A unified treatment of exactly solvable and quasi-exactly solvable quantum potentials

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Abstract

By exploiting the hidden algebraic structure of the Schrödinger Hamiltonian, namely the sl(2), we propose a unified approach of generating both exactly solvable and quasi-exactly solvable potentials. We obtain, in this way, two new classes of quasi-exactly solvable systems one of which is of periodic type while the other hyperbolic.

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Tracking down solvable quantum potentials has always aroused interest. Apart from being useful in the understanding of many physical phenomena, the importance of searching for them also stems from the fact that they very often provide a good starting point for undertaking perturbative calculations of more complex systems.

Solvable potentials can be broadly classified into two categories : the ones which are exactly solvable[1, 2, 3, 4](including the conditional ones[5, 6]) and others which are quasi-exactly solvable[7, 8, 9, 10]. A spectral problem is said to be exactly solvable(ES) if one can determine the whole spectrum analytically by a finite number of algebraic steps. Factorization hypothesis[11, 12], group-theoretical techniques with a spectrum-generating algebra[13, 14, 15] and use of integral transformations[16, 17] are some of the time-honoured procedures of constructing ES potentials[18]. On the other hand, there exist an infinite number of normal spectral problems which are not amenable to an exact treatment. These are the non-solvable (NS) ones. The quasi-exactly solvable(QES) class is the missing link[19, 20] between the ES and the NS potentials. Actually for a QES system we can only determine a part of the whole spectrum : this essentially means that in an infinite-dimensional space of states there exists a finite-dimensional subspace for which the Schrödinger equation admits partial algebraization.

However, in the literature, a common framework that brings together the ES and QES class is still lacking. The purpose of this letter is to fill this gap by exploiting the hidden dynamical symmetry of the Schrödinger equation. We show, in a straightforward way, that by subjecting the Schrödinger equation to some coordinate transformation and adopting for the underlying symmetry group the simplest choice namely the sl(2), it is possible to set up a master equation from which the ES and QES potentials readily follow. Following this line,

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we construct not only some of the well-known ES potentials which are with us for a long time but also uncover new families of QES potentials which hitherto have remained unnoticed.

Let us start with the following differential realization of the sl(2) generators $T^{\pm}, T^0$ given by

$$T^+ = \xi^2 \frac{d}{d\xi} - n\xi \,, \quad T^- = \frac{d}{d\xi} \,, \quad T^0 = \xi \frac{d}{d\xi} - \frac{1}{2} n \,, \tag{1}$$

where $\xi \in \mathbb{R}$ and n is some non-negative integer. These generators act on the represention space $\mathcal{P}_n$ of polynomials in ξ of degree not exceeding n and obey the commutation relations

$$[T^+, T^-] = -2T^0 \,, \qquad [T^0, T^{\pm}] = \pm T^{\pm} \,. \tag{2}$$

Let us assume that the quantum Hamiltonian is expressible as a quadratic combinations of the generators T^a with constant coefficients, that is

$$\mathcal{H} = - \sum_{a,b=0,\pm} C_{ab} T^a T^b - \sum_{a=0,\pm} C_a T^a - d(n) \,, \tag{3}$$

where C_{ab}, C_a are numerical parameters of which C_{ab} is symmetric and d is a suitably chosen constant that depends on n. Note that this functional dependence is single-valued for ES models, while it is multivalued for QES. In the latter case the range is $\{d_0(n), d_1(n), \dots d_n(n)\}$.

Now from (1) it is easy to see that $\mathcal{H}$ has the representation

$$\mathcal{H}(\xi) = - \sum_{j=2}^{4} B_j(\xi) \frac{d^{j-2}}{d\xi^{j-2}} \,. \tag{4}$$

The coefficients B_j's in (4) are the j-th degree polynomial in ξ :

$$\begin{aligned} B_4(\xi) &= C_{++}\xi^4 + 2C_{+0}\xi^3 + C_{00}\xi^2 + 2C_{0-}\xi + C_{--} \,, \\ B_3(\xi) &= \frac{1-n}{2} \frac{dB_4}{d\xi} + A_2(\xi) \,, \\ B_2(\xi) &= \frac{n(n-1)}{12} \frac{d^2 B_4}{d\xi^2} - \frac{n}{2} \frac{dA_2}{d\xi} + \frac{n(n+2)}{12} C_{00} + d(n) \,, \end{aligned} \tag{5}$$

where $C_{+-} = C_{-+} = 0$ because of the constancy of the Casimir operator and $A_2(\xi) = C_+\xi^2 + C_0\xi + C_-$.

