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COMPLEXIFIED PSUSY AND SSUSY INTERPRETATIONS OF SOME PT-SYMMETRIC HAMILTONIANS POSSESSING TWO SERIES OF REAL ENERGY EIGENVALUES

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Abstract

We analyze a set of three PT-symmetric complex potentials, namely harmonic oscillator, generalized Pöschl-Teller and Scarf II, all of which reveal a double series of energy levels along with the corresponding superpotential. Inspired by the fact that two superpotentials reside naturally in order-two parasupersymmetry (PSUSY) and second-derivative supersymmetry (SSUSY) schemes, we complexify their frameworks to successfully account for the three potentials.

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1 Introduction

In the literature, non-Hermitian Hamiltonians have received attention [1] from time to time because of their potential applications in scattering problems. Lately, a subclass of such Hamiltonians, containing operators invariant under joint actions of parity (P: $x \to -x$) and time reversal (T: $\mathrm{i} \to -\mathrm{i}$), has become a subject matter of considerable research interest [2]–[17]. An important reason for this is that PT invariance, in a number of cases, leads to energy eigenvalues that are real. Moreover, PT-invariant models share with the usual Hermitian ones many of the features that the latter admit of: namely, supersymmetrization [5, 6, 7], potential algebra [12, 15], quasi-solvability [3, 8, 11, 14], etc.

Recently, Znojil [9], by employing a simple complex shift of coordinate, pointed out that the PT-symmetric harmonic oscillator potential possesses two series of energy levels distinguishable by a quasi-parity parameter. Subsequently, we have also found [12] in an sl(2, $\mathbb{C}$) group theoretical context, that paired real energy levels exist for a PT-symmetric generalized Pöschl-Teller potential. The complexified Scarf II potential, which is also PT symmetric and emerges from the same sl(2, $\mathbb{C}$) algebra, displays a double tower of real energy levels as well.

The purpose of this paper is to bring together these potentials within the framework of an order-two parasupersymmetric (PSUSY) scheme and consequently interpret them in a second-derivative supersymmetric (SSUSY) setting. In the Hermitian context, both the procedures admit of two superpotentials. By complexifying them, we show that all the three potentials mentioned above come under the purview of PSUSY and SSUSY. In this way we establish that both PSUSY and SSUSY appear to be the most natural choice for describing occurrences of a double series of energy levels.

2 In pursuit of a complexified PSUSY

2.1 Underlying ideas of SUSY and PSUSY

The basic principles of SUSY [18, 19] and PSUSY [20, 21] in quantum mechanics (QM) are well known. In SUSYQM, the governing Hamiltonian is written in terms of a pair of supercharges Q and $\bar{Q}$, namely

$$H_s = Q\bar{Q} + \bar{Q}Q. \tag{2.1}$$

These supercharges are nilpotent and commute with H_s:

$$Q^2 = \bar{Q}^2 = 0, \qquad [H_s, Q] = [H_s, \bar{Q}] = 0. \tag{2.2}$$

The key role of Q ($\bar{Q}$) is that it operates on a bosonic state to transform it into a fermionic one and vice versa.

In the minimal version of SUSY [22], Q and $\bar{Q}$ are generally assumed to be represented by $Q = A\sigma_-$, $\bar{Q} = \bar{A}\sigma_+$, where A and $\bar{A}$ are taken to be first-derivative differential operators. So one works with

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} 0 & \bar{A} \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

$$A = \frac{d}{dx} + W(x), \qquad \bar{A} = -\frac{d}{dx} + W(x), \tag{2.4}$$

where $W(x)$ is the so-called superpotential. It is obvious from the above representations of Q and $\bar{Q}$ that H_s appears diagonal:

$$H_s = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}. \tag{2.5}$$

We can actually express H_+ and H_- in factorized forms in terms of A and $\bar{A}$,

$$H_+ = \bar{A}A = -\frac{d^2}{dx^2} + V_+(x) - E, \qquad H_- = A\bar{A} = -\frac{d^2}{dx^2} + V_-(x) - E, \tag{2.6}$$

at some arbitrary factorization energy E. In (2.6), $V_\pm(x)$ are

$$V_\pm(x) = W^2(x) \mp \frac{dW(x)}{dx} + E. \tag{2.7}$$

It may be noticed that the spectrum of H_s is doubly degenerate except possibly for the ground state. In the exact SUSY case to which we shall restrict ourselves here, the ground state at vanishing energy is nondegenerate. In the present notational set-up, it belongs to the H_+ component. Note that the double degeneracy of H_s is also implied by the intertwining relationships, which read

$$AH_+ = H_-A, \qquad H_+\bar{A} = \bar{A}H_-. \tag{2.8}$$

Relations (2.8) are indeed consistent with the definitions (2.6).

PSUSY of order two ($p = 2$), on the other hand, arises by imposing a symmetry between the standard bosonic and parafermionic states. As introduced by Rubakov and Spiridonov [20], the $p = 2$ PSUSY Hamiltonian H_{ps} is defined to obey the relations

$$Q^3 = 0, \qquad Q^2\bar{Q} + Q\bar{Q}Q + \bar{Q}Q^2 = 2QH_{ps}, \qquad [H_{ps}, Q] = 0, \tag{2.9}$$

along with their Hermitian conjugates.

In parallel to (2.3), the parasupercharges Q and $\bar{Q}$ can be assigned a matrix representation in a manner

$$(Q)_{ij} = \left[\frac{d}{dx} + W_j(x)\right]\delta_{i,j+1}, \quad (\bar{Q})_{ij} = \left[-\frac{d}{dx} + W_i(x)\right]\delta_{i+1,j}, \quad i,j = 1,2,3. \tag{2.10}$$

These read explicitly

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} 0 & \bar{A}_1 & 0 \\ 0 & 0 & \bar{A}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.11}$$

with

$$A_i = \frac{d}{dx} + W_i(x), \qquad \bar{A}_i = -\frac{d}{dx} + W_i(x), \qquad i = 1,2. \tag{2.12}$$

The PSUSY algebra (2.9) then leads to a diagonal form for H_{ps},

$$H_{ps} = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}, \tag{2.13}$$

provided

$$A_1\bar{A}_1 = \bar{A}_2A_2 - c, \tag{2.14}$$

where c is a constant. Translated in terms of the superpotentials, Eq. (2.14) expands to

$$W_2^2 - W_1^2 - \frac{dW_1}{dx} - \frac{dW_2}{dx} = c. \tag{2.15}$$

We thus have for H_1, H_2, and H_3,

$$\begin{aligned} H_1 &= \bar{A}_1 A_1 + c_1, \\ H_2 &= A_1 \bar{A}_1 + c_1 = \bar{A}_2 A_2 + c_2, \\ H_3 &= A_2 \bar{A}_2 + c_2, \end{aligned} \tag{2.16}$$

where the constants c_1 and c_2 satisfy $c_1 + c_2 = 0$ and $c_1 - c_2 = c$.

In summary, it is clear that whereas SUSY involves a single superpotential $W(x)$, PSUSY is described by two superpotentials $W_1(x)$ and $W_2(x)$. We now turn to the case of the PT-symmetric harmonic oscillator potential for a PSUSY analysis.

