\mathrm{Im}(ab^*)$ and $2i\,\mathrm{Im}(a'b'^*)$ obtained from the interaction vertices shown in Fig. 4 vanish. Let us remember that to get an EDM both P and CP symmetries should be broken. The origin we find for the cancellation of the 1-loop contribution arises from the fact that the orbifolding does not violate P symmetry, while it does break CP, since as is seen in (26) the extra space coordinates are invariant under the P transformation. At the first glance this argument seems to contradict the fact that the right- and left-handed currents coupled to the nonzero KK modes of the photon are not identical [see (50) and (54)]. We, however, realize that if we choose $V^{(m,n)}=\sigma_3U^{(m,n)\dagger}$, instead of $V^{(m,n)}=U^{(m,n)\dagger}$ in (36), the roles of right- and left-handed fermions are exchanged, compared with the case of (37). This implies that the contribution to the EDM should be invariant under the exchange $a\leftrightarrow b$ and therefore $\mathrm{Im}(ab^*)=\mathrm{Im}(ba^*)=0$.

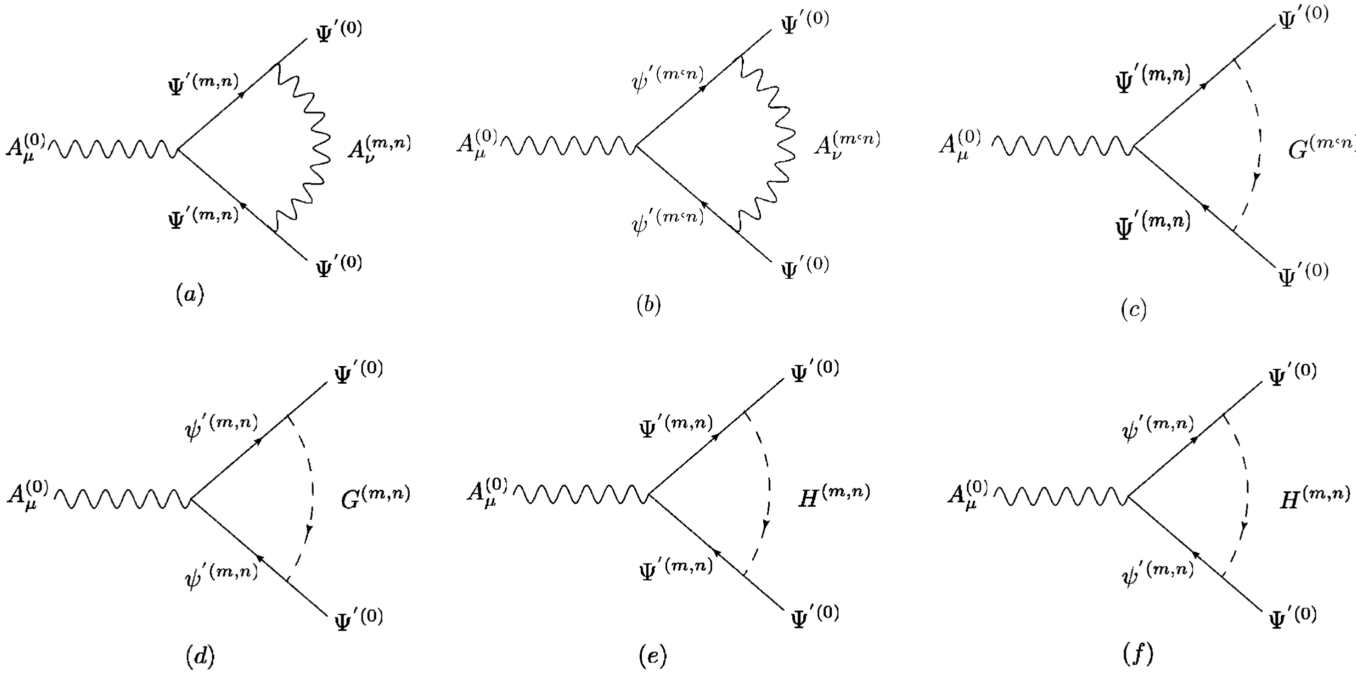

FIG. 6. Feynman diagrams for electron EDM at the 1-loop level.

As far as the vanishing contribution to the EDM has its origin in the *P* symmetry of the model, we anticipate that the EDM will not emerge even at the higher loop Feynman diagrams, though the explicit computations are desirable to settle the issue. Nevertheless, we still expect that EDM gets nonvanishing contribution as long as the *CP* symmetry is violated by the orbifolding, once the model is made realistic in a way it can incorporate the standard model, where *P* symmetry is broken.

IX. SUMMARY

In this paper we addressed the question of how *CP* violation is realized in the scenario of gauge-Higgs unification, where the interaction of the Higgs is governed by a gauge principle and therefore to get *CP* violating phases is a challenging issue.