However the coeffecient of $d^2/d\xi^2$ in (4) is not unity. To achieve this we introduce the mapping $u(\xi) = \int^{\xi} [B_4(\tau)]^{-1/2} d\tau$ thereby obtaining

$$\mathcal{H}(u) = -\frac{d^2}{du^2} + \frac{1}{2\sqrt{B_4}} \left(\frac{dB_4}{d\xi} - 2B_3 \right) \Bigg|_{\xi=\xi(u)} \frac{d}{du} - B_2 \Bigg|_{\xi=\xi(u)} \,. \tag{6}$$

To proceed further, we may look upon $\mathcal{H}(u)$ as a 'coordinate-transformed' Schrödinger Hamiltonian. Indeed let us consider a change of variable $x \to x(u)$ that transforms the Schrödinger wave function according to

$$\psi(x) \to g(x)\chi(u(x)) \,. \tag{7}$$

The standard Schrödinger equation (with $\hbar = 2m = 1$) with the potential $V(x)$

$$\left[-\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x) \tag{8}$$

is then recast to the form

$$-\frac{d^2\chi}{du^2} - \left[\frac{u''}{u'^2} + \frac{2g'}{u'g}\right]\frac{d\chi}{du} - \left[\frac{g''}{gu'^2} + \frac{E - V(x)}{u'^2}\right]\chi(u) = 0\,, \tag{9}$$

where the primes denote derivatives with respect to x.

The general nature of (9) enables one to touch upon those differential equations whose solutions are well-known. In particular those which are associated with special functions can be considered for the determination of solvable potentials. Here we take a different route by seeking a direct correspondance of (9) with the 'coordinate-transformed' Schrödinger Hamiltonian $\mathcal{H}$. We thus obtain

$$V(x) = \Bigg[E - \frac{u'''}{2u'} + \frac{3}{4}\left(\frac{u''}{u'}\right)^2 \\ -u'^2\left\{B_2 - \frac{1}{4}\left(2\frac{dB_3}{d\xi} - \frac{d^2B_4}{d\xi^2}\right) - \frac{1}{16B_4}(2B_3 - \frac{dB_4}{d\xi})(2B_3 - 3\frac{dB_4}{d\xi})\right\}_{\xi=\xi(u)}\Bigg]_{u=u(x)} \tag{10}$$

In this connection we may emphasize that whereas for ES models the energy levels form an infinite sequence, in the QES case there can be at most $(n+1)$ levels for each choice of n. It is thus appropriate to label the energy levels by an index j depending on n i. e. $j = j(n)$ that runs with n and takes values $0, 1, \ldots, n, \ldots \infty$ for the ES models but assumes only a finite number of values $\{0, 1, \ldots, n\}$ for QES.

In (10) the $B_j\,(j = 2, 3, 4)$ functions of sl(2) have to be suitably adjusted against the arbitrary function $u(x)$ of the coordinate transformation to have an acceptable form of the quantum potentials. For this the normalizability of the wave function is to be ensured. Note that in arriving at the form (10) we have eliminated the function $g(x)$. However, for a particular choice of $u(x)$, it can be determined from

$$g(x) = (u')^{-1/2}\exp\left[\frac{1}{2}\int^{u(x)}\left\{\frac{2B_3 - dB_4/d\xi}{2\sqrt{B_4}}\right\}_{\xi=\xi(u)} du\right]. \tag{11}$$

Knowing $u(x)$ and $g(x)$ the wave function can be found from (7).

Equation (10) is the central result of our paper : it opens up a new approach of generating ES and QES potentials.