2.2 PT-symmetric oscillator potential

The Hamiltonian [9]

$$H^{(\alpha)} = -\frac{d^2}{dx^2} + (x - \mathrm{i}\delta)^2 + \frac{\alpha^2 - \frac{1}{4}}{(x - \mathrm{i}\delta)^2}, \qquad \alpha > 0, \tag{2.17}$$

is easily seen to be PT symmetric: it can be obtained from the usual three-dimensional radial harmonic oscillator Hamiltonian by effecting a complex shift of coordinate $x \to x - \mathrm{i}\delta$, $\delta > 0$. The operator $H^{(\alpha)}$ is beset with a centrifugal-like core of strength $G = \alpha^2 - \frac{1}{4}$; nonetheless, the model proves to be exactly solvable on the entire real line for any $\alpha > 0$ like the linear harmonic oscillator (corresponding to $\alpha = 1/2$). Contrary to the latter, however, it has an unequal spectrum,

$$E_{qn}^{(\alpha)} = 4n + 2 - 2q\alpha, \qquad n = 0, 1, 2, \ldots, \tag{2.18}$$

if α is not integer, which we shall assume here. In (2.18), $q = \pm 1$ denotes the quasi-even (+) or quasi-odd (−) parity for the corresponding state. The accompanying eigenfunctions are expressible in terms of the standard orthogonal Laguerre polynomials:

$$\psi_{qn}^{(\alpha)}(x) \propto e^{-\frac{1}{2}(x - \mathrm{i}\delta)^2} (x - \mathrm{i}\delta)^{-q\alpha + \frac{1}{2}} L_n^{(-q\alpha)}[(x - \mathrm{i}\delta)^2]. \tag{2.19}$$

Before taking up the PSUSY study, it is interesting to discuss some of the SUSY aspects of $H^{(\alpha)}$. We see from (2.6) and (2.7) that there can be two independent[a] forms of the complex superpotentials associated with $H^{(\alpha)}$. These are

$$\begin{aligned} W^{(\alpha)}(x) &= x - \mathrm{i}\delta + \frac{\alpha - \frac{1}{2}}{x - \mathrm{i}\delta}, \qquad E = 2 - 2\alpha, && (2.20) \\ W'^{(\alpha)}(x) &= x - \mathrm{i}\delta - \frac{\alpha + \frac{1}{2}}{x - \mathrm{i}\delta}, \qquad E' = 2 + 2\alpha. && (2.21) \end{aligned}$$

In (2.20) and (2.21), E and E' stand for the corresponding factorization energies.

Let us consider $W^{(\alpha)}(x)$ first. Using (2.6) and (2.7), it follows readily that

$$V_+^{(\alpha)}(x) = V^{(\alpha)}(x), \qquad V_-^{(\alpha)}(x) = V^{(\alpha-1)}(x) + 2, \tag{2.22}$$

where $\alpha > 1$ and $V^{(\alpha)}(x)$ represents the potential in (2.17). Thus the partner Hamiltonians $H_\pm^{(\alpha)}$ acquire the forms

$$H_+^{(\alpha)} = H^{(\alpha)} - 2 + 2\alpha, \qquad H_-^{(\alpha)} = H^{(\alpha-1)} + 2\alpha. \tag{2.23}$$

Further, using the definitions $A = \frac{d}{dx} + W^{(\alpha)}(x)$ and $\bar{A} = -\frac{d}{dx} + W^{(\alpha)}(x)$, it is straightforward to verify that the operator A annihilates the ground state $\psi_{+0}^{(\alpha)}$:

$$A\psi_{+0}^{(\alpha)}(x) \propto \left(\frac{d}{dx} + x - \mathrm{i}\delta + \frac{\alpha - \frac{1}{2}}{x - \mathrm{i}\delta}\right) e^{-\frac{1}{2}(x-\mathrm{i}\delta)^2}(x - \mathrm{i}\delta)^{-\alpha+\frac{1}{2}} = 0. \tag{2.24}$$

So the spectra of H_+ and H_- read

$$\begin{aligned} \text{Spectrum of } H_+^{(\alpha)}: \quad & E_{+n}^{(\alpha)} - 2 + 2\alpha = 4n, \\ & E_{-n}^{(\alpha)} - 2 + 2\alpha = 4n + 4\alpha, && (2.25) \\ \text{Spectrum of } H_-^{(\alpha)}: \quad & E_{+n}^{(\alpha-1)} + 2\alpha = 4n + 4, \\ & E_{-n}^{(\alpha-1)} + 2\alpha = 4n + 4\alpha. && (2.26) \end{aligned}$$

[a]We can think of additional supersymmetries resulting from the choices

$$\begin{aligned} W''^{(\alpha)}(x) &= x - \mathrm{i}\delta + \frac{\alpha + \frac{1}{2}}{x - \mathrm{i}\delta}, \qquad E'' = -2\alpha, \\ W'''^{(\alpha)}(x) &= x - \mathrm{i}\delta - \frac{\alpha - \frac{1}{2}}{x - \mathrm{i}\delta}, \qquad E''' = 2\alpha, \end{aligned}$$

where E'' and E''' are the factorization energies. However, these supersymmetries are not new in that they can be obtained from $W^{(\alpha)}(x)$ and $W'^{(\alpha)}(x)$ by the replacement $\alpha \to \alpha+1$ or $\alpha \to \alpha-1$.

If however we consider $W'^{\alpha}(x)$ along with E' given by (2.21), then $V'^{(\alpha)}_{+}(x)$ and $V'^{(\alpha)}_{-}(x)$ become

$$V'^{(\alpha)}_{+}(x) = V^{(\alpha)}(x), \qquad V'^{(\alpha)}_{-}(x) = V^{(\alpha+1)}(x) + 2. \tag{2.27}$$

As such the corresponding component Hamiltonians $H'^{(\alpha)}_{+}$ and $H'^{(\alpha)}_{-}$ turn out to be

$$H'^{(\alpha)}_{+} = H^{(\alpha)} - 2 - 2\alpha, \qquad H'^{(\alpha)}_{-} = H^{(\alpha+1)} - 2\alpha. \tag{2.28}$$

The role of $W'^{(\alpha)}(x)$ is, however, quite different from $W^{(\alpha)}(x)$: it is the excited state $\psi^{(\alpha)}_{-0}$ that is annihilated by A' $(\equiv \frac{d}{dx} + W'^{(\alpha)}(x))$:

$$A'\psi^{(\alpha)}_{-0}(x) \propto \left(\frac{d}{dx} + x - \mathrm{i}\delta - \frac{\alpha + \frac{1}{2}}{x - \mathrm{i}\delta}\right) e^{-\frac{1}{2}(x-\mathrm{i}\delta)^2}(x - \mathrm{i}\delta)^{\alpha+\frac{1}{2}} = 0. \tag{2.29}$$

As a result, the spectra of $H'^{(\alpha)}_{+}$ and $H'^{(\alpha)}_{-}$ look much different from those in (2.25) and (2.26):

$$\begin{aligned} \text{Spectrum of } H'^{(\alpha)}_{+}: \quad & E^{(\alpha)}_{+n} - 2 - 2\alpha = 4n - 4\alpha, \\ & E^{(\alpha)}_{-n} - 2 - 2\alpha = 4n, \end{aligned} \tag{2.30}$$

$$\begin{aligned} \text{Spectrum of } H'^{(\alpha)}_{-}: \quad & E^{(\alpha+1)}_{+n} - 2\alpha = 4n - 4\alpha, \\ & E^{(\alpha+1)}_{-n} - 2\alpha = 4n + 4. \end{aligned} \tag{2.31}$$

With this background, we now proceed to discuss the PSUSY of the PT-symmetric oscillator Hamiltonian (2.17). Introducing a pair of complex superpotentials $W_1(x) = W^{(\alpha)}(x)$ and $W_2(x) = W'^{(\alpha-1)}(x)$ and taking $c_1 = -c_2 = -2\alpha$, we at once obtain from (2.16) the results:

$$H_1 = H^{(\alpha)}_{+} - 2\alpha = H^{(\alpha)} - 2, \qquad H_2 = H^{(\alpha)}_{-} - 2\alpha = H^{(\alpha-1)}, \qquad H_3 = H^{(\alpha)} + 2. \tag{2.32}$$

We are therefore led to the following PSUSY spectrum pattern:

$$\begin{aligned} \text{Spectrum of } H_1: \quad & E^{(\alpha)}_{+n} - 2 = 4n - 2\alpha, \\ & E^{(\alpha)}_{-n} - 2 = 4n + 2\alpha, \end{aligned} \tag{2.33}$$

$$\text{Spectrum of } H_2: \quad E_{+n}^{(\alpha-1)} = 4n - 2\alpha + 4,$$
$$E_{-n}^{(\alpha-1)} = 4n + 2\alpha, \tag{2.34}$$
$$\text{Spectrum of } H_3: \quad E_{+n}^{(\alpha)} + 2 = 4n - 2\alpha + 4,$$
$$E_{-n}^{(\alpha)} + 2 = 4n + 2\alpha + 4. \tag{2.35}$$

