A GENERAL SCHEME FOR THE EFFECTIVE-MASS SCHRÖDINGER EQUATION AND THE GENERATION OF THE ASSOCIATED POTENTIALS

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Abstract

A systematic procedure to study one-dimensional Schrödinger equation with a position-dependent effective mass (PDEM) in the kinetic energy operator is explored. The conventional free-particle problem reveals a new and interesting situation in that, in the presence of a mass background, formation of bound states is signalled. We also discuss coordinate-transformed, constant-mass Schrödinger equation, its matching with the PDEM form and the consequent decoupling of the ambiguity parameters. This provides a unified approach to many exact results known in the literature, as well as to a lot of new ones.

Running head: Effective-Mass Schrödinger Equation

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1 Introduction

In recent times, the concept of a position-dependent-effective-mass (PDEM) quantum Hamiltonian is rapidly gaining acceptance because of its increasing relevance in describing the motion of electrons in problems of compositionally graded crystals [1] (following our ability to fabricate semiconductor nanostructures), quantum dots [2], liquid crystals [3], etc. The appearance of PDEM is also well known in the energy density functional approach to the nuclear many-body problem [4] and its applications [5, 6] in the context of nonlocal terms of the accompanying potential. Other theoretical considerations where PDEM has been exploited include the derivation [7] of the underlying electron Hamiltonian from instantaneous Galilean invariance and implementation of the path integral techniques [8] to calculate the Green's function [9] for step and rectangular-barrier potentials and masses. Further, PDEM has proved to be appealing in the construction of acceptable quantum mechanical systems by seeking exact solutions of the Schrödinger equation [10, 11, 12, 13, 14, 15, 16, 17, 18] by extending the already existing methods of spectrum generating or potential algebras [19] and those of supersymmetric quantum mechanics [20, 21, 22, 23].

In this Letter we consider the PDEM formalism to address the issue of the generation of the corresponding potentials. We use the von Roos effective-mass kinetic energy operator [24], which has the advantage of an inbuilt Hermiticity, and we develop a PDEM scheme in very general terms that not only encompasses some of the previous ones but also brings out additional features not reported before.

First, we consider the problem of a free particle in the framework of a first-order intertwining relationship and show that in a suitable mass background it acts as a bound system.

Second, we observe that a Schrödinger equation with constant mass and having for its potential U can always be coordinate transformed in terms of some given mass function such that the resulting form matches with the one implied by PDEM. In this way the starting potential V of the PDEM scheme becomes expressible in terms of the coordinate-transformed U and the one induced by the effective kinetic energy operator $\hat{T}$. The latter is found to depend on two types of parameters: those entering the definition of the mass function and those coming from the ordering ambiguity of the momentum and mass operators.

Third, although it is clear that $\hat{T}$ provides a nontrivial contribution to the effective potential which a particle with a position-dependent mass experiences, it is not at all apparent how V stands out against the interplay of these ambiguity parameters vis-à-vis the ones of the coordinate-transformed U. In fact, as we shall see, in a PDEM problem the ambiguity parameters get decoupled in a natural way. This observation is new and of significance in that it allows V to become entirely identifiable with the coordinate-transformed U. The latter in the constant-mass limiting case goes over to a known form in the sense that it is either exactly solvable (ES) or quasi-exactly solvable (QES) or else conditionally exactly solvable (CES). We also discuss some consequences of our results.

2 General Strategy and the Free-particle Problem

The most commonly used effective-mass kinetic energy operator is the two-parameter form given by von Roos [24], which in one dimension reads

$$\hat{T} = \frac{1}{4}\left[m^{\alpha}(x)\,\hat{p}\,m^{\beta}(x)\,\hat{p}\,m^{\gamma}(x) + m^{\gamma}(x)\,\hat{p}\,m^{\beta}(x)\,\hat{p}\,m^{\alpha}(x)\right], \tag{1}$$

where $\hat{p}$ represents the momentum operator, $m(x)$ is the position-dependent mass and the parameters α, β, γ are subject to the condition $\alpha+\beta+\gamma=-1$. Because of the noncommutativity of the momentum and (position-dependent) mass operators, maintaining Hermiticity of the kinetic energy operator is not trivial. In (1), $\hat{T}$ has been specifically designed to be Hermitian but is by no means unique. However, it turns out that other plausible forms of $\hat{T}$ invariably reduce to one of the special cases of (1), apart from the fact that most of them are equivalent [1] up to a given level of accuracy.