To see how our scheme works in practice, let us first address to the problem of deriving some well-known ES potentials from (10). Without giving the details of our calculations which are straightforword we present the results in standard forms[21] :

- **Harmonic oscillator:** $u = x$, $B_4 = 1$, $B_3 = -\omega\xi$, $B_2 = n\omega$ [i. e. we choose $C_{--} = 1$, $d(n) = n\omega/2$, $C_0 = -\omega\ (\omega > 0)$, $C_{++} = C_{+0} = C_{00} = C_{0-} = C_+ = C_- = 0$.]

$$V(x) = \frac{1}{4}\omega^2x^2, \qquad E_j = (j + \frac{1}{2})\omega\,,$$

$$\psi_j(x) = \mathcal{N}_j \exp(-\frac{1}{4}\omega x^2) H_j(\sqrt{\frac{\omega}{2}} x), \quad j = 0, 1, \ldots, n, \ldots, \infty\,,$$

$\mathcal{N}_j$ being the normalization constant.

- **Morse:** $u = x$, $B_4 = \alpha^2\xi^2$, $B_3 = \alpha(\alpha - 2A)\xi + 2B\alpha$, $B_2 = -n^2\alpha^2 + 2An\alpha$ [i. e. $C_{00} = \alpha^2$, $C_{++} = C_{+0} = C_{0-} = C_{--} = C_+ = 0$, $C_0 = \alpha(n\alpha - 2A)$, $d(n) = An\alpha - 3n^2\alpha^2/4$, $C_- = 2B\alpha$.]

$$V(x) = B^2 \exp[-2\alpha x] - B(2A + \alpha) \exp[-\alpha x], \quad E_j = -(A - j\alpha)^2,$$

$$\psi_j(x) = \mathcal{N}_j \exp[(j\alpha - A)x] \exp[-\frac{B}{\alpha} e^{-\alpha x}] L_j^{(2\frac{A}{\alpha} - 2j)} \left(\frac{2B}{\alpha} e^{-\alpha x}\right), \quad j = 0, 1, \ldots, n, \ldots \infty\,,$$

$\mathcal{N}_j$ being the normalization constant.

- **Pöschl-Teller:** $u = x$, $B_4 = 4\alpha^2(\xi^2 - 1)$, $B_3 = 4\alpha\{(A + B + 2\alpha)\xi + B - A - \alpha\}$, $B_2 = \alpha^2(1 - 4n^2) + 2\alpha\{2n(A - B) + A + B\} + 4AB$ [i. e. $C_{00} = 4\alpha^2 = -C_{--}$, $C_{++} = C_{+0} = C_{0-} = C_+ = 0$, $d(n) = 4AB + \alpha^2(1 + 3n)(1 - n) + 2n\alpha(3A - B) + 2\alpha(A + B)$, $C_- = 4\alpha(B - A - \alpha)$, $C_0 = 4\alpha\{A + B + \alpha(n + 1)\}$.]

$$V(x) = B(B - \alpha)\text{cosech}^2\alpha x - A(A + \alpha)\text{sech}^2\alpha x, \quad E_j = -(A - B - 2j\alpha)^2,$$

$$\psi_j(x) = \mathcal{N}_j \sinh^{B/\alpha} \alpha x \cosh^{-A/\alpha} \alpha x\, P_j^{(\frac{B}{\alpha} - \frac{1}{2}, -\frac{A}{\alpha} - \frac{1}{2})}(\cosh 2\alpha x),\ j = 0, 1, \ldots, n, \ldots \infty\,,$$

$\mathcal{N}_j$ being the normalization constant.

- **Scarf II:** $u = x$, $B_4 = \alpha^2(\xi^2 + 1)$, $B_3 = \alpha(2A + 3\alpha)\xi + 2B\alpha$, $B_2 = \alpha(n+1)\{\alpha(1 - n) + 2A\}$[i. e. we choose $C_{00} = C_{--} = \alpha^2$, $C_0 = \alpha^2(n+2) + 2A\alpha$, $C_- = 2B\alpha$, $C_{++} = C_{+0} = C_{0-} = C_+ = 0$, $d(n) = \alpha^2 + A\alpha(3n + 2) + n\alpha^2(4 - 3n)/4$.]

$$V(x) = [B^2 - A(A + \alpha)]\text{sech}^2\alpha x + B(2A + \alpha)\text{sech}\alpha x \tanh \alpha x,$$

$$E_j = -(A - j\alpha)^2, \quad j = 0, 1, \ldots, n, \ldots \infty\,,$$

$$\psi_j(x) = \mathcal{N}_j \cosh^{-A/\alpha} \alpha x \exp[-\frac{B}{\alpha} \tan^{-1}(\sinh \alpha x)] P_j^{(-i\frac{B}{\alpha} - \frac{A}{\alpha} - \frac{1}{2}, i\frac{B}{\alpha} - \frac{A}{\alpha} - \frac{1}{2})}(i \sinh \alpha x)\,,$$

$\mathcal{N}_j$ being the normalization constant.

