

# A generalized non-Hermitian oscillator Hamiltonian, $\mathcal{N}$ -fold supersymmetry and position-dependent mass models

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## Abstract

A generalized non-Hermitian oscillator Hamiltonian is proposed that consists of additional linear terms which break  $\mathcal{PT}$ -symmetry explicitly. The model is put into an equivalent Hermitian form by means of a similarity transformation and the criterion of  $\mathcal{N}$ -fold supersymmetry with a position-dependent mass is shown to reside in it.

*Key words:*  $\mathcal{N}$ -fold supersymmetry, non-Hermitian Hamiltonian, position-dependent mass,  $\mathcal{PT}$ -symmetry

*PACS:* 03.65.Ca, 03.65.Fd, 11.30.Na, 11.30.Pb

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## 1 Introduction

The study of non-Hermitian quantum systems has been a subject of considerable interest in recent times. In particular, Swanson's scheme [1] of a generalized oscillator with the underlying Hamiltonian expressed in terms of the usual harmonic oscillator creation and annihilation operators  $\eta^\dagger$  and  $\eta$ , namely  $H = \omega(\eta^\dagger\eta + \frac{1}{2}) + \alpha\eta^2 + \beta(\eta^\dagger)^2$  with  $\omega, \alpha, \beta$  three real parameters such

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that  $\alpha \neq \beta$  and  $\Omega^2 = \omega^2 - 4\alpha\beta > 0$  has found much attention [2,3,4,5] in the literature in view of its implicit non-Hermiticity. In fact  $H$  is Hermitian only for the restricted case of  $\alpha = \beta$  but non-Hermitian otherwise. Actually  $H$  is  $\mathcal{PT}$ -symmetric as is readily demonstrated by applying the transformation properties of  $\mathcal{P} : \eta \rightarrow -\eta$  and  $\mathcal{T} : \eta \rightarrow \eta$  for all values of  $\alpha$  and  $\beta$ . Swanson's Hamiltonian is known to possess a real, positive and discrete spectrum in line with the conjecture of Bender and Boettcher [6].

Sometime ago, Jones [2] constructed a similarity transformation to point out that the above Hamiltonian admits of an equivalent Hermitian representation. Subsequently Musumbu *et al.* [3] showed by means of Bogoliubov transformations that a family of positive-definite metric operators exists for each of which the corresponding Hermitian counterpart could be worked out.

On the other hand, Bagchi *et al.* exploited [4] a hidden symmetry structure to expose the pseudo-Hermitian character [7,8,9,10,11] of  $H$  that allows access to a generalized quantum condition. This was followed up by Quesne [5,12] to take up an  $su(1,1)$  embedding of  $H$  and to obtain a family of positive definite metrics that facilitate seeking quasi-Hermitian supersymmetric (SUSY) theories (for references on SUSY quantum mechanics, see e.g. [13,14,15]).

A generalized quantum condition enables one [4] to connect those physical systems which are describable by a position dependence in mass by suitably representing the operator  $\eta$ . In this note, we show that such position dependent mass (PDM) models (for the recent development in PDM models, see e.g. [16,17,18] and the references cited therein) are necessarily endowed with a type A  $\mathcal{N}$ -fold SUSY [19,20,21] structure. As is well known, the latter is characterized by the anticommutators of fermionic operators which are polynomials of degree (at most)  $\mathcal{N}$  in bosonic operators. The framework of  $\mathcal{N}$ -fold SUSY (for the general aspects, see [22,23,24] and the references cited therein) has proved to be powerful to deal with one body quantum mechanical system admitting analytical solutions.

## 2 $\mathcal{N}$ -fold SUSY in Generalized Swanson's Models

To keep our discussion as general as possible, we address an extended Swanson's model defined by

$$H = \omega \left( \eta^\dagger \eta + \frac{1}{2} \right) + \alpha \eta^2 + \beta (\eta^\dagger)^2 + \gamma \eta + \delta \eta^\dagger, \quad (1)$$

where  $\omega, \alpha, \beta, \gamma, \delta$  are real parameters. It differs from the original Swanson's model in the presence of the linear terms which break  $\mathcal{PT}$ -symmetry explicitly.