On the other hand, analyses have been performed comparing experimental data with theoretical insights provided by various considerations such as, for example, the study of the envelope function for electrons in uniform or slowly graded crystals [1]. These, however, do not always reveal [25] precise and unambiguous estimates to the question of uniqueness of the parameters α, β, γ. Some of the appropriate single-band PDEM Hamiltonians are the ones of BenDaniel and Duke (BDD) [26] ($\alpha = 0$, $\beta = -1$), Bastard [27] ($\alpha = -1$, $\beta = 0$), Zhu and Kroemer (ZK) [28] ($\alpha = -\frac{1}{2}$, $\beta = 0$) and the redistributed model [29] ($\alpha = 0$, $\beta = -\frac{1}{2}$).

With the correspondence $\hat{p} \to -\mathrm{i}\hbar\frac{d}{dx}$ in (1), the time-independent Schrödinger equation reads

$$\begin{aligned} &\left\{-\frac{\hbar^2}{4}\left[m^\alpha(x)\frac{d}{dx}m^\beta(x)\frac{d}{dx}m^\gamma(x) + m^\gamma(x)\frac{d}{dx}m^\beta(x)\frac{d}{dx}m^\alpha(x)\right] + V(x)\right\}\psi(x) \\ &\quad = E\psi(x) \end{aligned} \tag{2}$$

for some given potential $V(x)$. On setting

$$m(x) = m_0 M(x), \tag{3}$$

where $M(x)$ is the dimensionless form of the mass function, along with $\hbar = 2m_0 = 1$, we can get rid of the ambiguity parameters by transferring them to the effective potential energy of the variable-mass system. Thus using the result

$$\begin{aligned} & M^\alpha \frac{d}{dx} M^\beta \frac{d}{dx} M^\gamma + M^\gamma \frac{d}{dx} M^\beta \frac{d}{dx} M^\alpha \\ & = \; 2\frac{d}{dx}\frac{1}{M}\frac{d}{dx} - (\beta+1)\frac{M''}{M^2} + 2[\alpha(\alpha+\beta+1)+\beta+1]\frac{M'^2}{M^3}, \end{aligned} \tag{4}$$

where a prime denotes derivative with respect to x, equation (2) acquires the form

$$H\psi(x) \equiv \left[-\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + V_{\rm eff}(x)\right]\psi(x) = E\psi(x), \tag{5}$$

in which the effective potential $V_{\rm eff}(x)$ is seen to depend on some mass terms:

$$V_{\rm eff}(x) = V(x) + \frac{1}{2}(\beta+1)\frac{M''}{M^2} - [\alpha(\alpha+\beta+1)+\beta+1]\frac{M'^2}{M^3}. \tag{6}$$

In the following, we shall be interested in bound-state eigenvalues E_n, $n = 0, 1, 2, \ldots$, and corresponding wavefunctions $\psi_n(x)$, $n = 0, 1, 2, \ldots$.

Let us consider the intertwining relationship

$$\eta H = H_1 \eta, \tag{7}$$

where H_1 has the same kinetic energy term as H and an associated potential $V_{1,\rm eff}(x)$. If the ground-state wavefunction ψ_0 of H is annihilated by the operator η, i.e., $\eta\psi_0 = 0$, the eigenvalues of H_1 are $E_{1,n} = E_{n+1}$, $n = 0, 1, 2, \ldots$, with corresponding wavefunctions $\eta\psi_{n+1}$, since from (7) it follows that $H_1(\eta\psi_{n+1}) = \eta H\psi_{n+1} = E_{n+1}(\eta\psi_{n+1})$ for $n = 0, 1, 2, \ldots$.