As a consequence of Eqs. (2.33)–(2.35), the spectrum of H_{ps} shows the features summarized below: if $N - 1 < \alpha < N$, where $N \in \{2, 3, \ldots\}$, then

$$\begin{aligned}
E_0 &= -2\alpha, \quad E_1 = -2\alpha + 4, \quad \ldots, E_{N-1} = -2\alpha + 4N - 4, \quad E_N = 2\alpha, \\
E_{N+1} &= -2\alpha + 4N, \quad E_{N+2} = 2\alpha + 4, \quad \ldots, E_{N+2m} = 2\alpha + 4m, \\
E_{N+2m+1} &= -2\alpha + 4N + 4m, \quad \ldots,
\end{aligned} \tag{2.36}$$

with degeneracies

$$\begin{aligned}
d_0 &= 1, \quad d_1 = 3, \quad \ldots, d_{N-1} = 3, \quad d_N = 2, \quad d_{N+1} = 3, \quad d_{N+2} = 3, \quad \ldots, \\
d_{N+2m} &= 3, \quad d_{N+2m+1} = 3, \quad \ldots.
\end{aligned} \tag{2.37}$$

In the Hermitian case, the spectrum of H_{ps} is known to be always three-fold degenerate at least starting from the second and higher excited states. From (2.37), we see that this is not true here. Note that H_3 is essentially a shifted H_1 and that the ground state is nondegenerate.

We now remark on the other possibility when we can identify $W_1'(x) = W'^{(\alpha)}(x)$ and $W_2'(x) = W^{(\alpha+1)}(x)$ with $c_1' = -c_2' = 2\alpha$. We obtain as a result

$$H_1' = H_+'^{(\alpha)} + 2\alpha = H^{(\alpha)} - 2, \qquad H_2' = H_-'^{(\alpha)} + 2\alpha = H^{(\alpha+1)}, \qquad H_3' = H^{(\alpha)} + 2. \tag{2.38}$$

The respective spectra of H_1', H_2', and H_3' are then

$$\text{Spectrum of } H_1': \quad E_{+n}^{(\alpha)} - 2 = 4n - 2\alpha,$$
$$E_{-n}^{(\alpha)} - 2 = 4n + 2\alpha, \tag{2.39}$$
$$\text{Spectrum of } H_2': \quad E_{+n}^{(\alpha+1)} = 4n - 2\alpha,$$
$$E_{-n}^{(\alpha+1)} = 4n + 2\alpha + 4, \tag{2.40}$$
$$\text{Spectrum of } H_3': \quad E_{+n}^{(\alpha)} + 2 = 4n - 2\alpha + 4,$$
$$E_{-n}^{(\alpha)} + 2 = 4n + 2\alpha + 4. \tag{2.41}$$

These yield the following spectrum of H_{ps}: if $N-1<\alpha<N$, where $N\in\{1,2,\ldots\}$, then

$$\begin{aligned} E_0 &= -2\alpha, \quad E_1=-2\alpha+4, \quad \ldots, E_{N-1}=-2\alpha+4N-4, \quad E_N=2\alpha, \\ E_{N+1} &= -2\alpha+4N, \quad E_{N+2}=2\alpha+4, \quad \ldots, E_{N+2m}=2\alpha+4m, \\ E_{N+2m+1} &= -2\alpha+4N+4m, \quad \ldots, \end{aligned} \tag{2.42}$$

with degeneracies

$$\begin{aligned} d_0 &= 2, \quad d_1=3, \quad \ldots, d_{N-1}=3, \quad d_N=1, \quad d_{N+1}=3, \quad d_{N+2}=3, \quad \ldots, \\ d_{N+2m} &= 3, \quad d_{N+2m+1}=3, \quad \ldots. \end{aligned} \tag{2.43}$$

In contrast to the previous case, here the ground state is doubly degenerate. However, similar to what we obtained before, the nature of degeneracies is not of the usual Hermitian type. Note that H_3 is again a shifted H_1. We therefore conclude that to get a shifted PT-symmetric oscillator, one has to resort to a complexified PSUSY of order two, contrary to what happens for the standard harmonic oscillator case where such a result is obtained in SUSYQM.

In the limiting cases where α becomes some integer N, the spectrum of $H^{(\alpha)}$ becomes equidistant and the need for quasi-parity disappears due to the phenomenon of unavoided level crossings without degeneracy [9]. We then recover a degeneracy pattern of the usual Hermitian type for the spectrum of H_{ps}, namely

$$d_0=1, \quad d_1=3, \quad \ldots, d_{N-1}=3, \quad d_N=3, \quad d_{N+1}=3, \quad d_{N+2}=3, \quad \ldots, \tag{2.44}$$

or

$$d_0=2, \quad d_1=3, \quad \ldots, d_{N-1}=3, \quad d_N=3, \quad d_{N+1}=3, \quad d_{N+2}=3, \quad \ldots, \tag{2.45}$$

for the choice (2.32) or (2.38), respectively. In both cases, the spectrum of H_{ps} is the same:

$$\begin{aligned} E_0 &= -2N, \quad E_1=-2N+4, \quad \ldots, E_{N-1}=2N-4, \quad E_N=2N, \\ E_{N+1} &= 2N+4, \quad E_{N+2}=2N+8, \quad \ldots, \end{aligned} \tag{2.46}$$

but for the former choice, N is restricted to the set $\{2,3,4,\ldots\}$, while for the latter it may take any value in $\{1,2,3,\ldots\}$.

2.3 PT-symmetric generalized Pöschl-Teller potential

The Hamiltonian for the PT-symmetric generalized Pöschl-Teller system is given by [12]

$$H^{(A,B)} = -\frac{d^2}{dx^2}+[B^2+A(A+1)]\,\text{cosech}^2\,\tau - B(2A+1)\,\text{cosech}\,\tau\coth\tau, \qquad \tau = x - \mathrm{i}\gamma, \tag{2.47}$$

where $-\frac{\pi}{4} \leq \gamma < 0$ or $0 < \gamma < \frac{\pi}{4}$, $B > A + \frac{1}{2} > 0$, and $A + \frac{1}{2}$ and B do not differ by an integer. Note that $H^{(A,B)}$ is invariant under the replacements $\left(A + \frac{1}{2}, B\right) \to \left(B, A + \frac{1}{2}\right)$.

We have recently shown [12], using sl(2,$\mathbb{C}$) as a tool, that the PT-symmetric Hamiltonian $H^{(A,B)}$ possesses two series of real energy eigenvalues according to

$$\begin{aligned} E_{+n}^{(A,B)} &= -\left(B - \tfrac{1}{2} - n\right)^2, \qquad n = 0, 1, \ldots, n_{+max}, \\ &\quad B - \tfrac{3}{2} \leq n_{+max} < B - \tfrac{1}{2}, \end{aligned} \tag{2.48}$$

$$\begin{aligned} E_{-n}^{(A,B)} &= -\left(A - n\right)^2, \qquad n = 0, 1, \ldots, n_{-max}, \\ &\quad A - 1 \leq n_{-max} < A, \end{aligned} \tag{2.49}$$

where $B > \frac{1}{2}$ and $A > 0$. Note that while the real counterpart of (2.47), obtained by setting $\gamma = 0$, is singular and so calls for its restriction to the half-line $(0, +\infty)$, the complexified potential as given above gets regularized on performing the shift $x \to x - \mathrm{i}\gamma$ and so may be considered on the entire real line. Note also that the coupling constants appearing in $H^{(A,B)}$ are all real.

Corresponding to the two series of energy levels (2.48) and (2.49), the eigenfunctions read

$$\psi_{+n}^{(A,B)} \propto (y-1)^{(A-B+1)/2}(y+1)^{-(A+B)/2} P_n^{(A-B+\frac{1}{2}, -A-B-\frac{1}{2})}(y), \tag{2.50}$$

$$\psi_{-n}^{(A,B)} \propto (y-1)^{(B-A)/2}(y+1)^{-(B+A)/2} P_n^{(B-A-\frac{1}{2}, -B-A-\frac{1}{2})}(y), \tag{2.51}$$

where $y = \cosh\tau$ and $P_n^{(\alpha,\beta)}(y)$ is a Jacobi polynomial.