- **Coulomb:** $u = 2\sqrt{x}\,(x > 0)$, $B_4 = 4\xi$, $B_3 = e^2\xi/(n + l + 1) + 8(l + 1)$, $B_2 = e^2(n + 2l + 2)/(n + l + 1)$ [i. e. $C_{0-} = 2$, $C_0 = e^2/(n + l + 1)\,(l \geq 0)$, $C_- = 2(4l + n + 3)$, $C_{++} = C_{+0} = C_{00} = C_{--} = C_+ = 0$, $d(n) = e^2(3n + 4l + 4)/2(n + l + 1)$.]

$$V(x) = -\frac{e^2}{x} + \frac{l(l + 1)}{x^2}\,(0 < x < \infty), \quad E_j = -\frac{e^4}{4(j + l + 1)^2},$$

$$\psi_j(x) = \mathcal{N}_j\, x^{l+1} \exp\left[-\frac{e^2 x}{2(n + l + 1)}\right] L_n^{(2l+1)} \left(\frac{e^2 x}{n + l + 1}\right),\ j = 0, 1, \ldots, n, \ldots \infty\,,$$

$\mathcal{N}_j$ being the normalization constant.

Having dealt successfully with the generation of ES potentials all of which have been well studied in the literature, let us turn to the problem of finding QES potentials from (10). In the following we present two new families of QES potentials, one of which is periodic while the other is hyperbolic.

A. Periodic model

Consider the transformation $u = x - a$ ($a \in \mathbb{R}$) along with the representation $B_4 = \beta^2(1-\xi^2)$ [this comes about if we set $C_{++} = C_{+0} = C_{0-} = 0$, $C_{00} = -\beta^2 = -C_{--}, \beta \neq 0$]. This particular form for B_4 immediately yields $\xi = \cos\beta u$ and facilitates generating a periodic QES system. Indeed, trialing with various choices of B_3 and B_2 we have found that in all four algebraizations exist each leading to a distinct QES family. Our results are

1. $B_3 = -\alpha\xi^2 - 2\beta^2\xi + \alpha \pm \beta^2$, $B_2 = n\alpha\xi + \frac{n(n+2)}{4}\beta^2 + d_j(n)$

   [i. e. $C_+ = -\alpha = \pm\beta^2 - C_-$, $C_0 = -(n+1)\beta^2, \alpha \neq 0$]

$$V_1(x) = -\frac{\alpha^2}{8\beta^2}\cos 2\beta(x-a) - \alpha(n+1)\cos\beta(x-a) - \frac{\beta^2}{4}$$

$$E_j = \frac{n(n+2)}{4}\beta^2 - \frac{\alpha^2}{8\beta^2} \mp \frac{\alpha}{2} + d_j(n), \quad j = 0, 1, \ldots, n\,,$$

$$\psi_j(x) = \sin\left\{(\delta_{k+})\frac{\pi}{2} + \beta\frac{x-a}{2}\right\}\exp[-\frac{\alpha}{\beta^2}\sin^2(\beta\frac{x-a}{2})]\sum_{r=0}^{n} b_j^{(r)}\cos^r\beta(x-a), (k=+,-).$$

2. $B_3 = \alpha\xi^2 - 2\beta^2\xi - \alpha \pm \beta^2$, $B_2 = -n\alpha\xi + \frac{n(n+2)}{4}\beta^2 + d_j(n)$

   [i. e. $C_+ = \alpha = \pm\beta^2 - C_-$, $C_0 = -(n+1)\beta^2, \alpha \neq 0$]

$$V_2(x) = -\frac{\alpha^2}{8\beta^2}\cos 2\beta(x-a) + \alpha(n+1)\cos\beta(x-a) - \frac{\beta^2}{4}$$

$$E_j = \frac{n(n+2)}{4}\beta^2 - \frac{\alpha^2}{8\beta^2} \pm \frac{\alpha}{2} + d_j(n), \quad j = 0, 1, \ldots, n\,,$$

$$\psi_j(x) = \sin\left\{(\delta_{k+})\frac{\pi}{2} + \beta\frac{x-a}{2}\right\}\exp[\frac{\alpha}{\beta^2}\sin^2(\beta\frac{x-a}{2})]\sum_{r=0}^{n} b_j^{(r)}\cos^r\beta(x-a),\ (k=+,-)$$