Choosing a first-derivative intertwining operator $\eta = A(x)\frac{d}{dx} + B(x)$ in (7), we are led to the restrictions

$$\begin{aligned} A(x) &= M^{-1/2}, & (8)\\ V_{\rm eff}(x) &= \lambda + B^2 - (AB)', & (9)\\ V_{1,\rm eff}(x) &= V_{\rm eff} + 2AB' - AA'', & (10) \end{aligned}$$

where λ denotes some integration constant.

It is instructive to consider the free-particle case $V(x) = V_0$ of (6). On comparing with (9), where we choose $\lambda = V_0$ and

$$B(x) = -\frac{1}{2}(\beta + 1)\frac{M'}{M^{3/2}}, \tag{11}$$

we get a constraint relation $\beta = -2\alpha - 1$ and equation (6) is found to be consistent with the PDEM Hamiltonians of BDD ($\alpha = 0$, $\beta = -1$) and ZK ($\alpha = -\frac{1}{2}$, $\beta = 0$). Specifically, $V_{\rm eff}$ and $V_{1,\rm eff}$ read

$$\begin{aligned} V_{\rm eff}(x) &= V_0 - \alpha\frac{M''}{M^2} + \alpha(\alpha+2)\frac{M'^2}{M^3}, & (12)\\ V_{1,\rm eff}(x) &= V_0 + \left(\alpha + \frac{1}{2}\right)\frac{M''}{M^2} + \left(\alpha + \frac{1}{2}\right)\left(\alpha - \frac{3}{2}\right)\frac{M'^2}{M^3}, & (13) \end{aligned}$$

which are interchanged under the transformation $\alpha \to -(\alpha + \frac{1}{2})$. In particular, for $\alpha = 0$,

$$V_{\rm eff}(x) = V_0, \qquad V_{1,\rm eff}(x) = V_0 + \frac{1}{2}\frac{M''}{M^2} - \frac{3}{4}\frac{M'^2}{M^3}, \tag{14}$$

while for $\alpha = -\frac{1}{2}$,

$$V_{\text{eff}}(x) = V_0 + \frac{1}{2}\frac{M''}{M^2} - \frac{3}{4}\frac{M'^2}{M^3}, \qquad V_{1,\text{eff}}(x) = V_0, \tag{15}$$

suggesting a duality between the BDD and ZK schemes. It is to be stressed that the results (8) – (15) are independent of any choice of $M(x)$.

To proceed further with the effective-mass Schrödinger equation (5) we need to have a precise example for $M(x)$. We can take for instance a deformed hyperbolic function

$$M(x) = \operatorname{sech}^2 qx, \qquad q > 0, \tag{16}$$

which depicts a solitonic profile. Another acceptable form for $M(x)$ is

$$M(x) = \left(1 + \frac{q}{1+x^2}\right)^2, \qquad q > 0, \tag{17}$$

which has yielded interesting connections [19] with the su(1,1) algebra. In (16) or (17), q may be treated as a deformation parameter so that when $q \to 0$, $M(x) \to 1$ in both the cases.

Let us take for concreteness the result (15), which is in conformity with the ZK scheme. We can write for the counterpart of (5) for H_1

$$-\frac{d}{dx}\left(\frac{1}{M}\frac{d\varphi_n}{dx}\right) = E'_{1,n}\varphi_n, \tag{18}$$

where $E'_{1,n} = E_{1,n} - V_0 = E_{n+1} - V_0 = E'_{n+1}$ and $\varphi_n(x) = \eta\psi_{n+1}(x)$ are the eigenfunctions of H_1. Using the specific example (16) for $M(x)$ and taking (8) and (11) into account, we can write these eigenfunctions as $\varphi_n(x) = \frac{d}{dx}(\cosh qx\, \psi_{n+1})$. Inserting this expression in (18), integrating once the resulting equation, changing $n+1$ into n and rewriting the result in terms of $\chi_n(t) = \cosh qx\, \psi_n(x)$, where $t = \tanh qx$, we find

$$(1-t^2)\frac{d^2\chi_n}{dt^2} - 2t\frac{d\chi_n}{dt} + \frac{E'_n}{q^2}\chi_n = 0. \tag{19}$$