Carrying out a standard SUSYQM analysis, we get for $A = \frac{d}{dx} + W^{(A,B)}(x)$ and $\bar{A} = -\frac{d}{dx} + W^{(A,B)}(x)$ the partner Hamiltonians

$$\begin{aligned} H_+^{(A,B)} &= \bar{A}A = -\frac{d^2}{dx^2} + V_+^{(A,B)}(x) - E, \\ H_-^{(A,B)} &= A\bar{A} = -\frac{d^2}{dx^2} + V_-^{(A,B)}(x) - E, \end{aligned} \tag{2.52}$$

where $V_\pm^{(A,B)}(x)$ are related to $W^{(A,B)}(x)$ as defined in (2.7). The superpotential $W^{(A,B)}$ is given by

$$W^{(A,B)}(x) = \left(B - \tfrac{1}{2}\right)\coth\tau - \left(A + \tfrac{1}{2}\right)\operatorname{cosech}\tau, \qquad E = -\left(B - \tfrac{1}{2}\right)^2, \tag{2.53}$$

E being the factorization energy.

It is simple to work out

$$V_+^{(A,B)}(x) = V^{(A,B)}(x), \qquad V_-^{(A,B)}(x) = V^{(A,B-1)}(x), \tag{2.54}$$

where $V^{(A,B)}$ is the potential in (2.47). Relations (2.54) imply as a consequence

$$H_+^{(A,B)} = H^{(A,B)} + \left(B - \tfrac{1}{2}\right)^2, \qquad H_-^{(A,B)} = H^{(A,B-1)} + \left(B - \tfrac{1}{2}\right)^2. \tag{2.55}$$

The nondegenerate ground state $\psi_{+0}^{(A,B)}$ is easily seen to be annihilated by the operator A:

$$\begin{aligned} A\psi_{+0}^{(A,B)} &\propto \left[\frac{d}{dx} + \left(B - \frac{1}{2}\right)\coth\tau - \left(A + \frac{1}{2}\right)\operatorname{cosech}\tau\right](y-1)^{(A-B+1)/2} \\ &\quad \times (y+1)^{-(A+B)/2} \\ &\propto (\sinh\tau)^{-1}\left[\left(y^2 - 1\right)\frac{d}{dy} + \left(B - \frac{1}{2}\right)y - \left(A + \frac{1}{2}\right)\right](y-1)^{(A-B+1)/2} \\ &\quad \times (y+1)^{-(A+B)/2} \\ &= 0, \end{aligned} \tag{2.56}$$

resulting in the following spectra of $H_\pm^{(A,B)}$:

$$\begin{aligned} \text{Spectrum of } H_+^{(A,B)}: \quad & E_{+n}^{(A,B)} + \left(B - \tfrac{1}{2}\right)^2 = n(2B - n - 1), \\ & E_{-n}^{(A,B)} + \left(B - \tfrac{1}{2}\right)^2 = \left(B - A + n - \tfrac{1}{2}\right) \\ & \times \left(B + A - n - \tfrac{1}{2}\right), \end{aligned} \tag{2.57}$$

$$\begin{aligned} \text{Spectrum of } H_-^{(A,B)}: \quad & E_{+n}^{(A,B-1)} + \left(B - \tfrac{1}{2}\right)^2 = (n+1)(2B - n - 2), \\ & E_{-n}^{(A,B-1)} + \left(B - \tfrac{1}{2}\right)^2 = \left(B - A + n - \tfrac{1}{2}\right) \\ & \times \left(B + A - n - \tfrac{1}{2}\right). \end{aligned} \tag{2.58}$$

Clearly from (2.57) and (2.58) we get the usual picture of unbroken SUSY.

Since the potential $V^{(A,B)}(x)$ is invariant under $A+\frac{1}{2} \leftrightarrow B$, we may as well have a second choice of the superpotential given by

$$W'^{(A,B)}(x) = A \coth\tau - B \operatorname{cosech}\tau, \qquad E' = -A^2, \tag{2.59}$$

where E' is the factorization energy. In this case, the wave function $\psi_{-0}^{(A,B)}$ is annihilated by the operator A', showing that an excited state at vanishing energy is suppressed:

$$\begin{aligned}
\text{Spectrum of } H_+'^{(A,B)}: \quad & E_{+n}^{(A,B)} + A^2 = \left(A - B + n + \tfrac{1}{2}\right)\left(A + B - n - \tfrac{1}{2}\right), \\
& E_{-n}^{(A,B)} + A^2 = n(2A - n), && (2.60) \\
\text{Spectrum of } H_-'^{(A,B)}: \quad & E_{+n}^{(A-1,B)} + A^2 = \left(A - B + n + \tfrac{1}{2}\right)\left(A + B - n - \tfrac{1}{2}\right), \\
& E_{-n}^{(A-1,B)} + A^2 = (n+1)(2A - n - 1). && (2.61)
\end{aligned}$$

Moving on to PSUSY we consider, as a first choice, the superpotentials $W_1(x)$ and $W_2(x)$ defined by

$$\begin{aligned}
W_1(x) &= W^{(A,B)}(x), \qquad W_2(x) = W'^{(A,B-1)}(x), \\
c_1 &= -c_2 = \tfrac{1}{2}\left[A^2 - \left(B - \tfrac{1}{2}\right)^2\right]. && (2.62)
\end{aligned}$$

We then get for the component Hamiltonians of H_{ps},

$$\begin{aligned}
H_1 &= H^{(A,B)} + \mathcal{E}, \qquad H_2 = H^{(A,B-1)} + \mathcal{E}, \qquad H_3 = H^{(A-1,B-1)} + \mathcal{E}, \\
\mathcal{E} &\equiv \tfrac{1}{2}\left[A^2 + \left(B - \tfrac{1}{2}\right)^2\right], && (2.63)
\end{aligned}$$

where (2.55) has been used. As a result, the following spectra of H_1, H_2, and H_3 emerge:

$$\begin{aligned}
\text{Spectrum of } H_1: \quad & E_{+n}^{(A,B)} + \mathcal{E} = -\left(B - \tfrac{1}{2} - n\right)^2 + \mathcal{E}, \\
& E_{-n}^{(A,B)} + \mathcal{E} = -(A - n)^2 + \mathcal{E}, && (2.64) \\
\text{Spectrum of } H_2: \quad & E_{+n}^{(A,B-1)} + \mathcal{E} = -\left(B - \tfrac{3}{2} - n\right)^2 + \mathcal{E}, \\
& E_{-n}^{(A,B-1)} + \mathcal{E} = -(A - n)^2 + \mathcal{E}, && (2.65) \\
\text{Spectrum of } H_3: \quad & E_{+n}^{(A-1,B-1)} + \mathcal{E} = -\left(B - \tfrac{3}{2} - n\right)^2 + \mathcal{E}, \\
& E_{-n}^{(A-1,B-1)} + \mathcal{E} = -(A - 1 - n)^2 + \mathcal{E}. && (2.66)
\end{aligned}$$

We therefore see that in going from H_1 to H_2, one suppresses the ground state of H_1 at an energy $\frac{1}{2}\left[A^2 - \left(B - \frac{1}{2}\right)^2\right] < 0$. Then in going from H_2 to H_3, one suppresses a state of H_2 at an energy $\frac{1}{2}\left[\left(B - \frac{1}{2}\right)^2 - A^2\right] > 0$. The latter is either an excited state or the ground state according to whether $B > A + \frac{3}{2}$ or $B < A + \frac{3}{2}$.

