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A new PT symmetric complex Hamiltonian with a real spectra

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Abstract

We construct an isospectral system in terms of a real and a complex potential to show that the underlying PT symmetric complex Hamiltonian possesses a real spectra which is shared by its real partner.

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Complex potentials have found a wide usage [1] in the literature especially in connection with scattering problems. Recently it has been emphasized [2] that by enforcing a PT-symmetry, one can obtain new classes of complex Hamiltonian which exhibit a real spectra of energy eigenvalues. The purpose of this letter is to bring to light a new complex Hamiltonian which is PT symmetric and possess a real energy spectra.

Consider potentials of the form $V^{(1),(2)} = U^2 \pm U'$ where U is complex function of x and a dash denotes a derivative with respect to x. Let us express U explicitly as $a(x) + ib(x)$, where $a(x)$ and $b(x)$ are certain real, continuously differentiable functions in R. We have ,

$$V^{(1),(2)} = (a^2 - b^2 \pm a') + i(2ab \pm b') \quad (1)$$

In the following we investigate the possibility when one of the potentials defined by (1) is real but the other is complex. To this end we choose, for the sake of concreteness, $V^{(2)}$ to be real thus resticting the function a to be given by

$$a = \frac{1}{2}\frac{b'}{b} \quad (2)$$

The consrtaint (2) gives for $V^{(1)}$ and $V^{(2)}$ the forms

$$V^{(1)} = (a^2 - b^2 + a') + 2ib' \quad (3a)$$

$$V^{(2)} = (a^2 - b^2 - a') \quad (3b)$$

Notice that $V^{(1)} = V^{(2)} + 2U'$ which seems to hint at a supersymmetric connection between $V^{(1)}$ and $V^{(2)}$; however, it should be borne in mind that the Hamiltonians for $V^{(1)}$ and $V^{(2)}$ cannot be made simultaneously hermitian.

Our task now is to demonstrate by appropriately choosing the functions a and b that the Hamiltonians $H^{(1),(2)} = -\frac{d^2}{dx^2} + V^{(1),(2)}$ yield a common real spectra thus forming an isospectral system. We propose the following example

$$a = -\frac{\mu}{2}\, tanh\ \mu x, \quad b = \lambda\ sech\ \mu x \quad (4)$$

where μ and λ are non-zero real parameters with $\mu \neq \lambda$. Note that the above representations of the functions a and b are consistent with the requirement (2).

Substitution of (4) in (3) gives for $V^{(1)}$ and $V^{(2)}$ the expressions

$$V^{(1)} = \frac{\mu^2}{4} - \mu^2 \left[\bar{\lambda}(\bar{\lambda} - 1) + 1\right]\ sech^2 \mu x - 2i\lambda\mu\ sech\ \mu x\ tanh\ \mu x \quad (5)$$

$$V^{(2)} \;=\; \frac{\mu^2}{4} - \mu^2 \;\bar{\lambda}\; (\bar{\lambda}-1)\; sech^2\mu x \tag{5b}$$

where $\bar{\lambda}(\bar{\lambda}-1) \;=\; \frac{\lambda^2}{\mu^2} - \frac{1}{4}$.

It is well known [3] that the non-zero energy levels for $V^{(2)}$ are given by

$$E_n^{(2)} \;=\; \frac{\mu^2}{4} - (\bar{\lambda}-1-n)^2\mu^2,\; n < \bar{\lambda}-1 \tag{6}$$

with $n = 0, 1, \cdots$. The associated eigenfunctions for the even and odd states are

$$\psi^{(2)}_{even}(x) \;=\; cosh^{\bar{\lambda}}\mu x \;{}_2F_1\left[\frac{1}{2}(\bar{\lambda}-1), \frac{1}{2}(\bar{\lambda}+1), \frac{1}{2} - sinh^2\mu x\right] \tag{7}$$

$$\psi^{(2)}_{odd}(x) \;=\; cosh^{\bar{\lambda}}\mu x \; sinh\;\mu x \;{}_2F_1\left(\frac{\bar{\lambda}}{2},\; \frac{\bar{\lambda}}{2}+1, \frac{3}{2}, -sinh^2\mu x\right) \tag{8}$$

The main point of this note is to expose the fact that the complex Hamiltonian $H^{(1)}$,in addition to the zero-energy level, mimicks the discrete real spectra (7) for a class of complex eigenfunctions. We illustrate this by considering the case $\frac{\lambda}{\mu} = -\frac{5}{2}$ for which $\bar{\lambda} = 3$. The relevant values of n then are 0 and 1, none of which, however, correspond to the zero-energy state for either $V^{(1)}$ or $V^{(2)}$. In Table 1 we furnish our results which have been obtained exploiting the interwining relations $\psi^{(1)}_{(n=0,1)} = \left(\frac{d}{dx} + U\right)\psi^{(2)}_{(n=0,1)}$. It is obvious from there that not only the energy eigenvalues for $V^{(1)}$ are real and match these of $V^{(2)}$ for $n = 0$ and 1 respectivly, but also that the corresponding eigenfunctions for both $V^{(1)}$ and $V^{(2)}$ have controllable asymptotic behaviour.

Further it is clear that the complex potential $V^{(1)}$ has a normalizable zero-energy state with the wave-function

$$\psi_0^{(1)} \alpha \sqrt{sech\;\mu x}\; e^{2i\frac{\lambda}{\mu}\; tan^{-1}(e^{\mu x})} \tag{9}$$

This is indeed a nice result considering the fact that the imaginary part of $V^{(1)}$ only contributes a phase factor in $\psi_0^{(1)}$.

To conclude, we have found a new PT symmetric complex potential whose energy levels are negative semi-definite and, excluding the zero-energy state, coinside with those of a known $sech^2$-potential. Results for different values of $\bar{\lambda}$ and other choices of a and b will be communicated in a future detailed publication.

References

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TABLE 1 : Eigenfunctions and eigenvalues of the potentials $V^{(1),(2)}$ for the case $\bar{\lambda} = 3$. See text for details. N_0 and $N_1^{(i)}$ and normalization constants.

Eigenfunctions	For the potential $V^{(1)}$ $(i = 1)$	For the potential $V^{(2)}$ $(i = 2)$	Energy eigenvalues
$\psi^{(i)}_{(n=0)}$	$N_0\ sech^2\mu x\ (tanh\ \mu x + i\ sech\ \mu x)$	$N_0\ sech^2\mu x$	$\frac{\mu^2}{4} - 4\mu^2$
$\psi^{(i)}_{(n=1)}$	$N_1^{(1)}\ sech\mu x\ (1 - \frac{5}{3}\ sech^2\mu x + i\ \frac{5}{3}\ sech^2\mu x\ sinh\mu x)$	$N_1^{(2)}\ sech^2\mu x\ sinh\mu x$	$\frac{\mu^2}{4} - \mu^2$

Footnote :

1 Under parity (P) $p \to -p$ and $x \to -x$ whereas under time reversal (T) $p \to -p$, $x \to -x$ and $i \to -i$.