Such an equation coincides with the equation for Legendre polynomials $\chi_n(t) = P_n(t)$, $n = 0, 1, 2, \ldots$, provided $E'_n = q^2 n(n+1)$.[1] From this, it follows that

[1] Actually the general solutions ${}_2F_1(-\nu, \nu+1; 1; (1-\tanh qx)/2)$ of Legendre's differential equation (19), corresponding to $E'_\nu = q^2\nu(\nu+1)$ ($\nu \neq n$), being not convergent for $x \to -\infty$, the polynomial solutions are the only acceptable ones.

apart from some normalization factors, $\psi_n(x) \propto \operatorname{sech} qx P_n(\tanh qx)$ and $\varphi_n(x) \propto \frac{d}{dx} P_{n+1}(\tanh qx)$. We therefore get for the first few normalizable solutions of H and H_1,

$$\begin{aligned}
\psi_0(x) &\propto \operatorname{sech} qx, \qquad E'_0 = 0, \\
\psi_1(x) &\propto \operatorname{sech} qx \tanh qx, \qquad \varphi_0(x) \propto \operatorname{sech}^2 qx, \qquad E'_1 = E'_{1,0} = 2q^2, \\
\psi_2(x) &\propto \operatorname{sech} qx (1 - \tfrac{3}{2} \operatorname{sech}^2 qx), \qquad \varphi_1(x) \propto \operatorname{sech}^2 qx \tanh qx, \\
& E'_2 = E'_{1,1} = 6q^2, \\
\psi_3(x) &\propto \operatorname{sech} qx \tanh qx (1 - \tfrac{5}{2} \operatorname{sech}^2 qx), \qquad \varphi_2(x) \propto \operatorname{sech}^2 qx (1 - \tfrac{5}{4} \operatorname{sech}^2 qx), \\
& E'_3 = E'_{1,2} = 12q^2, \\
\psi_4(x) &\propto \operatorname{sech} qx (1 - 5 \operatorname{sech}^2 qx + \tfrac{35}{8} \operatorname{sech}^4 qx), \qquad \varphi_3(x) \propto \operatorname{sech}^2 qx \tanh qx \\
& \times (1 - \tfrac{7}{4} \operatorname{sech}^2 qx), \qquad E'_4 = E'_{1,3} = 20q^2.
\end{aligned} \tag{20}$$

We conclude that a free particle placed in an appropriate mass background generates bound states in a manner as given above.

3 Coordinate Transformation

Equation (5) can be interpreted as a coordinate-transformed Schrödinger equation with the coordinate dependence arising from a suitable choice of mass function $M(x)$. To this end, let us consider a time-independent Schrödinger equation with constant mass under the action of a potential U,

$$\left[-\frac{d^2}{dy^2} + U(y; a) \right] \phi(y) = \epsilon \phi(y), \tag{21}$$

where a denotes collectively a set of coupling parameters, which may be present in U, and ϵ is the energy.

The change of variable

$$y = \lambda z(x) + \nu, \qquad \lambda, \nu \in \mathbb{R}, \tag{22}$$

$$z(x) = \int^x \sqrt{M(x')}\, dx', \tag{23}$$

transforms equation (21) into

$$\left[-\frac{1}{\sqrt{M}}\frac{d}{dx}\frac{1}{\sqrt{M}}\frac{d}{dx}+\lambda^2 U\Big(\lambda z(x)+\nu;a\Big)\right]\chi(x)=\lambda^2\epsilon\chi(x), \tag{24}$$

where $\chi(x)\equiv\phi(y(x))$.