For completeness, let us write down the spectrum of H_{ps}. If $B - N < A + \frac{1}{2} < B - N + 1$, where $N \in \{1, 2, 3, \ldots\}$, it reads

$$\begin{aligned}
E_0 &= \frac{1}{2}\left[A^2 - \left(B - \frac{1}{2}\right)^2\right], \qquad d_0 = 1, \\
E_1 &= E_0 + 2B - 2, \qquad d_1 = 3, \\
&\vdots \\
E_{N-1} &= E_0 + (N-1)(2B - N), \qquad d_{N-1} = 3, \\
E_N &= E_0 - A^2 + \left(B - \frac{1}{2}\right)^2, \qquad d_N = 2, \\
&\vdots \\
E_{N+2p+1} &= E_0 + (N+p)(2B - N - p - 1), \qquad d_{N+2p+1} = 3, \\
& \quad p = 0, 1, \ldots, p_{+max}, \\
E_{N+2p} &= E_0 - (A-p)^2 + \left(B - \frac{1}{2}\right)^2, \qquad d_{N+2p} = 3, \\
& \quad p = 1, 2, \ldots, p_{-max}.
\end{aligned} \tag{2.67}$$

In (2.67), d_i ($i = 0, 1, \ldots, N-1, N, \ldots, N+2p+1, N+2p$) is the degeneracy, $p_{+max} = n_{+max} - N$, and $p_{-max} = n_{-max}$.

We next consider the second choice of the superpotentials, namely

$$\begin{aligned}
W_1'(x) &= W'^{(A,B)}(x), \qquad W_2'(x) = W^{(A-1,B)}(x), \\
c_1' &= -c_2' = \frac{1}{2}\left[\left(B - \frac{1}{2}\right)^2 - A^2\right].
\end{aligned} \tag{2.68}$$

We obtain after a little algebra

$$H_1' = H^{(A,B)} + \mathcal{E}, \qquad H_2' = H^{(A-1,B)} + \mathcal{E}, \qquad H_3' = H^{(A-1,B-1)} + \mathcal{E}, \tag{2.69}$$

where $\mathcal{E}$ is the same as in (2.63). The spectra of H_1', H_2', and H_3' read

$$\text{Spectrum of } H_1': \qquad E_{+n}^{(A,B)} + \mathcal{E} = -\left(B - \frac{1}{2} - n\right)^2 + \mathcal{E},$$

$$E_{-n}^{(A,B)} + \mathcal{E} = -\left(A - n\right)^2 + \mathcal{E}, \tag{2.70}$$

$$\text{Spectrum of } H_2': \quad E_{+n}^{(A-1,B)} + \mathcal{E} = -\left(B - \tfrac{1}{2} - n\right)^2 + \mathcal{E},$$

$$E_{-n}^{(A-1,B)} + \mathcal{E} = -\left(A - 1 - n\right)^2 + \mathcal{E}, \tag{2.71}$$

$$\text{Spectrum of } H_3': \quad E_{+n}^{(A-1,B-1)} + \mathcal{E} = -\left(B - \tfrac{3}{2} - n\right)^2 + \mathcal{E},$$

$$E_{-n}^{(A-1,B-1)} + \mathcal{E} = -\left(A - 1 - n\right)^2 + \mathcal{E}. \tag{2.72}$$

We thus see that in going from H_1' to H_2', one suppresses an excited state of H_1' at an energy $\frac{1}{2}\left[\left(B - \frac{1}{2}\right)^2 - A^2\right] > 0$. Then in going from H_2' to H_3', one suppresses the ground state of H_2' at an energy $\frac{1}{2}\left[A^2 - \left(B - \frac{1}{2}\right)^2\right] < 0$. Further if $B - N < A + \frac{1}{2} < B - N + 1$, where $N \in \{1, 2, 3, \ldots\}$, then the spectrum of H_{ps} is the same as in the previous case, but the degeneracies are $d_0 = 2$, $d_1 = 3$, …, $d_{N-1} = 3$, $d_N = 1$, …, $d_{N+2p+1} = 3$, $d_{N+2p} = 3$.

In the limiting cases where $A + \frac{1}{2}$ and B differ by some integer, $H^{(A,B)}$ has a single series of energy levels due to the phenomenon of unavoided level crossings without degeneracy [12]. The PSUSY scheme then becomes similar to the usual one for Hermitian Hamiltonians.

2.4 PT-symmetric Scarf II potential

The Hamiltonian for the PT-symmetric Scarf II potential is given by [12]

$$H^{(A,B)} = -\frac{d^2}{dx^2} - \left[B^2 + A(A+1)\right] \operatorname{sech}^2 x + \mathrm{i}B(2A+1) \operatorname{sech} x \tanh x, \tag{2.73}$$

where $A > B - \frac{1}{2} > 0$ and $A - B + \frac{1}{2}$ is not an integer. The form (2.73) is PT symmetric; like the PT-symmetric generalized Pöschl-Teller Hamiltonian (2.47), it also exhibits invariance under exchange of the parameters $A + \frac{1}{2}$ and B. Full and detailed analyses of the various properties of (2.73) have already been given by us elsewhere [12] in connection with sl(2,$\mathbb{C}$) potential algebra. We have found that PT-symmetric Scarf II potential depicts two series of energy levels. These are

$$E_{+n}^{(A,B)} = -\left(A - n\right)^2, \qquad n = 0, 1, \ldots, n_{+max},$$

$$A-1 \le n_{+max} < A, \tag{2.74}$$

$$E_{-n}^{(A,B)} = -\left(B-\tfrac{1}{2}-n\right)^2, \qquad n=0,1,\ldots,n_{-max},$$
$$B-\tfrac{3}{2} \le n_{-max} < B-\tfrac{1}{2}. \tag{2.75}$$

The accompanying eigenfunctions read

$$\psi_{+n}^{(A,B)} \propto (\operatorname{sech} x)^A \exp[-\mathrm{i}B \arctan(\sinh x)] P_n^{(-A+B-\frac{1}{2},-A-B-\frac{1}{2})}(\mathrm{i}\sinh x), \tag{2.76}$$

$$\psi_{-n}^{(A,B)} \propto (\operatorname{sech} x)^{B-\frac{1}{2}} \exp\left[-\mathrm{i}\left(A+\tfrac{1}{2}\right)\arctan(\sinh x)\right] \times P_n^{(A-B+\frac{1}{2},-A-B-\frac{1}{2})}(\mathrm{i}\sinh x), \tag{2.77}$$

in terms of Jacobi polynomials.

We are now going to show that the model (2.73) possesses PSUSY. This can be easily established, as we did for the PT-symmetric oscillator and generalized Pöschl-Teller potentials, by demonstrating first that two superpotentials exist for it in the context of SUSY. PSUSY can then be contructed taking their help.

Indeed one can verify that two possible candidates of the superpotential are

$$W^{(A,B)}(x) = A\tanh x + \mathrm{i}B \operatorname{sech} x, \quad E=-A^2, \tag{2.78}$$

$$W'^{(A,B)}(x) = \left(B-\tfrac{1}{2}\right)\tanh x + \mathrm{i}\left(A+\tfrac{1}{2}\right)\operatorname{sech} x, \quad E' = -\left(B-\tfrac{1}{2}\right)^2, \tag{2.79}$$

where E and E' are the factorization energies. Note that (2.79) is obtainable from (2.78) under the replacements $A+\frac{1}{2} \leftrightarrow B$.