Let us adopt for  $\eta$  the most general first-order differential operator, namely

$$\eta = a(x) \frac{d}{dx} + b(x), \quad (2)$$

where  $a(x)$  and  $b(x)$  are arbitrary functions. As a result  $H$  becomes

$$H = -\tilde{\omega} \frac{d}{dx} a(x)^2 \frac{d}{dx} + b_1(x) \frac{d}{dx} + c_2(x), \quad (3)$$

where  $\tilde{\omega} = \omega - \alpha - \beta$ , and the functions  $b_1(x)$  and  $c_2(x)$  stand for

$$b_1(x) = (\alpha - \beta)a(x) (2b(x) - a'(x)) + (\gamma - \delta)a(x), \quad (4)$$

$$c_2(x) = (\tilde{\omega} + 2\alpha + 2\beta)b(x)^2 - (\tilde{\omega} + \alpha + 3\beta)a'(x)b(x) - (\tilde{\omega} + 2\beta)a(x)b'(x) + \beta (a(x)a''(x) + a'(x)^2) + (\gamma + \delta)b(x) - \delta a'(x) + \frac{\tilde{\omega} + \alpha + \beta}{2}. \quad (5)$$

Following a standard procedure, the non-Hermitian operator  $H$  can be transformed into an equivalent Hermitian form by means of the similarity transformation

$$h = \rho H \rho^{-1} = -\tilde{\omega} \frac{d}{dx} a(x)^2 \frac{d}{dx} + V_{\text{eff}}(x), \quad (6)$$

with the mapping function  $\rho$  given by

$$\rho = \exp \left( -\frac{1}{2\tilde{\omega}} \int dx \frac{b_1(x)}{a(x)^2} \right). \quad (7)$$

In Eq. (6), the effective potential in terms of  $a(x)$  and  $b(x)$  reads

$$V_{\text{eff}}(x) = \left( \frac{(\alpha - \beta)^2}{\tilde{\omega}} + \tilde{\omega} + 2\alpha + 2\beta \right) b(x) (b(x) - a'(x)) - (\tilde{\omega} + \alpha + \beta)a(x)b'(x) + \frac{\alpha + \beta}{2}a(x)a''(x) + \frac{1}{4} \left( \frac{(\alpha - \beta)^2}{\tilde{\omega}} + 2\alpha + 2\beta \right) a'(x)^2 + \left( \frac{(\alpha - \beta)(\gamma - \delta)}{\tilde{\omega}} + \gamma + \delta \right) \left( b(x) - \frac{a'(x)}{2} \right) + \frac{(\gamma - \delta)^2}{4\tilde{\omega}} + \frac{\tilde{\omega} + \alpha + \beta}{2}. \quad (8)$$

Let us consider the case when the commutator of  $\eta$  and  $\eta^\dagger$  is a constant, that is,

$$[\eta, \eta^\dagger] = 2a(x)b'(x) - a(x)a''(x) = 1. \quad (9)$$

This is equivalent to the relation

$$b(x) = \int \frac{dx}{2a(x)} + \frac{a'(x)}{2}. \quad (10)$$

Then, the effective potential (8) is expressed solely in terms of  $a(x)$  as

$$V_{\text{eff}}(x) = a_1 \left( \int \frac{dx}{2a(x)} \right)^2 + a_2 \int \frac{dx}{2a(x)} - \frac{\tilde{\omega}}{4} \left( 2a(x)a''(x) + a'(x)^2 \right) + \lambda, \quad (11)$$

where  $a_1$ ,  $a_2$ , and  $\lambda$  denote the following constants:

$$a_1 = \frac{(\alpha - \beta)^2}{\tilde{\omega}} + \tilde{\omega} + 2\alpha + 2\beta, \quad (12)$$

$$a_2 = \frac{(\alpha - \beta)(\gamma - \delta)}{\tilde{\omega}} + \gamma + \delta, \quad (13)$$

$$\lambda = \frac{(\gamma - \delta)^2}{4\tilde{\omega}}. \quad (14)$$

The connection to PDM systems can now be identified by applying the following set of transformations

$$a(x) = \frac{1}{\sqrt{2\tilde{\omega}m(x)}}, \quad u(x) = \int dx \sqrt{m(x)}. \quad (15)$$

Written in terms of the equivalent Hermitian Hamiltonian  $h$  gives:

$$h = -\frac{d}{dx} \frac{1}{2m(x)} \frac{d}{dx} + \frac{\tilde{\omega}}{2} a_1 u(x)^2 + \sqrt{\frac{\tilde{\omega}}{2}} a_2 u(x) + \frac{m''(x)}{8m(x)^2} - \frac{7m'(x)^2}{32m(x)^3} + \lambda. \quad (16)$$

It readily follows from (16) that  $h$  has  $\mathcal{N}$ -fold SUSY. More precisely,  $h$  belongs to type A  $\mathcal{N}$ -fold SUSY PDM Hamiltonian, case I, discussed in [21].

### 3 Further Generalization

What happens if the commutator of  $\eta$  and  $\eta^\dagger$  is not a constant? To observe it, consider the general form of type A  $\mathcal{N}$ -fold SUSY PDM Hamiltonians given by [21]

$$H_{\mathcal{N}}^\pm = -\frac{d}{dx} \frac{1}{2m(x)} \frac{d}{dx} + V_{\mathcal{N}}^\pm(u) + \frac{m''(x)}{8m(x)^2} - \frac{7m'(x)^2}{32m(x)^3}, \quad (17)$$

where

$$V_{\mathcal{N}}^\pm = \frac{Q(z)^2}{2f'(u)^2} - \frac{\mathcal{N}^2 - 1}{24} \left( \frac{2f'''(u)}{f'(u)} - \frac{3f''(u)^2}{f'(u)^2} \right) \pm \frac{\mathcal{N}}{2} \left( \frac{f''(u)}{f'(u)^2} Q(z) - Q'(z) \right) - R \Big|_{z=f(u)}. \quad (18)$$

In (18),  $R$  is a constant,  $Q(z)$  is a polynomial of (at most) second degree in  $z$ , and  $f(u)$  is one of the following complex functions which characterize the different cases of type A  $\mathcal{N}$ -fold SUSY (cf. Section 6 and Table 1 in [21]):

$$u, \quad u^2, \quad e^{2\sqrt{\nu}u}, \quad \cosh 2\sqrt{\nu}u, \quad \wp(u). \quad (19)$$

Using (15), the type A  $\mathcal{N}$ -fold SUSY PDM Hamiltonian (17) acquires a form resembling a transformed oscillator model as follows:

$$H_{\mathcal{N}}^{\pm} = -\tilde{\omega} \frac{d}{dx} a(x)^2 \frac{d}{dx} + V_{\mathcal{N}}^{\pm}(u) - \frac{\tilde{\omega}}{4} (2a(x)a''(x) + a'(x)^2). \quad (20)$$