To compare (24) with (5), one needs to write the term containing the second-order derivative in the former to transform in the same way as in the latter. For such a purpose, we exploit the property $\frac{1}{\sqrt{M}}\frac{d}{dx}=\frac{d}{dx}\frac{1}{\sqrt{M}}+\frac{1}{2}\frac{M'}{M^{3/2}}$ and substitute $\chi(x)=M^{-1/4}(x)\psi(x)$ to reset (24) as

$$\left[-\frac{d}{dx}\frac{1}{M}\frac{d}{dx}+\frac{M''}{4M^2}-\frac{7M'^2}{16M^3}+\lambda^2 U\Big(\lambda z(x)+\nu;a\Big)\right]\psi(x)=\lambda^2\epsilon\psi(x). \tag{25}$$

Equations (5) and (25) are seen to coincide (with $E=\lambda^2\epsilon$) provided V_{eff} is identified as

$$V_{\text{eff}}(x)=\lambda^2 U\Big(\lambda z(x)+\nu;a\Big)+\frac{M''}{4M^2}-\frac{7M'^2}{16M^3}. \tag{26}$$

From (6) and (26), we thus arrive at an important result

$$V(x)=V_1(x;\alpha,\beta)+V_2(x;a), \tag{27}$$

where $V_1(x;\alpha,\beta)$ and $V_2(x;a)$ are

$$V_1(x;\alpha,\beta) = \left[\alpha(\alpha+\beta+1)+\beta+\frac{9}{16}\right]\frac{M'^2}{M^3}-\frac{1}{4}(2\beta+1)\frac{M''}{M^2}, \tag{28}$$

$$V_2(x;a) = \lambda^2 U\Big(\lambda z(x)+\nu;a\Big). \tag{29}$$

We see at once that for any solution $(\epsilon,\phi(y))$ of the constant-mass Schrödinger equation, we can derive a corresponding one $(E,\psi(x))$ of the effective-mass Schrödinger equation with $E=\lambda^2\epsilon$ and the potential $V(x)$ given by the set (27) – (29).

4 Analysis of $V(x)$

Of the two parts of $V(x)$, one involves the ambiguity parameters of the $\hat{T}$ operator, which we call $V_1(x;\alpha,\beta)$, while the other (i.e. $V_2(x;a)$) is a coordinate-transformed piece obtained as a result of coordinate transforming U according to (22) and (23).

4.1 The potential $V_1(x;\alpha,\beta)$

Unrestricted α, β parameters are often of much use to study the general class of solutions for the PDEM Schrödinger equation and to understand their role when the system undergoes a transition from a smooth potential and mass step to an abrupt situation [11]. In spite of this, an interesting question is whether the presence of V_1 can be eliminated from V for special choices of α, β and/or the mass function $M(x)$. The answer turns out to be in the affirmative as our analysis below will presently confirm.

Writing $V_1(x;\alpha,\beta)$ as

$$V_1(x;\alpha,\beta) = f(\alpha,\beta)\frac{M'^2}{M^3} - g(\beta)\frac{M''}{M^2}, \tag{30}$$

where $f(\alpha,\beta) = \left(\alpha+\frac{1}{4}\right)^2 + \frac{1}{2}(\alpha+1)(2\beta+1)$, $g(\beta) = \frac{1}{4}(2\beta+1)$, we notice that V_1 vanishes for $\beta = -\frac{1}{2}$ and $\alpha = -\frac{1}{4}$, corresponding to any smooth mass function $M(x)$. The value $\beta = -\frac{1}{2}$ is consistent with the redistributed model [29], which is a special case of (1), as noted earlier. Although the accompanying zero value of α is different from $-\frac{1}{4}$, the PDEM kinetic energy operators are known [30, 31] to be equivalent over a wide range of α values for suitable width of the transition region.