While corresponding to (2.78) we derive

$$V_+^{(A,B)}(x) = V^{(A,B)}(x), \qquad V_-^{(A,B)}(x) = V^{(A-1,B)}(x), \tag{2.80}$$

where $V^{(A,B)}(x)$ is the potential of (2.73), Eq. (2.79) yields the pair

$$V_+'^{(A,B)}(x) = V^{(A,B)}(x), \qquad V_-'^{(A,B)}(x) = V^{(A,B-1)}(x). \tag{2.81}$$

The associated partner Hamiltonians for (2.80) and (2.81) are

$$H_+^{(A,B)} = H^{(A,B)} + A^2, \qquad H_-^{(A,B)} = H^{(A-1,B)} + A^2, \tag{2.82}$$

$$H_+'^{(A,B)} = H^{(A,B)} + \left(B-\tfrac{1}{2}\right)^2, \qquad H_-'^{(A,B)} = H^{(A,B-1)} + \left(B-\tfrac{1}{2}\right)^2. \tag{2.83}$$

Further, defining operators $A = \frac{d}{dx} + W^{(A,B)}(x)$ and $A' = \frac{d}{dx} + W'^{(A,B)}(x)$, it is a simple exercise to check that the states $\psi_{+0}^{(A,B)}$ and $\psi_{-0}^{(A,B)}$ are annihilated by A and A', respectively. The spectra of $H_\pm^{(A,B)}$ and $H_\pm'^{(A,B)}$ turn out to be

$$
\begin{aligned}
\text{Spectrum of } H_+^{(A,B)}: \quad & E_{+n}^{(A,B)} + A^2 = n(2A-n), \\
& E_{-n}^{(A,B)} + A^2 = \left(A - B + n + \tfrac{1}{2}\right) \\
& \times \left(A + B - n - \tfrac{1}{2}\right), \qquad (2.84) \\
\text{Spectrum of } H_-^{(A,B)}: \quad & E_{+n}^{(A-1,B)} + A^2 = (n+1)(2A-n-1), \\
& E_{-n}^{(A-1,B)} + A^2 = \left(A - B + n + \tfrac{1}{2}\right) \\
& \times \left(A + B - n - \tfrac{1}{2}\right), \qquad (2.85) \\
\text{Spectrum of } H_+'^{(A,B)}: \quad & E_{+n}^{(A,B)} + \left(B - \tfrac{1}{2}\right)^2 = \left(B - A + n - \tfrac{1}{2}\right) \\
& \times \left(B + A - n - \tfrac{1}{2}\right), \\
& E_{-n}^{(A,B)} + \left(B - \tfrac{1}{2}\right)^2 = n(2B-n-1), \qquad (2.86) \\
\text{Spectrum of } H_-'^{(A,B)}: \quad & E_{+n}^{(A,B-1)} + \left(B - \tfrac{1}{2}\right)^2 = \left(B - A + n - \tfrac{1}{2}\right) \\
& \times \left(B + A - n - \tfrac{1}{2}\right), \\
& E_{-n}^{(A,B-1)} + \left(B - \tfrac{1}{2}\right)^2 = (n+1)(2B-n-2). \qquad (2.87)
\end{aligned}
$$

While (2.84) and (2.85) show the conventional unbroken SUSY picture, (2.86) and (2.87) point to an unusual scenario: an excited state at vanishing energy is suppressed.

Equipped with the above SUSY machinery, we define the following pair of superpotentials for $p = 2$ PSUSY:

$$
\begin{aligned}
W_1(x) &= W^{(A,B)}(x), \qquad W_2(x) = W'^{(A-1,B)}(x), \\
c_1 &= -c_2 = \tfrac{1}{2}\left[\left(B - \tfrac{1}{2}\right)^2 - A^2\right]. \qquad (2.88)
\end{aligned}
$$

Then it follows from (2.16), (2.82), and (2.83) that

$$
\begin{aligned}
H_1 &= H^{(A,B)} + \mathcal{E}, \qquad H_2 = H^{(A-1,B)} + \mathcal{E}, \qquad H_3 = H^{(A-1,B-1)} + \mathcal{E}, \\
\mathcal{E} &\equiv \tfrac{1}{2}\left[A^2 + \left(B - \tfrac{1}{2}\right)^2\right]. \qquad (2.89)
\end{aligned}
$$

The spectra of H_1, H_2, and H_3 are

$$\begin{aligned}
\text{Spectrum of } H_1: \quad & E_{+n}^{(A,B)} + \mathcal{E} = -\left(A-n\right)^2 + \mathcal{E}, \\
& E_{-n}^{(A,B)} + \mathcal{E} = -\left(B-\tfrac{1}{2}-n\right)^2 + \mathcal{E}, \qquad (2.90)\\
\text{Spectrum of } H_2: \quad & E_{+n}^{(A-1,B)} + \mathcal{E} = -\left(A-1-n\right)^2 + \mathcal{E}, \\
& E_{-n}^{(A-1,B)} + \mathcal{E} = -\left(B-\tfrac{1}{2}-n\right)^2 + \mathcal{E}, \qquad (2.91)\\
\text{Spectrum of } H_3: \quad & E_{+n}^{(A-1,B-1)} + \mathcal{E} = -\left(A-1-n\right)^2 + \mathcal{E}, \\
& E_{-n}^{(A-1,B-1)} + \mathcal{E} = -\left(B-\tfrac{3}{2}-n\right)^2 + \mathcal{E}. \qquad (2.92)
\end{aligned}$$

From (2.90)–(2.92) we find that when going from H_1 to H_2, one suppresses the ground state of H_1 at an energy $\frac{1}{2}\left[\left(B-\frac{1}{2}\right)^2 - A^2\right] < 0$. Then when going from H_2 to H_3, one suppresses a state of H_2 at an energy $\frac{1}{2}\left[A^2 - \left(B-\frac{1}{2}\right)^2\right] > 0$. Such a state is an excited or the ground state according to whether $A > B+\frac{1}{2}$ or $A < B+\frac{1}{2}$. In general, if $A-N < B-\frac{1}{2} < A-N+1$, where $N \in \{1,2,3,\ldots\}$, then the spectrum of H_{ps} is

$$\begin{aligned}
E_0 &= \tfrac{1}{2}\left[\left(B-\tfrac{1}{2}\right)^2 - A^2\right], \qquad d_0 = 1,\\
E_1 &= E_0 + 2A - 1, \qquad d_1 = 3,\\
&\vdots\\
E_{N-1} &= E_0 + (N-1)(2A+1-N), \qquad d_{N-1} = 3,\\
E_N &= E_0 + A^2 - \left(B-\tfrac{1}{2}\right)^2, \qquad d_N = 2,\\
&\vdots\\
E_{N+2p+1} &= E_0 + (N+p)(2A-N-p), \qquad d_{N+2p+1} = 3,\\
& \quad p = 0, 1, \ldots, p_{+max},\\
E_{N+2p} &= E_0 + A^2 - \left(B-\tfrac{1}{2}-p\right)^2, \qquad d_{N+2p} = 3,\\
& \quad p = 1, 2, \ldots, p_{-max}, \qquad (2.93)
\end{aligned}$$

where d_i ($i = 0,\ 1,\ \ldots,\ N-1,\ N,\ \ldots,\ N+2p+1,\ N+2p$) is the degeneracy, $p_{+max} = n_{+max} - N$, and $p_{-max} = n_{-max}$.

Keeping in mind the invariance of (2.73) under $A + \frac{1}{2} \leftrightarrow B$, we can also define another set of superpotentials

$$\begin{aligned} W_1'(x) &= W'^{(A,B)}(x), \qquad W_2'(x) = W^{(A,B-1)}(x), \\ c_1' &= -c_2' = \frac{1}{2}\left[A^2 - \left(B - \frac{1}{2}\right)^2\right]. \end{aligned} \tag{2.94}$$

Then using (2.83) and (2.82), we get

$$H_1' = H^{(A,B)} + \mathcal{E}, \qquad H_2' = H^{(A,B-1)} + \mathcal{E}, \qquad H_3' = H^{(A-1,B-1)} + \mathcal{E}, \tag{2.95}$$

implying the following spectra:

$$\begin{aligned} \text{Spectrum of } H_1'\text{:} \quad & E_{+n}^{(A,B)} + \mathcal{E} = -\left(A - n\right)^2 + \mathcal{E}, \\ & E_{-n}^{(A,B)} + \mathcal{E} = -\left(B - \tfrac{1}{2} - n\right)^2 + \mathcal{E}, \end{aligned} \tag{2.96}$$

$$\begin{aligned} \text{Spectrum of } H_2'\text{:} \quad & E_{+n}^{(A,B-1)} + \mathcal{E} = -\left(A - n\right)^2 + \mathcal{E}, \\ & E_{-n}^{(A,B-1)} + \mathcal{E} = -\left(B - \tfrac{3}{2} - n\right)^2 + \mathcal{E}, \end{aligned} \tag{2.97}$$

$$\begin{aligned} \text{Spectrum of } H_3'\text{:} \quad & E_{+n}^{(A-1,B-1)} + \mathcal{E} = -\left(A - 1 - n\right)^2 + \mathcal{E}, \\ & E_{-n}^{(A-1,B-1)} + \mathcal{E} = -\left(B - \tfrac{3}{2} - n\right)^2 + \mathcal{E}. \end{aligned} \tag{2.98}$$

We thus see that in going from H_1' to H_2', one suppresses an excited state of H_1' at an energy $\frac{1}{2}\left[A^2 - \left(B - \frac{1}{2}\right)^2\right] > 0$. Then when going from H_2' to H_3', one suppresses the ground state of H_2' at an energy $\frac{1}{2}\left[\left(B - \frac{1}{2}\right)^2 - A^2\right] < 0$. If $A - N < B - \frac{1}{2} < A - N + 1$, where $N \in \{1, 2, 3, \ldots\}$, then the spectrum of H_{ps} is the same as in the previous case of (2.88), but the degeneracies are $d_0 = 2$, $d_1 = 3$, …, $d_{N-1} = 3$, $d_N = 1$, …, $d_{N+2p+1} = 3$, $d_{N+2p} = 3$.