As a simple and nontrivial example we examined a 6-dimensional U(1) model compactified on an 2-dimensional orbifold T^2/Z_4. First we extended an argument of how four-dimensional *CP* transformation is related to the complex structure of the extra space and showed that the adopted Z_4 orbifolding is incompatible with such defined *CP* symmetry and therefore leads to *CP* violation. Next, we confirmed the expectation by extensively studying the interaction vertices derived from the overlap integrals over the extra space coordinates of mode functions. We could get *CP* violating phases which do not vanish even after the possible rephasing of the relevant fields. For completeness, we derived a rephasing invariant *CP* violating parameter, following a similar argument in the Kobayashi-Maskawa model which led to the Jarlskog parameter.

As a typical example of *CP* violating observable we made a brief comment on the EDM of electron in our model. It turned out that at the 1-loop level, the EDM gets no contributions. The origin of the vanishing EDM in our model was argued to be the fact that the orbifolding does not break the *P* symmetry, while both of *P* and *CP* symmetries should be broken to get a nonvanishing EDM. Nevertheless, the EDM is expected to get a nonvanishing contribution as long as the *CP* symmetry is violated by the orbifolding, once the model is made realistic in a way that incorporates the standard model where *P* symmetry is broken. The chiral theory with *P* violation will be realized, once we start from six-dimensional Weyl fermion with definite eigenvalue of Γ_7, instead of six-dimensional Dirac fermion, though in that case we have to ensure the cancellation of six-dimensional anomaly [9] by suitably choosing the matter content.

An interesting candidate of such realistic higher dimensional gauge theory of the type discussed in this paper may be the theory based on the "gauge-Higgs unification" scenario. The scenario was proposed long time ago [11–14], where the Higgs field is identified with the zero mode of an extra spatial component of higher dimensional gauge fields. It has been revived as one of the attractive scenarios solving the hierarchy problem without invoking supersymmetry [15]. This is based on the observation that the quantum correction to the Higgs mass is finite and insensitive to the ultraviolet (UV) cutoff of the theory thanks to the higher dimensional local gauge symmetry, in spite of the fact that higher dimensional gauge theories are generally regarded as nonrenormalizable. Since then, many interesting works based on this scenario have appeared in the literature from various points of view [16–39].

Strictly speaking, the U(1) model discussed in this paper is not a model of gauge-Higgs unification, as the extra space component of gauge field does not have a zero mode that behaves as a Higgs field. We, however, believe that the discussions of the mechanism of CP violation extended in this paper holds in general for the models of gauge-Higgs unification with larger gauge symmetries including that of the standard model, since the mechanism is based on the manner of compactification and does not depend on the choice of the gauge group. It, however, should be pointed out that the introduction of brane-localized fields and their interactions with bulk fields may be needed to make the theory realistic [30]. The localized mixing mass parameter may become another source of CP violation, independent of the mechanism of CP violation due to the compactification discussed in this paper.

It is interesting to note that the proposed mechanism of CP violation due to the Z_4 orbifolding does not need flavor or generation structure, as our U(1) model incorporates only 1 generation, i.e. the electron. The CP violation is achieved through the interactions including nonzero KK modes. From such a point of view, our mechanism of CP violation is quite different from that in the Kobayashi-Maskawa model. It will be an interesting and important question how the mechanism of CP violation can be extended when we include multiple generations. Once the generations are introduced we will be able to discuss other well-known CP violating observables caused by flavor changing neutral current processes, such as ϵ in the neutral kaon system or CP asymmetries in B meson decays.