For $\beta \neq -\frac{1}{2}$, the vanishing of $V_1(x;\alpha,\beta)$ also takes place for the following mass dependence

$$f \neq g: \qquad M \sim x^{\xi}, \qquad \xi = \left(1 - \frac{f}{g}\right)^{-1}, \tag{31}$$

$$f = g: \qquad M \sim e^{\pm kx}, \qquad k \in \mathbb{R}^+, \tag{32}$$

i.e., either for power and inverse power laws or for exponentially rising and falling masses. For the kinetic energy operators of BDD [26], Bastard [27], and ZK [28], ξ turns out to be $-\frac{4}{3}$, $-\frac{4}{5}$, -4, respectively. In such cases, $M(x)$ is singular for $x \to 0$. In general (i.e., notwithstanding the vanishing of V_1), the case $f = g$ is disfavoured by the above schemes because the underlying constraint equation $\left(\alpha+\frac{1}{4}\right)^2 + \frac{1}{2}\left(\alpha+\frac{1}{2}\right)(2\beta+1) = 0$ is not satisfied by the corresponding parameter values. However, exponential dependence on the position coordinate of the mass has been considered [22] viable in semiconductor quantum well structures and also in problems of exact solvability in the supersymmetric context.

4.2 The potential $V_2(x; a)$

The comprehensive nature of our scheme enables us to obtain rather straightforwardly the relevant forms of V_2 should U for the constant-mass Schrödinger equation be available. To show how our scheme works in practice, we consider the following illustration of the Scarf I potential, which is well known to be ES on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\begin{aligned} U(x; A, B) &= [A(A-1) + B^2] \sec^2 x - B(2A-1) \sec x \tan x - A^2, \\ &\quad 0 < B < A - 1, \end{aligned} \tag{33}$$

$$\epsilon_n = (A+n)^2 - A^2, \qquad n = 0, 1, 2, \ldots. \tag{34}$$

Using (23), (29) and say, for instance, (17), we easily find that in the presence of mass deformation the argument x in (33) is modified according to $x \to \lambda z(x) + \nu$, $z(x) \equiv x + q \tan^{-1} x$. As a result, $V_2(x; A, B)$ can be written as

$$V_2(x; A, B) = [A(A-1) + B^2] \sec^2[z(x)] - B(2A-1) \sec[z(x)] \tan[z(x)] - A^2, \tag{35}$$

where along with $\nu = 0$, we have set $\lambda = 1$ so that the energy levels are the same as given by (34), i.e., $E_n = \epsilon_n$.

A graphical representation of (35) is shown in figure 1, where we have considered the effects of q-deformed mass function (17) for various q values and compared them with the undeformed (i.e. $q = 0$) case (33). It is to be noticed that $V_2(x; A, B)$ is not singular provided

$$-\frac{1}{q}\left(\frac{\pi}{2} + x\right) < \tan^{-1} x < \frac{1}{q}\left(\frac{\pi}{2} - x\right). \tag{36}$$

Hence it is defined on the interval $(-x_q, x_q)$, where $x_q \in \left(0, \frac{\pi}{2}\right)$ is the solution of the transcendental equation $\tan^{-1} x_q = \frac{1}{q}\left(\frac{\pi}{2} - x_q\right)$. As q increases, x_q decreases from

$x_0 = \frac{\pi}{2}$ to $x_\infty = 0$. One finds, for instance, $x_{1/10} \simeq 1.47335$, $x_{1/2} \simeq 1.14446$, and $x_1 \simeq 0.860334$ for the curves displayed in figure 1.