Comparing (20) with (6), we immediately see that the operator  $h$  is of type A  $\mathcal{N}$ -fold SUSY if only  $b(x)$  is a function of  $a(x)$  and  $u(x)$  such that  $V_{\text{eff}}$  depends on  $a(x)$  just as the last term in (20). This is shown to be possible if and only if  $b(x)$  is assigned the following form:

$$b(x) = B_0(u) + B_2(u)a'(x), \quad (21)$$

where  $B_0(u)$  and  $B_2(u)$  are for the moment arbitrary functions. Indeed substituting (21) in (8), one obtains

$$V_{\text{eff}}(x) = F_0(u) + F_1(u)a'(x) + F_2(u)a(x)a''(x) + F_3(u)a'(x)^2, \quad (22)$$

where

$$F_0(u) = a_1 B_0(u)^2 + a_2 B_0(u) - \frac{\tilde{\omega} + \alpha + \beta}{\sqrt{2\tilde{\omega}}} B_0'(u) + \lambda + \frac{\tilde{\omega} + \alpha + \beta}{2}, \quad (23)$$

$$F_1(u) = \left( a_1 B_0(u) + \frac{a_2}{2} \right) (2B_2(u) - 1) - \frac{\tilde{\omega} + \alpha + \beta}{\sqrt{2\tilde{\omega}}} B_2'(u), \quad (24)$$

$$F_2(u) = -(\tilde{\omega} + \alpha + \beta) B_2(u) + \frac{\alpha + \beta}{2}, \quad (25)$$

$$F_3(u) = a_1 \left( B_2(u)^2 - B_2(u) + \frac{1}{4} \right) - \frac{\tilde{\omega}}{4}. \quad (26)$$

We are thus led to a necessary condition for  $h$  to be of type A  $\mathcal{N}$ -fold SUSY summarized by the following set of equations:

$$F_1(u) = 0, \quad 2F_2(u) = 4F_3(u) = -\tilde{\omega}. \quad (27)$$

It is straightforward to observe that the non-trivial solution to (27) is

$$B_2(u) = \frac{1}{2}, \quad (28)$$

and we have the desired form

$$b(x) = B_0(u) + \frac{a'(x)}{2}. \quad (29)$$

In this case, the commutator of  $\eta$  and its transposition is not a constant but can be an arbitrary function of  $u(x)$ :

$$[\eta, \eta^\dagger] = 2a(x)b'(x) - a(x)a''(x) = \sqrt{\frac{2}{\tilde{\omega}}} B_0'(u). \quad (30)$$

If we put  $B_0(u) = \sqrt{\tilde{\omega}/2} u$ , we reproduce all the previous results, namely, (9)–(16). With the condition (29) satisfied, the equivalent Hermitian Hamiltonian operator  $h$  now becomes

$$h = -\tilde{\omega} \frac{d}{dx} a(x)^2 \frac{d}{dx} + F_0(u) - \frac{\tilde{\omega}}{4} (2a(x)a''(x) + a'(x)^2). \quad (31)$$

Finally, comparing with (20), we obtain the necessary and sufficient condition for the operator  $h$  to be type A  $\mathcal{N}$ -fold SUSY as

$$F_0(u) = a_1 B_0(u)^2 + a_2 B_0(u) - a_3 B_0'(u) + a_4 = V_{\mathcal{N}}^\pm(u), \quad (32)$$

where  $a_3$  and  $a_4$  are given by

$$a_3 = -\frac{\tilde{\omega} + \alpha + \beta}{\sqrt{2\tilde{\omega}}}, \quad a_4 = \lambda + \frac{\tilde{\omega} + \alpha + \beta}{2}. \quad (33)$$

To derive a solution to Eq. (32), it is convenient to convert it to a standard form of Riccati equation

$$\phi'(u) = a_1 a_3^{-2} \phi(u)^2 + a_2 a_3^{-1} \phi(u) - V_{\mathcal{N}}^\pm(u) + a_4, \quad (34)$$

by the substitution  $B_0(u) = a_3^{-1} \phi(u)$ . Linearizing (34) with the help of introducing  $\phi(u) = -a_3^2 \psi'(u) / a_1 \psi(u)$  gives

$$-\psi''(u) + \frac{a_2}{a_3} \psi'(u) + \frac{a_1}{a_3^2} (V_{\mathcal{N}}^\pm(u) - a_4) \psi(u) = 0. \quad (35)$$