Whenever $A - B + \frac{1}{2}$ goes to an integer, we observe the same collapse of the double series of energy levels [12] and restoration of the usual PSUSY scheme as in the two previous subsections.

3 In pursuit of a complexified SSUSY

3.1 Underlying ideas of SSUSY

SSUSY is an extended supersymmetric theory having a second-derivative realization of the differential operators A and $\bar{A}$ [23]–[28]. SSUSY schemes find interesting applicability to non-trivial quantum mechanical problems, which include coupled channel problems and those related to transparent matrix potentials. SSUSY is not guided by a Schrödinger form of the Hamiltonian operator, but instead by a quasi-Hamiltonian K, which is a fourth-order differential operator. However, under certain conditions, K can be related to the square of the Schrödinger Hamiltonian: indeed this feature has been exploited to arrive at models of PSUSY by glueing two ordinary SUSY systems [23].

Consider supercharges involving second derivatives ($\partial \equiv d/dx$):

$$\mathcal{A}^+ = \partial^2 - 2p(x)\partial + b(x), \tag{3.1}$$

$$\mathcal{A}^- = \partial^2 + 2p(x)\partial + 2p'(x) + b(x), \tag{3.2}$$

where $p(x)$ and $b(x)$ are arbitrary functions. Let us introduce the following operators built out of $\mathcal{A}^+$ and $\mathcal{A}^-$:

$$Q^+ = \begin{pmatrix} 0 & 0 \\ \mathcal{A}^- & 0 \end{pmatrix}, \qquad Q^- = \begin{pmatrix} 0 & \mathcal{A}^+ \\ 0 & 0 \end{pmatrix}. \tag{3.3}$$

In analogy with (2.1), we can think of a quasi-Hamiltonian K defined by

$$K = Q^+Q^- + Q^-Q^+. \tag{3.4}$$

Clearly K is a fourth-order differential operator.

We can also construct another operator H from two Schrödinger-like Hamiltonians $h^{(1)}$ and $h^{(2)}$:

$$H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix}, \tag{3.5}$$

$$h^{(1,2)} = -\partial^2 + V^{(1,2)}, \tag{3.6}$$

such that H commutes with $Q^\pm$:

$$\left[H, Q^\pm\right] = 0. \tag{3.7}$$

From (3.3), (3.5), and (3.7), we are led to

$$\mathcal{A}^- h^{(1)} = h^{(2)} \mathcal{A}^-, \qquad \mathcal{A}^+ h^{(2)} = h^{(1)} \mathcal{A}^+. \tag{3.8}$$

These are intertwining relationships similar to the supersymmetric ones (2.8).

Using the representations (3.1) and (3.2), we can exploit (3.8) to obtain constraints among the functions $p(x)$, $b(x)$, and the potentials $V^{(1,2)}(x)$:

$$b = -p' + p^2 - \frac{p''}{2p} + \left(\frac{p'}{2p}\right)^2 + \frac{d}{4p^2}, \tag{3.9}$$

$$V^{(1,2)} = \mp 2p' + p^2 + \frac{p''}{2p} - \left(\frac{p'}{2p}\right)^2 - \frac{d}{4p^2} - a, \tag{3.10}$$

where d and a are integration constants and the primes denote derivatives with respect to x.

We next address to what is known as polynomial SUSY. Here the quasi-Hamiltonian K is taken to be a quadratic in H:

$$K = H^2 + 2\alpha H + \beta = (H + a)^2 + d, \tag{3.11}$$

where α, β are constants, and $a = \alpha$, $d = \beta - \alpha^2$. A PSUSY model can be developed [23] by choosing $a = 0$, for which

$$K = H^2 + d. \tag{3.12}$$

Factorization of K requires d to be a perfect square in the form $d = \frac{c^2}{4}$ for $d > 0$ or $d = -\frac{c^2}{4}$ for $d < 0$. Andrianov *et al.* [24, 25] call $d < 0$ a reducible algebra and $d > 0$ an irreducible one. In the reducible case, we can imagine the existence of an intermediate Hamiltonian that behaves like a superpartner to both $h^{(1)}$ and $h^{(2)}$. Alternatively, this triplet of Hamiltonians furnishes a model for PSUSY.

In the following we will be interested in the reducible case only and write

$$\begin{aligned} K &= H^2 - \tfrac{c^2}{4} \\ &= \begin{pmatrix} \left(h^{(1)} + \frac{c}{2}\right)\left(h^{(1)} - \frac{c}{2}\right) & 0 \\ 0 & \left(h^{(2)} - \frac{c}{2}\right)\left(h^{(2)} + \frac{c}{2}\right) \end{pmatrix}. \end{aligned} \tag{3.13}$$

We also know from (3.4) and (3.3) that

$$K = \begin{pmatrix} \mathcal{A}^+\mathcal{A}^- & 0 \\ 0 & \mathcal{A}^-\mathcal{A}^+ \end{pmatrix}. \tag{3.14}$$

Our immediate problem will be to reconcile (3.13) and (3.14). To this end, we factorize $\mathcal{A}^+$ and $\mathcal{A}^-$ as

$$\begin{aligned} \mathcal{A}^+ &= q_1^+ q_2^+ = (-\partial + W_1)(-\partial + W_2), \\ \mathcal{A}^- &= q_2^- q_1^- = (\partial + W_2)(\partial + W_1). \end{aligned} \tag{3.15}$$

We thus run into a pair of superpotentials W_1 and W_2 in SSUSY quite naturally. Next choosing a constraint

$$q_2^+ q_2^- - \tfrac{c}{2} = q_1^- q_1^+ + \tfrac{c}{2}, \tag{3.16}$$

we at once see that we can express

$$\begin{aligned} \mathcal{A}^+\mathcal{A}^- &= \left(q_1^+ q_1^- + \tfrac{c}{2} + \tfrac{c}{2}\right)\left(q_1^+ q_1^- + \tfrac{c}{2} - \tfrac{c}{2}\right), \\ \mathcal{A}^-\mathcal{A}^+ &= \left(q_2^- q_2^+ - \tfrac{c}{2} - \tfrac{c}{2}\right)\left(q_2^- q_2^+ - \tfrac{c}{2} + \tfrac{c}{2}\right). \end{aligned} \tag{3.17}$$

Eq. (3.17) suggests that we can interpret

$$h^{(1)} = q_1^+ q_1^- + \tfrac{c}{2}, \qquad h^{(2)} = q_2^- q_2^+ - \tfrac{c}{2}. \tag{3.18}$$

Hence (3.13) and (3.14) can be reconciled.