3. $B_3 = \pm\alpha\xi^2 - 3\beta^2\xi \mp \alpha$, $B_2 = \frac{n(n+4)}{4}\beta^2 + d_j(n) \mp n\alpha\xi$

   [i. e. $C_+ = \pm\alpha = -C_-$, $C_0 = -(n+2)\beta^2, \alpha \neq 0$]

$$V_3(x) = -\frac{\alpha^2}{8\beta^2}\cos 2\beta(x-a) \pm \alpha(n+\frac{3}{2})\cos\beta(x-a) - \frac{\beta^2}{4}$$

$$E_j = \frac{n(n+4)+3}{4}\beta^2 - \frac{\alpha^2}{8\beta^2} + d_j(n), \quad j = 0, 1, \ldots, n\,,$$

$$\psi_j(x) = \sin\beta(x-a)\exp[\pm\frac{\alpha}{\beta^2}\sin^2\beta\frac{x-a}{2})]\sum_{r=0}^{n} b_j^{(r)}\cos^r\beta(x-a).$$

4. $B_3 = \pm\alpha\xi^2 - \beta^2\xi \mp \alpha$, $B_2 = \mp n\alpha\xi + \frac{n^2}{4}\beta^2 + d_j(n)$

   [i. e. $C_+ = \pm\alpha = -C_-$, $C_0 = -n\beta^2, \alpha \neq 0$]

$$V_4(x) = -\frac{\alpha^2}{8\beta^2}\cos 2\beta(x-a) \pm \alpha(n+\frac{1}{2})\cos\beta(x-a) - \frac{\beta^2}{4}$$

$$E_j = \frac{n^2-1}{4}\beta^2 - \frac{\alpha^2}{8\beta^2} + d_j(n), \quad j = 0, 1, \ldots, n\,,$$

$$\psi_j(x) = \exp[\pm\frac{\alpha}{\beta^2}\sin^2\beta\frac{x-a}{2}]\sum_{r=0}^{n} b_j^{(r)}\cos^r\beta(x-a).$$

The above potentials are new and appear in the true spirit of QES. It should be noted that the particular class corresponding to $n = 0$ for $V_1(x)$ has been studied for understanding nonaveraged properties of disordered systems[22]. The coefficients $b_j^{(r)}, r = 1, 2, \ldots n$ and $d_j(n)$ appearing in the Bloch wave functions $\psi_j(x)$ and band-edge energies are to be calculated from (9) for a given n. It is found that number of levels in the algebraic sector is $2|m|$, where m is the coefficient of $\alpha\cos\beta(x-a)$ in the potentials.

B. Generalized double-well potential

With $u = x - a (a \in \mathbb{R})$, we next adopt for B_4 the choice $B_4 = 4\gamma^2(\xi^2 - 1)$[i. e. if we set $C_{++} = C_{+0} = C_{0-} = 0$, $C_{00} = 4\gamma^2 = -C_{--}, \gamma \neq 0$]. ξ turns out to be in the hyperbolic form $\xi = \cosh 2\gamma u$. As before we carry out trials with B_3 and B_2 to arrive at the following four types of algebraizations:

1. $B_3 = 2\gamma^2\eta\xi^2 + 8\gamma^2\xi + 2\gamma^2(\pm 2 - \eta)$, $B_2 = -n\gamma^2(2\eta\xi + n + 2) + d_j(n)$

   [i. e. $C_+ = 2\gamma^2\eta = \pm 4\gamma^2 - C_-$, $C_0 = 4\gamma^2(n+1), \eta \neq 0$]

$$V_1(x) = \frac{\gamma^2\eta^2}{8}\cosh 4\gamma(x-a) + 2\eta\gamma^2(n+1)\cosh 2\gamma(x-a) - \frac{\gamma^2\eta^2}{8}$$

$$E_j = -[(n+1)^2 \pm \eta]\gamma^2 + d_j(n), \quad j = 0, 1, \ldots, n\,,$$

$$\begin{aligned}\psi_j(x) \quad = \quad & [(\delta_{k+})\sinh\gamma(x-a) + (\delta_{k-})\cosh\gamma(x-a)]\,\exp[\frac{\eta}{4}\cosh 2\gamma(x-a)] \\ & \times \sum_{r=0}^{n} b_j^{(r)}\cosh^r 2\gamma(x-a), \quad (k = +, -).\end{aligned}$$