Lastly, to eliminate the first derivative term, we make an ansatz

$$\psi(u) = \exp\left(\frac{a_2}{2a_3} u\right) \hat{\psi}(u), \quad (36)$$

which leads to

$$\left[ -\frac{1}{2} \frac{d^2}{du^2} + \frac{a_1}{2a_3^2} V_{\mathcal{N}}^\pm(u) - \frac{a_1 a_4}{2a_3^2} + \frac{a_2^2}{8a_3^2} \right] \hat{\psi}(u) = 0. \quad (37)$$

Equation (37) is nothing but a Schrödinger equation subject to the potential  $a_1 V_{\mathcal{N}}^\pm(u) / 2a_3^2$ .

An interesting possibility then emerges when  $a_1 = 2a_3^2$ , or equivalently from (12) and (33), when  $\alpha\beta = 0$ . In this case, the Schrödinger operator in (37)

itself has type A  $\mathcal{N}$ -fold SUSY (with the constant mass  $m = 1$ ). Since  $\mathcal{N}$ -fold SUSY is essentially equivalent to quasi-solvability [23], the operator is always quasi-solvable (and is solvable if it does not substantially depend on  $\mathcal{N}$ ), which means that we can obtain (at most)  $\mathcal{N}$  analytic (local) solutions to Eq. (37) in a closed form having the following representation:

$$\hat{\psi}(u) = P_{\mathcal{N}-1}(f(u))e^{-\mathcal{W}_{\mathcal{N}}(u)}, \quad (38)$$

where  $P_n$  is a polynomial of (at most)  $n$ th degree,  $\mathcal{W}_{\mathcal{N}}$  is a (generalized) superpotential, and  $f(u)$  is a function which characterizes one of the cases of type A  $\mathcal{N}$ -fold SUSY (see, Eq.(19)). From (37) and (38), we arrive at the following specific representation of  $B_0(u)$ :

$$B_0(u) = -\frac{a_3 \hat{\psi}'(u)}{a_1 \hat{\psi}(u)} - \frac{a_2}{2a_1} = -\frac{a_3}{a_1} \left[ \frac{f'(u)P'_{\mathcal{N}-1}(f(u))}{P_{\mathcal{N}-1}(f(u))} - \mathcal{W}'_{\mathcal{N}}(u) \right] - \frac{a_2}{2a_1}. \quad (39)$$

Here we note that since the Schrödinger equation (37) is just a differential equation but not an eigenvalue problem, the normalizability of the solutions including (38) is irrelevant.

It is also important to note that in any case of  $a_1 \neq 2a_3^2$  (or  $\alpha\beta \neq 0$ ) the Schrödinger operator in (37) is in general not quasi-solvable since any change of overall multiplicative factor of the potential  $V_{\mathcal{N}}^{\pm} \rightarrow \rho V_{\mathcal{N}}^{\pm}$  breaks quasi-solvability, the fact that is originated from the form invariance of the quasi-solvable potentials under the scale transformation  $u \rightarrow \rho u$  (cf. Section 4 in [25]). However, solvability of the operator remains under the transformation  $V_{\mathcal{N}}^{\pm} \rightarrow \rho V_{\mathcal{N}}^{\pm}$  since the effect can be absorbed by the rescaling of the variable  $u$  without violating solvability. Hence, an analytic solution of (37) is still obtainable when the polynomial  $Q(z)$  in (18) is at most first-degree in  $z$  for the cases I-IV of type A  $\mathcal{N}$ -fold SUSY. Note also that the formula (39) continues to be valid. It would be an interesting problem to examine whether we could obtain the complete list of possible explicit forms of  $B_0(u)$ .

## Acknowledgements

One of us (BB) gratefully acknowledges Prof. C.-H. Chen's hospitality during his visit at the National Cheng Kung University, Taiwan where this work was carried out. This work was partially supported by the National Cheng Kung University under the grant No. OUA:95-3-2-071 (TT).

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