From (3.18), we further have

$$\begin{aligned} h^{(1)} &= (-\partial + W_1)(\partial + W_1) + \frac{c}{2} \\ &= -\partial^2 + V^{(1)}(x), \end{aligned} \tag{3.19}$$

$$\begin{aligned} h^{(2)} &= (\partial + W_2)(-\partial + W_2) - \frac{c}{2} \\ &= -\partial^2 + V^{(2)}(x), \end{aligned} \tag{3.20}$$

reflecting

$$V^{(1)}(x) = W_1^2 - \frac{dW_1}{dx} + \frac{c}{2}, \qquad V^{(2)}(x) = W_2^2 + \frac{dW_2}{dx} - \frac{c}{2}. \tag{3.21}$$

To tie up, we confront (3.10) with the expressions (3.21). The results for $a = 0$ are

$$W_1 = -\frac{2p' + c}{4p} + p, \qquad W_2 = \frac{2p' + c}{4p} + p. \tag{3.22}$$

We are thus led to explicit forms of the two superpotentials W_1 and W_2 in terms of the function $p(x)$ only.

Note that there exists, notionally, an intermediate Hamiltonian h, which is superpartner to both $h^{(1)}$ and $h^{(2)}$:

$$h^{(1)} = q_1^+ q_1^- + \tfrac{c}{2}, \qquad h = q_1^- q_1^+ + \tfrac{c}{2}, \qquad h^{(2)} = q_2^- q_2^+ - \tfrac{c}{2}. \tag{3.23}$$

Due to the constraint (3.16), we can express h as

$$h = q_1^- q_1^+ + \tfrac{c}{2} = q_2^+ q_2^- - \tfrac{c}{2}. \tag{3.24}$$

The constraint (3.16), when exposed in terms of the superpotentials W_1 and W_2, reads

$$W_2^2 - W_1^2 - \frac{dW_1}{dx} - \frac{dW_2}{dx} = c. \tag{3.25}$$

Eq. (3.25) coincides with (2.15).

3.2 PT-symmetric oscillator potential

First of all, we notice that $h^{(1)}$, h, $h^{(2)}$ defined above go over to H_1, H_2, H_3 of (2.13), respectively, provided we identify q_1^+, q_2^+ with $\bar{A}_1$, $\bar{A}_2$, and the constants c_1, c_2 with $c/2$, $-c/2$, respectively. The latter certainly hold since in (2.16) we have taken $c_1 + c_2 = 0$.

Setting now $c = -4\alpha$ and $p(x) = x - \mathrm{i}\delta$, it is trivial to see that W_1 and W_2 in (3.22) get complexified and are mapped to the expressions of $W^{(\alpha)}(x)$ and $W'^{(\alpha-1)}(x)$ given by (2.20) and (2.21), respectively.

On the other hand, if we set $c = +4\alpha$, then W_1 and W_2 in (3.22) are mapped to $W'^{(\alpha)}(x)$ and $W^{(\alpha+1)}(x)$ given by (2.21) and (2.20), respectively.

Concerning the constraint relation (3.25), we observe that it holds both for $c = -4\alpha$ and $c = +4\alpha$ if the corresponding expressions for W_1 and W_2 are plugged in.

Finally, from the point of view of SSUSY we can associate with PSUSY Hamiltonian (2.13) two distinct SUSY Hamiltonians given by

$$H_s^{(1)} = \begin{pmatrix} \bar{A}_1 A_1 + \frac{c}{2} & 0 \\ 0 & A_1 \bar{A}_1 + \frac{c}{2} \end{pmatrix}, \qquad H_s^{(2)} = \begin{pmatrix} \bar{A}_2 A_2 - \frac{c}{2} & 0 \\ 0 & A_2 \bar{A}_2 - \frac{c}{2} \end{pmatrix}, \quad (3.26)$$

where $c = \pm 4\alpha$. Conversely, we could arrive at the $p = 2$ PSUSY form for the Hamiltonian by glueing $H_s^{(1)}$ and $H_s^{(2)}$ given by (3.26).

In the following we show that the results of the PT-symmetric generalized Pöschl-Teller and Scarf II potentials are similar to those just obtained for the PT-symmetric oscillator one.

3.3 PT-symmetric generalized Pöschl-Teller potential

With the first choice of superpotentials coming from the analysis carried out for the PT-symmetric generalized Pöschl-Teller problem earlier, namely

$$\begin{aligned} W_1 &= W^{(A,B)} = \left(B - \tfrac{1}{2}\right) \coth \tau - \left(A + \tfrac{1}{2}\right) \operatorname{cosech} \tau, \\ W_2 &= W'^{(A,B-1)} = A \coth \tau - (B - 1) \operatorname{cosech} \tau, \end{aligned} \quad (3.27)$$

it is easy to see that (3.27) fits into the scheme (3.22) for the combination

$$\begin{aligned} p(x) &= \tfrac{1}{2} \left(A + B - \tfrac{1}{2}\right) (\coth \tau - \operatorname{cosech} \tau), \\ c &= \left(A + B - \tfrac{1}{2}\right) \left(A - B + \tfrac{1}{2}\right). \end{aligned} \quad (3.28)$$

If we consider instead the second choice

$$\begin{aligned} W_1 &= W'^{(A,B)} = A \coth \tau - B \operatorname{cosech} \tau, \\ W_2 &= W^{(A-1,B)} = \left(B - \tfrac{1}{2}\right) \coth \tau - \left(A - \tfrac{1}{2}\right) \operatorname{cosech} \tau, \end{aligned} \quad (3.29)$$

we need only to interchange A and $B - \frac{1}{2}$ in (3.28). Thus $p(x)$ is left unchanged while c just changes sign:

$$\begin{aligned} p(x) &= \tfrac{1}{2} \left(A + B - \tfrac{1}{2}\right) (\coth \tau - \operatorname{cosech} \tau), \\ c &= -\left(A + B - \tfrac{1}{2}\right) \left(A - B + \tfrac{1}{2}\right). \end{aligned} \quad (3.30)$$

3.4 PT-symmetric Scarf II potential

Here our first choice of superpotentials comes from (2.78) and (2.79):

$$\begin{aligned} W_1 &= W^{(A,B)} = A \tanh x + \mathrm{i} B \operatorname{sech} x, \\ W_2 &= W'^{(A-1,B)} = \left(B - \tfrac{1}{2}\right) \tanh x + \mathrm{i} \left(A - \tfrac{1}{2}\right) \operatorname{sech} x. \end{aligned} \tag{3.31}$$

On seeking consistency with (3.22), we are led to the solutions

$$\begin{aligned} p(x) &= \tfrac{1}{2} \left(A + B - \tfrac{1}{2}\right) (\tanh x + \mathrm{i} \operatorname{sech} x), \\ c &= -\left(A + B - \tfrac{1}{2}\right) \left(A - B + \tfrac{1}{2}\right). \end{aligned} \tag{3.32}$$

The second choice of superpotentials pertains to

$$\begin{aligned} W_1 &= W'^{(A,B)} = \left(B - \tfrac{1}{2}\right) \tanh x + \mathrm{i} \left(A + \tfrac{1}{2}\right) \operatorname{sech} x, \\ W_2 &= W^{(A,B-1)} = A \tanh x + \mathrm{i}(B-1) \operatorname{sech} x. \end{aligned} \tag{3.33}$$

This corresponds to an interchange of A and $B - \frac{1}{2}$ in the first choice (3.31). The function $p(x)$ remains the same but c changes sign:

$$\begin{aligned} p(x) &= \tfrac{1}{2} \left(A + B - \tfrac{1}{2}\right) (\tanh x + \mathrm{i} \operatorname{sech} x), \\ c &= \left(A + B - \tfrac{1}{2}\right) \left(A - B + \tfrac{1}{2}\right). \end{aligned} \tag{3.34}$$

4 Summary

To summarize our results, we note that in all the three potentials considered by us, namely the PT-symmetric harmonic oscillator, generalized Pöschl-Teller, and Scarf II potentials, we found order-two PSUSY and SSUSY appropriate mediums to account for their double series of energy levels. Taking cue from the SUSY results, we found possible to confront the expressions for the relevant superpotential by an appropriate series of energy levels. These superpotentials, in turn, not only allow developing PSUSY models, but also adjust nicely with the constraint relations relevant to the SSUSY construction. In this way, the potentials considered by us can be interpreted in terms of PSUSY and SSUSY schemes.

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