Conditionally exactly solvable potential and dual transformation in quantum mechanics

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Abstract

We comment that the conditionally exactly solvable potential of Dutt *et al* (1995 *J. Phys. A: Math. Gen.* **28** L107) and the exactly solvable potential from which it is derived form a dual system.

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Conditionally exactly solvable (CES) potentials have received considerable attention in recent times [1, 2, 3, 4, 5]. The main feature of such potentials is that one or more coupling constants in them are fixed to a specific value. There have been instances of CES potentials running into inconsistencies with threshold boundary conditions [3, 4], but there do exist some which possess valid asymptotic behaviour. One such acceptable class is the one proposed by Dutt *et al* [2] sometime ago and which reads [1]

$$V(y) = \frac{A}{1+e^{-2y}} - \frac{B}{(1+e^{-2y})^{1/2}} - \frac{3}{4(1+e^{-2y})^2} \tag{1}$$

where $y \in (-\infty, \infty)$ and A, B are some real parameters defining the shape of the potential. Note the presence of the fixed numerical value $-\frac{3}{4}$ for one of the coupling constants in $V(y)$ that provides its identification as a CES. Potential (1) has for its associated eigenfunctions

$$\psi_n(y) = z^{\frac{1}{4}}(z-1)^{-\left(\frac{c}{2}-\frac{B}{4c}\right)}(z+1)^{-\left(\frac{c}{2}-\frac{B}{4c}\right)} P_n^{\left(\frac{B}{2c}-c,-\frac{B}{2c}-c\right)}(z), \tag{2}$$

where $z = 1 + e^{-2y}$ and c is related to the energy eigenvalue $-\epsilon_n$ as $c = n + \frac{1}{2} + \sqrt{\epsilon_n}$. Actually $\sqrt{\epsilon_n}$ satisfies a complicated cubic equation but it has been observed [5] that only one of its roots is compatible with the normalizability condition.

Some remarks are in order concerning the derivation of the eigenfunctions (2) and the energy eigenvalue equation. The Schrödinger equation is subjected to a coordinate transformation and the transformation function is chosen in such a manner that corresponding to an exactly solvable (ES) potential one has a new analytically solvable one. It turns out that for a half-line-full-line mapping function, one can generate the CES potential for some known shape-invariant potential as an input. The available energy eigenvalues and eigenfunctions of the latter then furnish the corresponding ones for the former.

The purpose of this comment is to establish that the CES potential (1) and its accompanying shape-invariant ES potential are actually dual partners in the sense that the corresponding time-independent Schrödinger equations are mapped to each other under appropriate space transformations called the dual transformations [6, 7, 8]. The latter are known to relate some apparently unconnected problems both in classical and quantum mechanics. The one-dimensional harmonic oscillator and the Coulomb problem [9], the latter and the isotropic oscillator [10, 11], the Pöschl-Teller and infinite potential well problems [6] are some examples of dual systems (DS).

[1] There is also another class of CES potentials proposed in [2] but by a redefinition of parameters [5] it can be made equivalent to (1) and so is not considered here.

In the present context, let us write down the following set of DS given by the pair of Schrödinger equations (with $\hbar = 2m = 1$)

$$\left[-\frac{d^2}{dx^2} + \lambda\left(\frac{dy}{dx}\right)^2 + \nu\frac{dy}{dx}\right]\psi = \mu\psi \tag{3}$$

$$\left[-\frac{d^2}{dy^2} - \frac{1}{2}\{x,y\} - \mu\left(\frac{dx}{dy}\right)^2 + \nu\frac{dx}{dy}\right]\phi = -\lambda\phi \tag{4}$$

where μ and $-\lambda$ are the energies, ν is a constant, $\{x,y\}$ is the Schwarzian derivative, which can be written as

$$\{x,y\} = -\frac{1}{y'^2}\left[\frac{d}{dx}\left(\frac{y''}{y'}\right) - \frac{1}{2}\left(\frac{y''}{y'}\right)^2\right] \tag{5}$$

the primes denoting derivatives with respect to the variable x. The wave functions ψ and ϕ are related in the manner

$$\psi = \left(\frac{dx}{dy}\right)^{1/2}\phi. \tag{6}$$

In DS as above, the energy and the coupling constant μ and λ exchange roles. Further DS are meaningful when the potential and its partner are expressible as

$$W(x) = \lambda\left(\frac{dy}{dx}\right)^2 + \nu\frac{dy}{dx} \tag{7}$$

$$U(y) = -\mu\left(\frac{dx}{dy}\right)^2 + \nu\frac{dx}{dy} - \frac{1}{2}\{x,y\}. \tag{8}$$

Generalizations to include integral or even fractional powers in the derivatives in (7) and (8) are straightforward.

We now set $dy/dx = \coth x$, i.e., $y = \log\sinh x$: in other words, for $y \in (-\infty,\infty)$ the variable $x \in (0,\infty)$. It implies from (7) that the Schrödinger equation (3) has the potential $W(x) = \tilde{W}(x) + \alpha(\alpha-1)$, where

$$\tilde{W}(x) = -2\beta\coth x + \alpha(\alpha-1)\,\mathrm{cosech}^2 x \qquad x \in (0,\infty) \tag{9}$$

which is ES (for $\beta > \alpha^2$, $\alpha > 0$) and shape invariant as well. The energy eigenvalues are given by [12]

$$E_n = -\left(\frac{\beta}{\alpha+n}\right)^2 - (\alpha+n)^2 \qquad n = 0,1,2,\ldots. \tag{10}$$

Indeed the correspondence with (3) is provided by the following identifications

$$\begin{aligned} \lambda &= \alpha(\alpha-1) \\ \nu &= -2\beta \\ \mu &= E_n + \alpha(\alpha-1). \end{aligned} \tag{11}$$

We next enquire into the dual potential $U(y)$. It is easy to work out $\{x, y\}$ from (5) as

$$\{x, y\} = -\left(\operatorname{sech}^2 x \tanh^2 x + \operatorname{sech}^2 x - \frac{1}{2}\operatorname{sech}^4 x\right). \tag{12}$$

Hence we find from (8) and (4) that $U(y) = \tilde{U}(y) + \frac{1}{4}$, where

$$\tilde{U}(y) = \left[\left(\frac{1}{2} - \mu\right)\tanh^2 x + \nu \tanh x - \frac{3}{4}\tanh^4 x\right]_{x=\sinh^{-1}(e^y)} \tag{13}$$

$$\epsilon_n = \alpha(\alpha-1) + \frac{1}{4}. \tag{14}$$

The relation between the energies $-\epsilon_n$ and E_n turns out to be

$$\epsilon_n + E_n = \mu + \frac{1}{4}. \tag{15}$$

On elimination of the parameter α, from (10), (11) and (14) we then get a cubic equation in $\sqrt{\epsilon_n}$, similar to that given in [2]. Further, in terms of the variable $y \in (-\infty, \infty)$, $\tilde{U}(y)$ in (13) translates to

$$\tilde{U}(y) = \frac{\frac{1}{2} - \mu}{1 + e^{-2y}} + \frac{\nu}{(1 + e^{-2y})^{1/2}} - \frac{3}{4(1 + e^{-2y})^2} \qquad y \in (-\infty, \infty) \tag{16}$$

which is identical to the potential in (1) for $\mu = \frac{1}{2} - A$ and $\nu = -B$. Finally, the wave functions for $\tilde{U}(y)$ can be obtained from (6) and yield the same form as in (2).

To conclude we have demonstrated that the CES potential of Dutt *et al* and its ES partner form a DS.

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