A similar procedure can be used by taking as inputs the ES potentials known in quantum mechanics [32, 33] or generalizations thereof. In this way, we can get in a unified way many of the exact results that have been derived elsewhere by some ad hoc techniques [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

As two new examples, let us mention

- Kratzer potential

$$\begin{aligned} V_2(x;\gamma) &= 4\gamma^2 - \frac{3}{16z^2(x)} - \frac{\gamma}{z(x)}, \qquad \gamma > 0, \\ E_n &= 4\gamma^2 - \frac{4\gamma^2}{(2n+1)^2}, \qquad n = 1, 3, 5, \ldots, \end{aligned}$$

- Symmetric QES potential

$$\begin{aligned} V_2(x;A) &= \frac{1}{4}A^2\sinh^2[z(x)] - A\cosh[z(x)] + \frac{1}{2}A + \frac{1}{4}, \qquad A > 0, \\ E_0 &= 0, \qquad E_1 = A, \end{aligned}$$

where we have taken $\nu = 0$ for simplicity and we have fixed $\lambda = 1$ without any loss of generality.

We remark that one of the couplings in Kratzer is fixed at $-\frac{3}{16}$ and so not free. This entitles it to be classified as a CES potential. Further we observe that the run of n is restricted to odd values only because physically acceptable wavefunctions then have the correct $x^{3/4}$ behaviour at the origin (for a discussion of wavefunction behaviour for strongly singular potentials see [34, 35] and references quoted therein). The symmetric QES potential [36], on the other hand, is a special case of Razavy potential [37] with two known eigenstates. It is needless to say that broader classes of potentials, such as the ones obtained in [38, 39, 40, 41, 42, 43, 44], may also be mass deformed following the approach prescribed in this paper.

5 Conclusion

We studied in this Letter a PDEM quantum Hamiltonian in one dimension guided by the kinetic energy operator $\hat{T}$ of von Roos. The free-particle problem is analyzed first in the presence of a sech^2-mass background and generation of bound states is noted by exploiting a first-order intertwining relationship. The non-trivial nature of this result should be stressed. In a second step, the accompanying potential $V(x)$ of the PDEM Hamiltonian is found to reduce to two terms — one involving the ambiguity parameters of $\hat{T}$ and the other emerging from the coordinate transformation of the constant-mass Schrödinger equation. The advantage of our scheme is that, for a given mass function, a knowledge of some constant-mass Schrödinger potential allows a full access to the associated potential in the effective-mass Hamiltonian. This provides a unified treatment of all mass-deformed potentials corresponding to ES, QES or CES potentials known in the constant-mass case. We illustrated our results by an appropriate choice of the mass function as applied to the well-known Scarf I potential.

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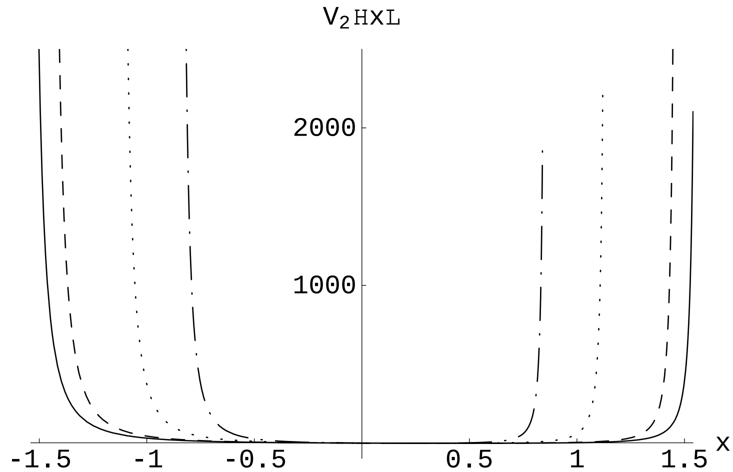
Figure Caption

Fig. 1. Comparison between the potential $V_2(x;3,1)$ of Eq. (35) for $q = 0.1$ (dashed line), $q = 0.5$ (dotted line), or $q = 1$ (dot-dashed line) and the Scarf I potential $U(x;3,1)$ (solid line).



V2HxL
2000
1000
-1.5
-1
-0.5
0.5
1
1.5
x