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[1] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
[2] A. Strominger and E. Witten, Commun. Math. Phys. **101**, 341 (1985).
[3] C. S. Lim, Phys. Lett. B **256**, 233 (1991).
[4] T. Kobayashi and C. S. Lim, Phys. Lett. B **343**, 122 (1995).
[5] B. Grzadkowski and J. Wudka, Phys. Rev. Lett. **93**, 211603 (2004).
[6] Y. Adachi, C. S. Lim, and N. Maru, Phys. Rev. D **80**, 055025 (2009).
[7] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985); Z. Phys. C **29**, 491 (1985).
[8] H. Georgi, A. K. Grant, and G. Hailu, Phys. Lett. B **506**, 207 (2001); B. A. Dobrescu and E. Ponton, J. High Energy Phys. 03 (2004) 071; G. Burdman, B. A. Dobrescu, and E. Ponton, J. High Energy Phys. 02 (2006) 033; E. Ponton and L. Wang, J. High Energy Phys. 11 (2006) 018.
[9] B. A. Dobrescu and E. Poppitz, Phys. Rev. Lett. **87**, 031801 (2001).
[10] K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D **6**, 2923 (1972).
[11] N. S. Manton, Nucl. Phys. **B158**, 141 (1979).
[12] D. B. Fairlie, Phys. Lett. **82B**, 97 (1979); J. Phys. G **5**, L55 (1979).
[13] Y. Hosotani, Phys. Lett. **126B**, 309 (1983); **129B**, 193 (1983); Ann. Phys. (N.Y.) **190**, 233 (1989).
[14] I. Antoniadis, Phys. Lett. B **246**, 377 (1990).
[15] H. Hatanaka, T. Inami, and C. S. Lim, Mod. Phys. Lett. A **13**, 2601 (1998).
[16] M. Kubo, C. S. Lim, and H. Yamashita, Mod. Phys. Lett. A **17**, 2249 (2002).
[17] G. Burdman and Y. Nomura, Nucl. Phys. **B656**, 3 (2003).
[18] C. Csáki, C. Grojean, and H. Murayama, Phys. Rev. D **67**, 085012 (2003).
[19] I. Gogoladze, Y. Mimura, and S. Nandi, Phys. Lett. B **560**, 204 (2003); I. Gogoladze, T. Li, Y. Mimura, and S. Nandi, Phys. Rev. D **72**, 055006 (2005).
[20] C. A. Scrucca, M. Serone, and L. Silvestrini, Nucl. Phys. **B669**, 128 (2003).
[21] N. Haba, Y. Hosotani, Y. Kawamura, and T. Yamashita, Phys. Rev. D **70**, 015010 (2004).
[22] Y. Hosotani, S. Noda, and K. Takenaga, Phys. Rev. D **69**, 125014 (2004); Phys. Lett. B **607**, 276 (2005).
[23] G. Martinelli, M. Salvatori, C. A. Scrucca, and L. Silvestrini, J. High Energy Phys. 10 (2005) 037.
[24] C. Biggio and M. Quiros, Nucl. Phys. **B703**, 199 (2004).
[25] G. Panico and M. Serone, J. High Energy Phys. 05 (2005) 024.
[26] G. Cacciapaglia, C. Csaki, and S. C. Park, J. High Energy Phys. 03 (2006) 099.
[27] G. Panico, M. Serone, and A. Wulzer, Nucl. Phys. **B739**, 186 (2006); **B762**, 189 (2007).
[28] N. Maru and K. Takenaga, Phys. Rev. D **72**, 046003 (2005); Phys. Lett. B **637**, 287 (2006); Phys. Rev. D **74**, 015017 (2006).
[29] C. S. Lim and N. Maru, Phys. Rev. D **75**, 115011 (2007).
[30] K. Agashe, R. Contino, and A. Pomarol, Nucl. Phys. **B719**, 165 (2005); K. Agashe and R. Contino, Nucl. Phys. **B742**, 59 (2006); K. Agashe, R. Contino, L. Da Rold, and A. Pomarol, Phys. Lett. B **641**, 62 (2006); R. Contino, L. Da Rold, and A. Pomarol, Phys. Rev. D **75**, 055014 (2007).
[31] K. y. Oda and A. Weiler, Phys. Lett. B **606**, 408 (2005).
[32] Y. Hosotani and M. Mabe, Phys. Lett. B **615**, 257 (2005);

Y. Hosotani, S. Noda, Y. Sakamura, and S. Shimasaki, Phys. Rev. D **73**, 096006 (2006); Y. Sakamura and Y. Hosotani, Prog. Theor. Phys. **118**, 935 (2007); Y. Hosotani, K. Oda, T. Ohnuma, and Y. Sakamura, Phys. Rev. D **78**, 096002 (2008).

[33] M. Carena, E. Ponton, J. Santiago, and C. E. M. Wagner, Nucl. Phys. **B759**, 202 (2006); Phys. Rev. D **76**, 035006 (2007); A. D. Medina, N. R. Shah, and C. E. M. Wagner, Phys. Rev. D **76**, 095010 (2007); M. Carena, A. D. Medina, B. Panes, N. R. Shah, and C. E. M. Wagner, Phys. Rev. D **77**, 076003 (2008); M. Carena, A. D. Medina, N. R. Shah, and C. E. M. Wagner, Phys. Rev. D **79**, 096010 (2009).

[34] H. Hatanaka, arXiv:0712.1334.

[35] C. S. Lim and N. Maru, Phys. Lett. B **653**, 320 (2007).

[36] A. Falkowski, Phys. Rev. D **77**, 055018 (2008).

[37] N. Maru and N. Okada, Phys. Rev. D **77**, 055010 (2008); N. Maru, Mod. Phys. Lett. A **23**, 2737 (2008).

[38] C. S. Lim and N. Maru, arXiv:0904.0304.

[39] N. Haba, Y. Sakamura, and T. Yamashita, J. High Energy Phys. 07 (2009) 020.