2. $B_3 = 2\gamma^2(-\eta\xi^2 + 4\xi + \eta \pm 2)$, $B_2 = n\gamma^2(2\eta\xi - n - 2) + d_j(n)$

   [i. e. $C_+ = -2\gamma^2\eta = \pm 4\gamma^2 - C_-$, $C_0 = 4\gamma^2(n+1), \eta \neq 0$]

$$V_2(x) = \frac{\gamma^2\eta^2}{8}\cosh 4\gamma(x-a) - 2\eta\gamma^2(n+1)\cosh 2\gamma(x-a) - \frac{\gamma^2\eta^2}{8}$$

$$E_j = -[(n+1)^2 \mp \eta]\gamma^2 + d_j(n), \quad j = 0, 1, \ldots, n\,,$$

$$\psi_j(x) = [(\delta_{k+})\sinh\gamma(x-a) + (\delta_{k-})\cosh\gamma(x-a)]\exp[-\frac{\eta}{4}\cosh 2\gamma(x-a)]$$

$$\times \sum_{r=0}^{n} b_j^{(r)} \cosh^r 2\gamma(x-a), \quad (k = +, -)\,.$$

3. $B_3 = 2\gamma^2(\mp\eta\xi^2 + 2\xi \pm \eta],\, , B_2 = n\gamma^2(\pm 2\eta\xi - n) + d_j(n)$

[i. e. $C_+ = \mp 2\gamma^2\eta = -C_-$, $C_0 = 4n\gamma^2, \eta \neq 0$]

$$V_3(x) = \frac{\gamma^2\eta^2}{8}\cosh 4\gamma(x-a) \mp 2\eta\gamma^2(n+\frac{1}{2})\cosh 2\gamma(x-a) - \frac{\gamma^2\eta^2}{8}$$

$$E_j = -n^2\gamma^2 + d_j(n), \quad j = 0, 1, \ldots, n\,,$$

$$\psi_j(x) = \exp[\mp\frac{\eta}{4}\cosh 2\gamma(x-a)] \sum_{r=0}^{n} b_j^{(r)} \cosh^r 2\gamma(x-a).$$

4. $B_3 = 2\gamma^2(\mp\eta\xi^2 + 6\xi \pm \eta)$, $B_2 = n\gamma^2(\pm 2\eta\xi - n - 4) + d_j(n)$

[i. e. $C_+ = \mp 2\gamma^2\eta = -C_-$, $C_0 = 4\gamma^2(n+2), \eta \neq 0$]

$$V_4(x) = \frac{\gamma^2\eta^2}{8}\cosh 4\gamma(x-a) \mp 2\eta\gamma^2(n+\frac{3}{2})\cosh 2\gamma(x-a) - \frac{\gamma^2\eta^2}{8}$$

$$E_j = -(n+2)^2\gamma^2 + d_j(n), \quad j = 0, 1, \ldots, n\,,$$

$$\psi_j(x) = \sinh 2\gamma(x-a)\exp[\mp\frac{\eta}{4}\cosh 2\gamma(x-a)] \sum_{r=0}^{n} b_j^{(r)} \cosh^r 2\gamma(x-a).$$

The potentials V_1, V_2, V_3, V_4 may be looked upon as hyperbolic counterparts to those of the periodic model. These generalize the bistable potential studied in the context of homonuclear diatomic molecule[23]. Our potentials are also of interest in spin-boson and spin-spin interacting models where similar hyperbolic forms are known to exist[24]. As before the coefficients $b_j^{(r)}$ in the wave functions and $d_j(n)$ in the energies are determined for a given n from (9). The number of analytically known levels is $2|t|$, t being the coefficient of $2\eta\gamma^2\cosh 2\gamma(x-a)$.

To summarize, we have presented a unified approach of generating ES and QES potentials by exploiting the hidden sl(2) symmetry of the Schrödinger equation and setting up a master equation. Our scheme is especially suitable for generating new types of solvable potentials as we have demonstrated for the QES cases.

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